



A NEW APPROACH OF THE FERMAT-TORRICELLI CONFIGURATION (III)

TEMISTOCLE BÎRSAN

ABSTRACT. We continue the study on the *Fermat-Torricelli configuration* started in [2] and continued in [3]. Given a triangle ABC , instead of the six equilateral triangles constructed (inside or outside) on its sides, we consider three equilateral triangles AA_1A_2 , BB_1B_2 , CC_1C_2 with the same orientation as the given triangle and such that $A_1, A_2 \in BC$, $B_1, B_2 \in CA$, $C_1, C_2 \in AB$ (Fig. 1). The close connection of the circle $\mathcal{C}(H, R)$ with this configuration is highlighted. Two points, P and F^* , are considered on the Fermat axis F_+F_- , and several properties of them in relation to the configuration are indicated. The role of the line HF^* is also shown. The coordinates of the points P and F^* , and the equation of the line HF^* in homogeneous barycentric coordinates are found.

This work continues the study initiated in [2] and [3]. Within the Fermat-Torricelli configuration, some new geometric objects are introduced and studied. More precisely, the connection of the circle $\mathcal{C}(H, R)$ with the Fermat-Torricelli configuration is studied and, on the other hand, two points, namely P and F^* , are defined and studied.

1. NOTATION AND PRELIMINARIES

Consider an arbitrary triangle ABC . Let A_+, A_- be the apexes of the equilateral triangles built on the side BC outside and inside, respectively; define B_+, B_- and C_+, C_- cyclically. The *classic Fermat-Torricelli configuration* starts with points $A, B, C, A_+, A_-, B_+, B_-, C_+, C_-$. The circumcircles of equilateral triangles A_+BC and A_-BC are denoted by $\mathcal{T}_+^a$ and $\mathcal{T}_-^a$ – *Torricelli circles* relative to side BC ; define the Torricelli circles $\mathcal{T}_+^b, \mathcal{T}_-^b, \mathcal{T}_+^c, \mathcal{T}_-^c$ cyclically. On this subject there is a vast specialized literature - treatises, textbooks and papers: [1], [6], [5], [10], [7], etc.

In [2] and [3] new properties of this configuration are highlighted starting from another six-point system: $A, B, C, A_1, A_2, B_1, B_2, C_1, C_2$, where $A_1, A_2 \in BC$ so that triangle AA_1A_2 is equilateral and with the same orientation as the given triangle, and $B_1, B_2; C_1, C_2$ are defined similarly (Fig 1). In the new approach to the Fermat-Torricelli configuration, in addition to the starting triangles $AA_1A_2, BB_1B_2, CC_1C_2$, an important role is played by the triangles $AB_2C_1, BC_2A_1, CA_2B_1$ and their duals $AB_1C_2, BC_1A_2, CA_1B_2$. Also, we denote by O_a, O_b, O_c the centers of the equilateral triangles $AA_1A_2, BB_1B_2, CC_1C_2$, respectively; $O_aO_bO_c$ is called the *orthocentroidal triangle* of the given one.

We recall some simple properties of the Fermat-Torricelli configuration established in [2] that will be useful in what follows. They can be easily proven by the reader.

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1.1 each of the triangles AB_2C_1 , BC_2A_1 , CA_2B_1 ; AB_1C_2 , BC_1A_2 , CA_1B_2 is *inversely similar* to the triangle ABC , and two of them are *directly similar* ([2], Proposition 4.1);

1.2 the triangles ABC and $O_aO_bO_c$ are *inversely similar*;

1.3 the points A, H, O_a are *collinear* ([2], Corollary 4.1);

1.4 the lines B_2C_1 and B_1C_2 are *parallel*, and each of them is *antiparallel* to BC .

Properties similar to 1.3 and 1.4 are also true for the sides CA and AB .

The position of points $A_1, A_2, B_1, B_2, C_1, C_2$, on the sides depends on the shape of triangle ABC . Thus, to prove a statement it is often necessary to consider several cases regarding the given triangle. Since the arguments used for one case can be adapted to the other cases, we will limit ourselves to proving only one, the one that appears in the associated figure. We can assume, without loss of generality, that $A > B > C$. In [2] and [3] the cases that influence the shape of the triangle ABC and implicitly the Fermat-Torricelli configuration are discussed.

Hereafter, the circumcircle of a triangle XYZ will be denoted (XYZ) . Denote by $\mathcal{C}$ the circumcircle of triangle ABC and Δ its area. *Conway's symbols* ([8], [11], [13]) are frequently used.

2. TRIANGLES AA_1A_2 , BB_1B_2 , CC_1C_2 AND CIRCLE $\mathcal{C}(H, R)$

Consider the side lines of triangles AA_1A_2 , BB_1B_2 and CC_1C_2 , different from the side lines of the given triangle, namely AA_1 , AA_2 , BB_1 , BB_2 and CC_1 , CC_2 . The number of intersection points of these lines, different from the vertices of the given triangle, is twelve. In this section we will highlight the properties for only six of them; the properties of the remaining points will be studied in the next section.

Let's consider the following points (Fig. 1):

$$A' = BB_1 \cap CC_1, B' = CC_1 \cap AA_1, C' = AA_1 \cap BB_1; \quad (2.1)$$

$$A'' = BB_2 \cap CC_2, B'' = CC_2 \cap AA_2, C'' = AA_2 \cap BB_2. \quad (2.2)$$

We will see that there is a close connection between these points and the circle $\mathcal{C}(H, R)$.

The position of these points is specified in the following two sentence.

Proposition 2.1. *We have: (i) $B', C'' \in (AB_2C_1)$, $C', A'' \in (BC_2A_1)$, $A', B'' \in (CA_2B_1)$; (ii) (dual) $B'', C' \in (AB_1C_2)$, $C'', A' \in (BC_1A_2)$, $A'', B' \in (CA_1B_2)$ (Fig. 2).*

Proof. We will only prove the statement $B', C'' \in (AB_2C_1)$; the others are proven in the same way.

In triangle $A_1B'C$ we have: $\widehat{B'} = \widehat{A_1B'C} = \pi - \widehat{B'A_1C} - \widehat{B'CA_1} = \pi - \frac{\pi}{3} - \widehat{B'CA_1} = \frac{2\pi}{3} - \widehat{C_1CB}$.

But, in triangle C_1CB we have: $\widehat{C_1CB} = \pi - \widehat{BC_1C} - \widehat{CBC_1} = \pi - \frac{\pi}{3} - B = \frac{2\pi}{3} - B$. So, we obtain:

$$\widehat{B'} = \frac{2\pi}{3} - \left(\frac{2\pi}{3} - B \right) = B, \text{ that is } \widehat{B'} = B. \text{ Then, } \widehat{AB'C_1} = \pi - B. \text{ This and the fact that } \widehat{AB_2C_1} = B$$

(Section 1, (1.1) or [2], Proposition 4.1, (i)) leads to $\widehat{AB'C_1} + \widehat{AB_2C_1} = \pi$. Therefore, the quadrilateral $AB'C_1B_2$ is cyclic and hence $B' \in (AB_2C_1)$.

On the other hand, in triangle $BC''A_2$ we have: $\widehat{C''} = \pi - \frac{\pi}{3} - \widehat{A_2BC''} = \frac{2\pi}{3} - \widehat{A_2BB_2} = \frac{2\pi}{3} - \widehat{CBB_2}$.

Since $\widehat{CBB_2} = \pi - \frac{\pi}{3} - C$ (triangle CBB_2), we obtain $\widehat{C''} = C$. Because $\widehat{AC_1B_2} = C$ (Section 1, (1.1)),

we obtain $\widehat{C''} = \widehat{AC_1B_2}$, i.e. $\widehat{AC''B_2} = \widehat{AC_1B_2}$. So, $C'' \in (AB_2C_1)$. $\square$

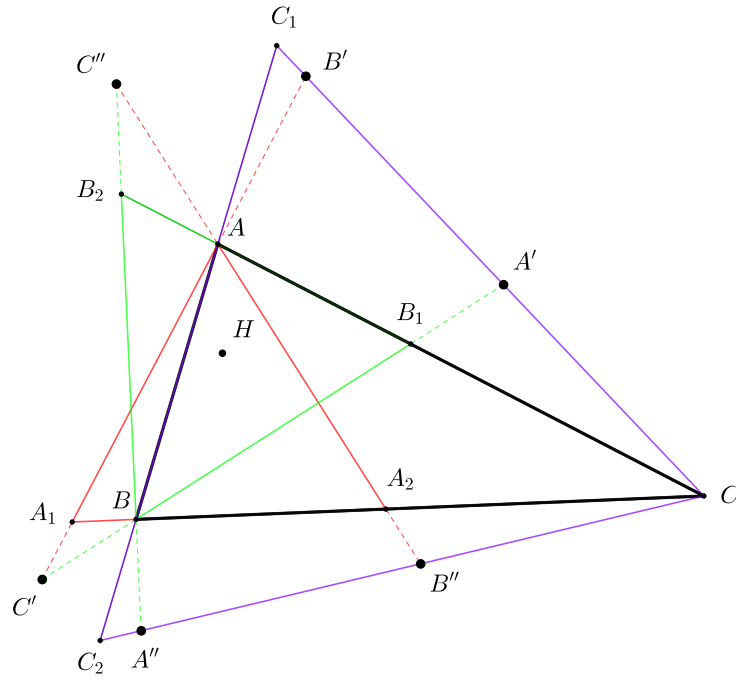


Fig. 1 Triangles AA_1A_2 , BB_1B_2 , CC_1C_2 and points $A', A'', B', B'', C', C''$

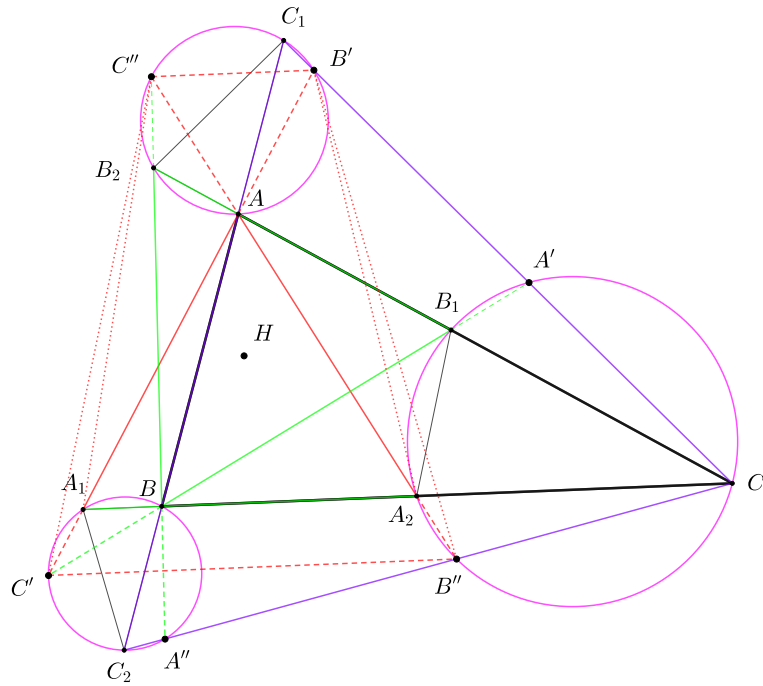


Fig. 2 Points $A', A'', B', B'', C', C''$ and circles (AB_2C_1) , (BC_2A_1) , (CA_2B_1)

Proposition 2.2. *The following statements are true: (i) triangles $AB'C''$, $BC'A''$ and $CA'B''$ are equilateral;*

(ii) (dual) triangle $AB''C'$, $BC''A'$ and $CA''B'$ are equilateral (Fig. 2).

Proof. (i) First, we have: $\widehat{B'AC''} = \frac{\pi}{3}$ (because triangle AA_1A_2 is equilateral). In the circle (AB_2C_1) , we have: $\widehat{B'C''A} = \widehat{B'C_1A} = \widehat{CC_1C_2} = \frac{\pi}{3}$ (because triangle CC_1C_2 is equilateral). Thus, $\widehat{B'AC''} = \widehat{B'C''A} = \frac{\pi}{3}$ and, therefore, triangle $AB'C''$ is equilateral.

(ii) Now, let's show that triangle $AB''C'$ is equilateral. The powers of A with regard to the circles (BC_2A_1) and $(BC_2B''A_2)$ are $AA_1 \cdot AC' = AB \cdot AC_2$ and $AB \cdot AC_2 = AA_2 \cdot AB''$, respectively. Then, $AA_1 \cdot AC' = AA_2 \cdot AB''$ and, therefore, the triangles AA_1A_2 and $AB''C'$ are similar. So, $AB''C'$ is equilateral. $\square$

Remark 2.1. *The previous proposition is a more complete form of Lemma 5.1 in [2]. According to this proposition, the six points considered at the beginning of the section: $A', A'', B', B'', C', C''$ determine six equilateral triangles associated with the given triangle.*

Corollary 2.1. *We have: (i) $BC \parallel B'C'' \parallel C'B''$;*

(ii) quadrilaterals $B'C''A_1A_2$, $B'C''C'B''$, $A_1A_2B''C'$ are isosceles trapezoids, and the analogues of these statements (Fig. 2).

Proof. From Proposition 2.2, (i) or (ii), it immediately follows the statement (i) of the corollary and the fact that the quadrilaterals $B'C''A_1A_2$, $B'C''C'B''$, $A_1A_2B''C'$ are trapezoids. It is easy to show that these trapezoids have equal diagonals, so they are isosceles. $\square$

Next, we will see that the points $A', A'', B', B'', C', C''$ are concyclic and we will specify who is the circle they are on.

Proposition 2.3. *We have $A', A'', B', B'', C', C'' \in \mathcal{C}(H, R)$ (the circle with center H and radius R).*

Proof. According to Corollary 2.1, (ii), quadrilaterals $B'C''C'B''$ and $C'A''A'C''$ are isosceles trapezoids, so they are cyclic (Fig. 3). Let us show that the circles $(B'C''C'B'')$ and $(C'A''A'C'')$ coincide. They have in common the points C' and C'' . It suffices to show that the point A' is also common to them. Indeed, since $A'C'' \parallel AC$ (Corollary 2.1, (i)), we obtain that $\widehat{C'A'C''} = \widehat{C'B_1A}$, so $\widehat{C'A'C''} = \widehat{BB_1A} = \widehat{BB_1B_2} = \frac{\pi}{3}$. Similarly, from $C'B'' \parallel BC$ it follows that $\widehat{C'B''C''} = \widehat{BA_2C''} = \widehat{BA_2A} = \frac{\pi}{3}$. Combining, we find that $\widehat{C'A'C''} = \widehat{C'B''C''} = \frac{\pi}{3}$, that is C', C'', A', A'' are concyclic. Then, the six points are concyclic.

Now, let us show that the circumcenter of the hexagon determined by these six points is H . Indeed, since $B'C''C'B''$ is an isosceles trapezoid (Corollary 2.1, (ii)), the chords $B'C''$ and $C'B''$ have the same perpendicular bisector - the axis d_a of the trapezoid. According to Proposition 2.2, triangles $AB'C''$, $AB''C'$ are equilateral and $C'B'' \parallel BC$; so, the points A and H lie on d_a . Similarly, the axes d_b and d_c of the trapezoids $C'A''A'C''$ and $A'B''A''B'$ pass through H .

All that remains is to find the radius of the hexagon's circumcircle. For this, we apply the cosine formula to triangle HAB' (Fig. 3):

$$(HB')^2 = HA^2 + (AB')^2 - 2AH \cdot AB' \cos \widehat{HAB'}.$$
 (2.3)

But we have

$$HA = 2R \cos A,$$
 (2.4)

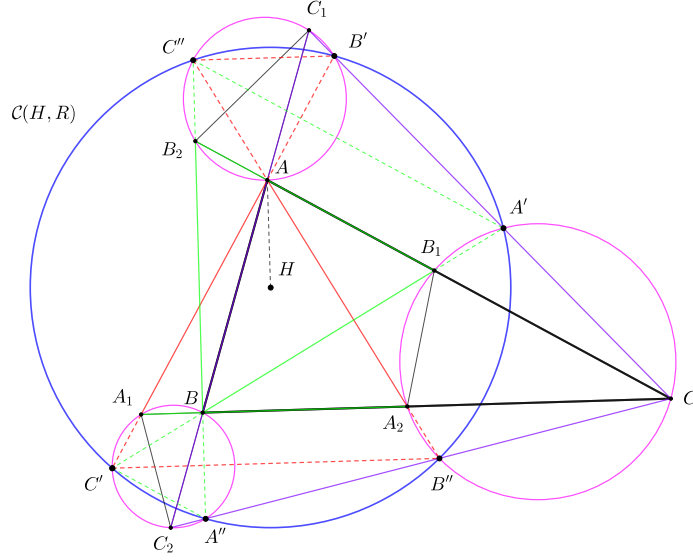


Fig. 3 Points $A', A'', B', B'', C', C''$ and circle $\mathcal{C}(H, R)$

$$\cos \widehat{HAB'} = -\cos \widehat{A_1AH} = -\cos \frac{\pi}{6} = -\frac{\sqrt{3}}{2}, \quad (2.5)$$

and also, taking into account the formula $r_+^a = \frac{2\sqrt{3}}{3}R \sin\left(A - \frac{\pi}{3}\right)$ for the radius r_+^a of the circle (AB_2C_1) ([2], Proposition 4.7),

$$\begin{aligned} AB' &= 2r_+^a \sin \frac{\pi}{3} = \sqrt{3}r_+^a = 2R \sin\left(A - \frac{\pi}{3}\right) \\ &= R\left(\sin A - \sqrt{3} \cos A\right). \end{aligned} \quad (2.6)$$

Using (2.4), (2.5), (2.6), equation (2.3) is rewritten as

$$\begin{aligned} (HB')^2 &= R^2 \left[4\cos^2 A + \left(\sin A - \sqrt{3} \cos A\right)^2 + 2\sqrt{3} \cos A \left(\sin A - \sqrt{3} \cos A\right) \right] \\ &= R^2, \end{aligned}$$

from where $HB' = R$. The proof is complete. $\square$

Proposition 2.4. *Triangles ABC , $A'B'C'$, $A''B''C''$ are equal and have the same orientation (Fig. 3).*

Proof. First we will show that these triangles are similar. We do this only for triangles ABC and $A'B'C'$. Indeed, we have $\widehat{B'} = \widehat{C'B'A'} = \pi - \widehat{AB'C_1}$. On the other hand, in the circle (AB_2C_1) we have $\widehat{AB'C_1} = \pi - \widehat{AB_2C_1}$. Hence $\widehat{B'} = \widehat{AB_2C_1}$. But, according to (Section1,(1.1)), $\widehat{AB_2C_1} = B$. Therefore, $\widehat{B'} = B$. Similarly, in the circle (BC_2A_1) , we have $\widehat{C'} = \widehat{A_1C'B} = \widehat{A_1C_2B} = C$, from where $\widehat{C'} = C$. Since $\widehat{B'} = B$ and $\widehat{C'} = C$, it follows that triangles $A'B'C'$ and ABC are similar. Triangles ABC , $A'B'C'$, $A''B''C''$ are equal because they are similar and are inscribed in the equal circles $\mathcal{C}(O, R)$ and $\mathcal{C}(H, R)$. $\square$

Lemma 2.1. *Quadrilaterals ABA_2B_1 , ABA_1B_2 and $ABC'C''$ are cyclic (Fig. 4).*

Proof. Obviously, the first two are inscribed in the Torricelli circles $\mathcal{T}_-^c$ and $\mathcal{T}_+^c$, respectively. For $ABC'C''$ to be cyclic, it suffices to show that $\widehat{AC'B} = \widehat{AC''B}$. But $\widehat{AC'B} = \widehat{B'C'A'}$ and because $\widehat{B'C'A'} = C$ (Proposition 2.4), it follows that $\widehat{AC'B} = C$. Similarly, $\widehat{AC''B} = \widehat{B''C''A''} = C$. So, we obtain $\widehat{AC'B} = \widehat{AC''B} = C$, and the quadrilateral $ABC'C''$ is cyclic. $\square$

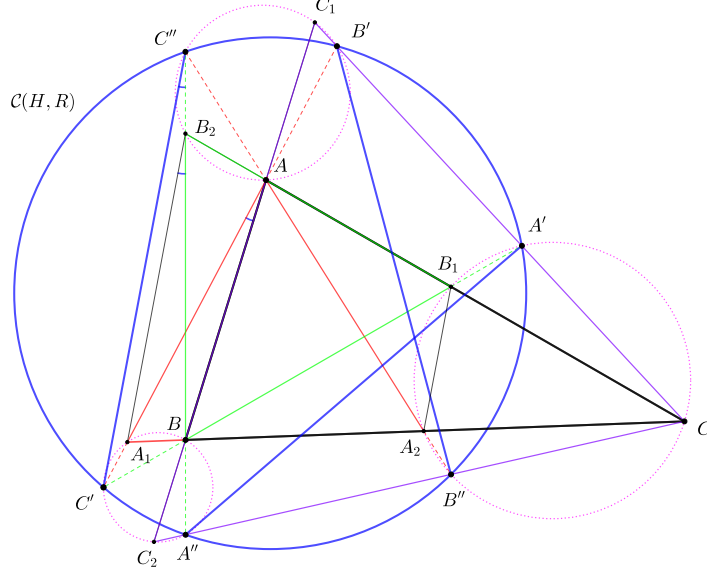


Fig. 4 The chords $A'A''$, $B'B''$, $C'C''$ of the circle $\mathcal{C}(H, R)$

Proposition 2.5. (i) $A'A'' \parallel B_1C_2 \parallel B_2C_1$, and $B'B'' \parallel C_1A_2 \parallel C_2A_1$, $C'C'' \parallel A_1B_2 \parallel A_2B_1$;
(ii) the chords $A'A''$, $B'B''$ and $C'C''$ of the circle $\mathcal{C}(H, R)$ are equal and have the length $R\sqrt{3}$ (Fig. 4).

Proof. (i) We will only prove the statement $C'C'' \parallel A_1B_2 \parallel A_2B_1$; the other two are shown in the same way. By Lemma 2.1, the quadrilaterals ABA_2B_1 and ABA_1B_2 are cyclic; hence, the lines A_2B_1 and A_1B_2 are antiparallel to AB , and therefore $A_2B_1 \parallel A_1B_2$.

For $C'C'' \parallel A_1B_2$ it is necessary and sufficient that $\widehat{C'C''B} = \widehat{A_1B_2B}$. Because $\widehat{C'C''B} = \widehat{C'AB}$ ($ABC'C''$ is cyclic) and $\widehat{A_1B_2B} = \widehat{A_1AB}$ (ABA_1B_2 is cyclic) $= \widehat{C'AB}$ it follows that $\widehat{C'C''B} = \widehat{A_1B_2B}$. So, statement (i) is proven.

(ii) According to Corollary 2.1, quadrilaterals $B'C''C'B''$ and $C'A''A'C''$ are isosceles trapezoids, with $B'B'' = C''C'$ and $C''C' = A''A'$, respectively. Hence, $A'A'' = B'B'' = C'C''$.

On the circle $\mathcal{C}(H, R)$, we consider the positive orientation of the arcs. Since $B'B'' = C'C''$, we have $\widehat{B''B'} = \widehat{C''C'}$ (Fig. 4). Also, $\widehat{A_1AA_2} = \frac{\pi}{3}$ implies $\widehat{C'B''} + \widehat{B'C''} = \frac{2\pi}{3}$. Therefore, $2\pi = \widehat{B''B'} + \widehat{C''C'} + (\widehat{C'B''} + \widehat{B'C''}) = 2\widehat{B''B'} + \frac{2\pi}{3}$, from where $\widehat{B''B'} = \frac{1}{2} \left(2\pi - \frac{2\pi}{3} \right) = \frac{2\pi}{3}$. Therefore, $B''B'$ can be the side of an equilateral triangle inscribed in the circle $\mathcal{C}(H, R)$ and, therefore, we have $A'A'' = B'B'' = C'C'' = R\sqrt{3}$. The proof is complete. $\square$

Remark 2.2. From the above, the points $A', B', C'; A'', B'', C''$ defined by (2.1) and (2.2) can also be entered in the following ways:

1. $A' = BB_1 \cap (CA_2B_1)$, $A'' = BB_2 \cap (BC_2A_1)$, etc. (i.e., these points are the intersections of the side lines of triangles AA_1A_2 , BB_1B_2 , CC_1C_2 with the complementary circles (AB_2C_1) , (BC_2A_1) , (CA_2B_1));
2. $A' = (CA_2B_1) \cap \mathcal{C}(H, R)$, $A'' = (BC_2A_1) \cap \mathcal{C}(H, R)$, etc. (i.e., these points are the intersections of the complementary circles with the circle $\mathcal{C}(H, R)$).

3. TRIANGLES AA_1A_2 , BB_1B_2 , CC_1C_2 AND CARNOT CIRCLES

We remind that the Carnot circle of triangle ABC relative to the side AB is the circle (ABH) ; similar relative to sides BC and CA . These circles are the reflections of the circumcircle $\mathcal{C}(O, R)$ in the sides of the triangle ABC ; hence, they are equal to $\mathcal{C}(O, R)$ and, consequently, are also equal to $\mathcal{C}(H, R)$ (Fig. 5). Carnot circle are closely related to orthocentric systems ([6], [5]). The centers of the Carnot circles and their antipodes will be named as in Fig. 5.

Proposition 3.1. (i) The circles $(ABC'C'')$, $(BCA'A'')$, $(CAB'B'')$ are the Carnot circles relative to the sides AB , BC , CA , respectively.

(ii) Their centers as well as their antipodes are located on the circle $\mathcal{C}(H, R)$.

Proof. (i) We will only show that $(ABC'C'')$ is the Carnot circle relative to AB . So, let's show that $H \in (ABC'C'')$. Indeed, it is known that $\widehat{AHB} = \pi - C$. Also, we have $\widehat{AC''B} = \widehat{B''C''A''} = C$ (Proposition 2.4) Then, $\widehat{AHB} + \widehat{AC''B} = \pi$, hence $H \in (ABC'C'')$.

(ii) These properties are known and trivial. Indeed, since the Carnot circles are equal to $\mathcal{C}(H, R)$ and pass through its center H , it follows that their centers lie on the circle $\mathcal{C}(H, R)$. The second part is obvious. $\square$

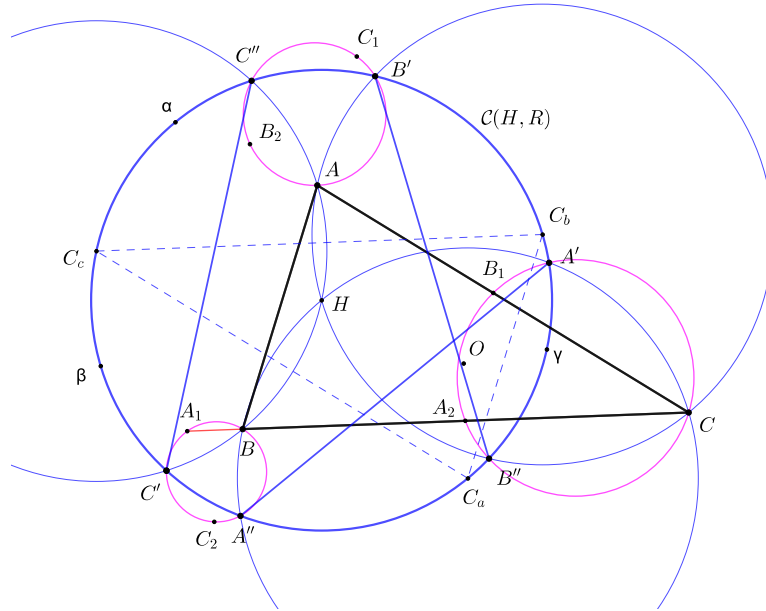


Fig. 5 $\mathcal{C}(H, R)$ and the Carnot circles

The following statement will be useful in the next section.

Proposition 3.2. *Triangles ABC , $C_a C_b C_c$, and $\alpha\beta\gamma$ are equal, similarly oriented, and their corresponding sides are parallel.*

Proof. These triangles are obtained from each other through simple geometric transformations. Indeed, we have:

$$\alpha\beta\gamma = \mathcal{S}_H(C_a C_b C_c), \quad (3.1)$$

where $\mathcal{S}_H$ denotes the symmetry of the plane with respect to point H , and

$$C_a C_b C_c = \mathcal{H}(O_9, -1)(ABC), \quad (3.2)$$

where $\mathcal{H}(O_9, -1)$ denotes the homothety with center O_9 and ratio -1 .

The statements of the proposition result directly from formulas (3.1), (3.2) and the properties of these two geometric transformations. $\square$

Corollary 3.1. *Triangles ABC , $A'B'C'$, $A''B''C''$, $C_a C_b C_c$, $\alpha\beta\gamma$ are equal and have the same orientation.*

Proof. From the previous proposition and Proposition 2.4. $\square$

Proposition 3.3. *Triangles $\alpha A' A''$, $\beta B' B''$, $\gamma C' C''$ are equilateral.*

Proof. Let us only show that triangle $\gamma C' C''$ is equilateral. According to Proposition 2.5(ii), the length of side $C' C''$ is $R\sqrt{3}$. On the other hand, $C' C''$ is the common chord of the circle $\mathcal{C}(H, R)$ and the Carnot circle $\mathcal{C}(C_c, R)$ (Proposition 2.3 and Proposition 3.1(i)) (Fig. 5). These circles being equal, the perpendicular bisector of the chord $C' C''$ passes through their centers H and C_c . Moreover, according to Prop.3.1(ii), on this line is also the antipode γ of the point $C_c \in \mathcal{C}(H, R)$. Consequently, triangle $\gamma C' C''$ is equilateral. $\square$

4. RADICAL AXES OF THE CIRCLES $(AA_1 A_2)$, $(BB_1 B_2)$, $(CC_1 C_2)$

We continue the study initiated in the previous section by introducing six new points also defined as the intersection of side lines $AA_1, AA_2, BB_1, BB_2, CC_1, CC_2$; namely, we consider the following points (Fig. 6):

$$\alpha_1 = BB_1 \cap CC_2, \beta_1 = CC_1 \cap AA_2, \gamma_1 = AA_1 \cap BB_2, \quad (4.1)$$

$$\alpha_2 = BB_2 \cap CC_1, \beta_2 = CC_2 \cap AA_1, \gamma_2 = AA_2 \cap BB_1. \quad (4.2)$$

Unlike $A', A'', B', B'', C', C''$, these points do not belong to circle $\mathcal{C}(H, R)$, but they will determine a point on this circle.

Proposition 4.1. *The lines $\alpha_1 \alpha_2$, $\beta_1 \beta_2$, $\gamma_1 \gamma_2$ are the radical axes of the circles $(AA_1 A_2)$, $(BB_1 B_2)$, $(CC_1 C_2)$.*

Proof. We will show that the radical axis of the circles $(AA_1 A_2)$, $(BB_1 B_2)$ is $\gamma_1 \gamma_2$ (Fig. 6). Indeed, the powers of point γ_1 with respect to circles $(AA_1 A_2)$ and $(BB_1 B_2)$ are given by

$$p_{\gamma_1}(AA_1 A_2) = \gamma_1 A \cdot \gamma_1 A_1 \quad \text{and} \quad p_{\gamma_1}(BB_1 B_2) = \gamma_1 B \cdot \gamma_1 B_2. \quad (4.3)$$

On the other hand, the quadrilateral $ABA_1 B_2$ is inscribed in the Torricelli circle $\mathcal{T}_+^c$ and we have:

$$\gamma_1 A \cdot \gamma_1 A_1 = \gamma_1 B \cdot \gamma_1 B_2. \quad (4.4)$$

From (4.3) and (4.4) it follows that $p_{\gamma_1}(AA_1 A_2) = p_{\gamma_1}(BB_1 B_2)$, i.e. γ_1 is on the radical axis of the circles $(AA_1 A_2)$ and $(BB_1 B_2)$.

Using the quadrilateral ABA_2B_1 (inscribed in the Torricelli circle $\mathcal{T}_-^c$), with the same arguments we show that the point γ_2 has the same power with respect to the circles (AA_1A_2) and (BB_1B_2) .

Consequently, the line $\gamma_1\gamma_2$ is the radical axis of the circles (AA_1A_2) and (BB_1B_2) . The remaining statements are shown in the same way. $\square$

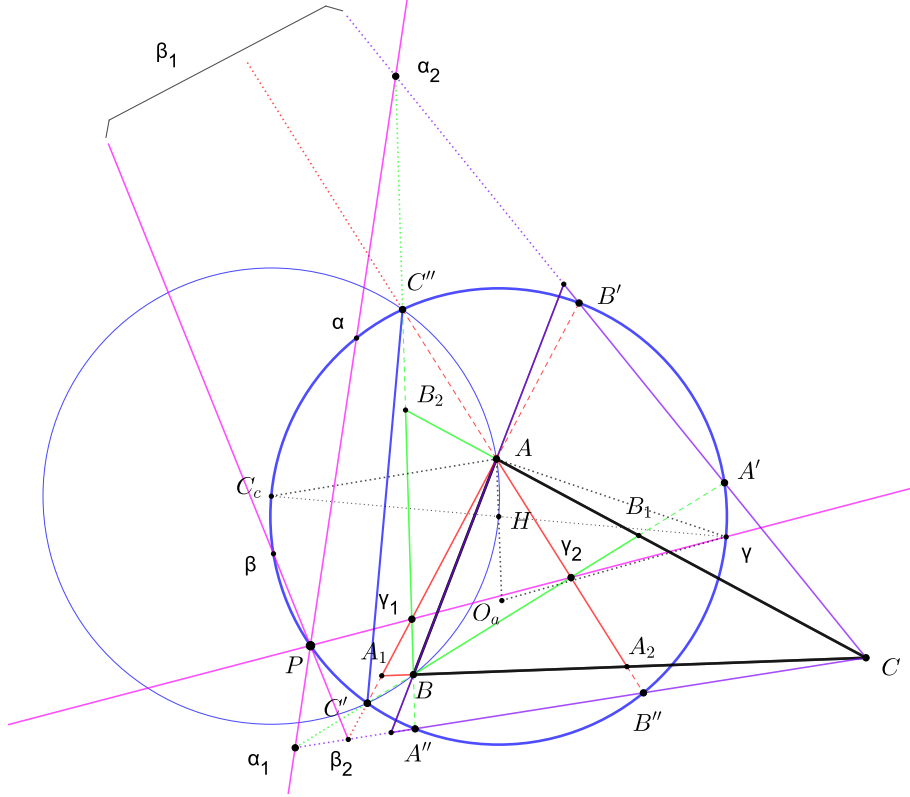


Fig. 6 Radical axes of circles (AA_1A_2) , (BB_1B_2) , (CC_1C_2)

Recall that points α, β, γ were defined as the antipodes of the centers of the Carnot circles in the circle $\mathcal{C}(H, R)$. Proposition 3.3 states a property of these points. We will highlight a new and useful property in what follows.

Proposition 4.2. *The radical axes $\alpha_1\alpha_2, \beta_1\beta_2, \gamma_1\gamma_2$ pass through the point α, β, γ , respectively.*

Proof. We will only show that $\gamma \in \gamma_1\gamma_2$. Taking into account Proposition 4.1, it is sufficient to show that the point γ has the same power with respect to the circles (AA_1A_2) and (BB_1B_2) , i.e. it is sufficient to show that

$$p_\gamma(AA_1A_2) = p_\gamma(BB_1B_2). \quad (4.5)$$

Let's calculate $p_\gamma(AA_1A_2)$; we have:

$$p_\gamma(AA_1A_2) = (\gamma O_a)^2 - (AO_a)^2,$$

and so

$$p_\gamma(AA_1A_2) = (\gamma O_a)^2 - \left(\frac{2}{3}h_a\right)^2 \quad (4.6)$$

(h_a - the altitude from A of the given triangle, O_a - the center of the equilateral triangle AA_1A_2). Applying the median theorem to triangle $AC_c\gamma$ (Fig. 6), we obtain

$$2AH^2 = (AC_c)^2 + A\gamma^2 - \frac{1}{2}(C_c\gamma)^2,$$

and taking into account that $AH = 2R \cos A$, $AC_c = R$, $C_c\gamma = 2R$, we find

$$\gamma A^2 = R^2(8 \cos^2 A + 1). \quad (4.7)$$

Now, applying Stewart's theorem to triangle γAO_a and point $H \in AO_a$ (Fig. 6), we have:

$$(\gamma O_a)^2 \cdot AH + (\gamma A)^2 \cdot HO_a - (\gamma H)^2 \cdot AO_a = AH \cdot HO_a \cdot AO_a. \quad (4.8)$$

But, $\gamma H = R$, $HO_a = AO_a - AH = \frac{2}{3}h_a - 2R \cos A$. Using these formulas and (4.7), the equation (4.8) allows us to find γO_a by direct calculation. We will find

$$(\gamma O_a)^2 = \left(\frac{2}{3}h_a\right)^2 + R^2(8 \cos^2 A + 1) - 4Rh_a \cos A. \quad (4.9)$$

From (4.6) and (4.9) we obtain

$$p_\gamma(AA_1A_2) = R^2(8 \cos^2 A + 1) - 4Rh_a \cos A.$$

Since $h_a = 2R \sin B \sin C$, this equation is equivalent to

$$p_\gamma(AA_1A_2) = R^2(8 \cos^2 A + 1 - 8 \cos A \sin B \sin C)$$

or, using the formula $1 - 4 \cos A \sin B \sin C = -\cos 2A + \sin 2B + \sin 2C$,

$$p_\gamma(AA_1A_2) = R^2(3 + 2 \sum \cos 2A). \quad (4.10)$$

Similarly, we have

$$p_\gamma(BB_1B_2) = R^2(3 + 2 \sum \cos 2A). \quad (4.11)$$

Obviously, equation (4.5) is implied by (4.10) and (4.11). Finally, the point γ lies on the radical axis $\gamma_1\gamma_2$. $\square$

We need the following elementary result.

Lemma 4.1. ([6], pp. 281-282) *The lines through the vertices of a triangle, perpendicular to the corresponding sides of a triangle inversely similar to it, are concurrent on its circumcircle.*

Proposition 4.3. *The radical center of the circles (AA_1A_2) , (BB_1B_2) , (CC_1C_2) lies on the circle $\mathcal{C}(H, R)$ (Fig. 7).*

Proof. Triangle $\alpha\beta\gamma$ is inscribed in the circle $\mathcal{C}(H, R)$. Moreover, the triangles $\alpha\beta\gamma$ and ABC are equal and have the same orientation (Corollary 3.1). Also, triangle $\alpha\beta\gamma$ is inversely similar to orthocentroidal triangle $O_aO_bO_c$ (Corollary 3.1 and Section 1, 1.2). By the previous lemma, it follows that the perpendicular through the points α, β , and γ to the lines O_bO_c, O_cO_a and respectively O_aO_b are concurrent on the circle $\mathcal{C}(H, R)$.

We know that $\gamma \in \gamma_1\gamma_2$ (Proposition 4.2), and the line $\gamma_1\gamma_2$ is the radical axis of the circles $(BB_1B_2), (CC_1C_2)$ (Proposition 4.1). Then, the radical axis $\gamma_1\gamma_2$ is precisely the perpendicular through γ to the O_aO_b .

It is clear that the radical axes $\alpha_1\alpha_2$ and $\beta_1\beta_2$ have similar properties. By Lemma 4.1, the three axes are concurrent on $\mathcal{C}(H, R)$, i.e. the radical center of the circles $(AA_1A_2), (BB_1B_2), (CC_1C_2)$ lies on $\mathcal{C}(H, R)$. $\square$

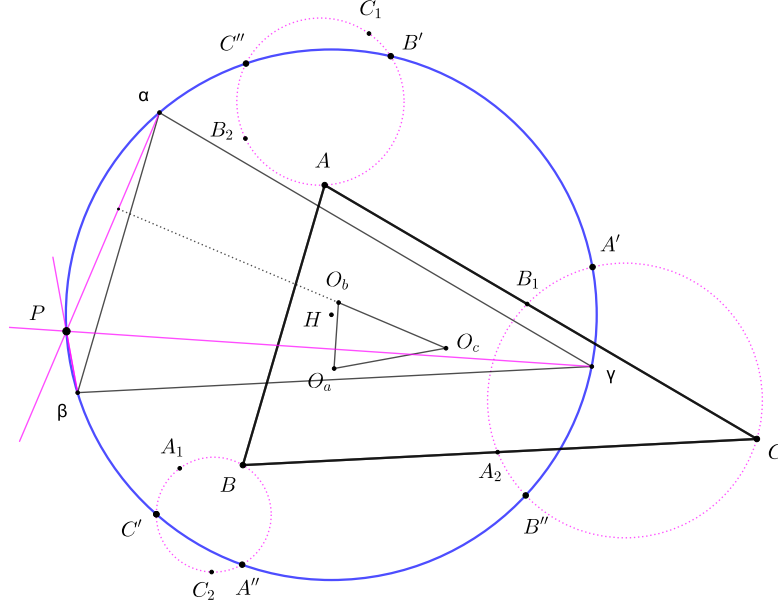


Fig. 7 Radical center of circles (AA_1A_2) , (BB_1B_2) , (CC_1C_2)

Remark 4.1. Let us denote by P the radical center of the circles (AA_1A_2) , (BB_1B_2) , (CC_1C_2) , i.e. $P = \alpha_1\alpha_2 \cap \beta_1\beta_2 \cap \gamma_1\gamma_2$. Then, $P \in \mathcal{C}(H, R)$ and we can say that P is the Tarry point of triangle $\alpha\beta\gamma$ or the Steiner point of triangle $C_aC_bC_c$ (Fig. 7).

Remark 4.2. The radical axes of the circles (AA_1A_2) , (BB_1B_2) , (CC_1C_2) intersect the circle $\mathcal{C}(H, R)$ in three points α, β, γ (different from P - their radical center) that form a triangle equal to ABC , similarly oriented, and with the corresponding parallel sides (Fig. 7).

5. TRIPLETS OF TRIANGLES IN PERSPECTIVE

In this section we will consider some triplets of triangles associated with triangle ABC and highlight their properties of perspectivity.

First, we consider and examine the triplet $(ABC, \alpha_1\beta_1\gamma_1, \alpha_2\beta_2\gamma_2)$ (see (4.1) and (4.2)).

Proposition 5.1. We have:

- (i) the triangles ABC , $\alpha_1\beta_1\gamma_1$, $\alpha_2\beta_2\gamma_2$ are in perspective two by two;
- (ii) P , F_+ , F_- are the centers of perspective of pairs of triangles $(\alpha_1\beta_1\gamma_1, \alpha_2\beta_2\gamma_2)$, $(\alpha_2\beta_2\gamma_2, ABC)$, $(ABC, \alpha_1\beta_1\gamma_1)$, respectively.

Proof. Homologous lines of triangles $\alpha_1\beta_1\gamma_1$ and $\alpha_2\beta_2\gamma_2$ are $\alpha_1\alpha_2, \beta_1\beta_2, \gamma_1\gamma_2$ (Fig. 8). But, these lines are the radical axes of circles (AA_1A_2) , (BB_1B_2) , (CC_1C_2) . According to Proposition 4.1, they are concurrent in P . Thus, P is the center of perspective of the triangles $\alpha_1\beta_1\gamma_1$ and $\alpha_2\beta_2\gamma_2$.

Homologous lines of triangles $\alpha_2\beta_2\gamma_2$ and ABC are $A\alpha_2, B\beta_2, C\gamma_2$. Let us show that these lines coincide with AF_+, BF_+, CF_+ and it will follow that they are concurrent in F_+ . We will only show that $A\alpha_2$ coincides with AF_+ . Since AF_+ is the common chord of the Torricelli circles $\mathcal{T}_+^b$ and $\mathcal{T}_+^c$, it is sufficient to show that point α_2 has equal powers with respect to these circles. We therefore have to show that $\alpha_2B_2 \cdot \alpha_2B = \alpha_2C_1 \cdot \alpha_2C$. But, this equality results from the similarity of the triangles $\alpha_2B_2C_1$ and α_2CB (since B_2C_1 is antiparallel to the BC , Section 1, 1.4).

Similarly, it is shown that F_- is the center of perspective of triangles ABC and $\alpha_1\beta_1\gamma_1$. $\square$

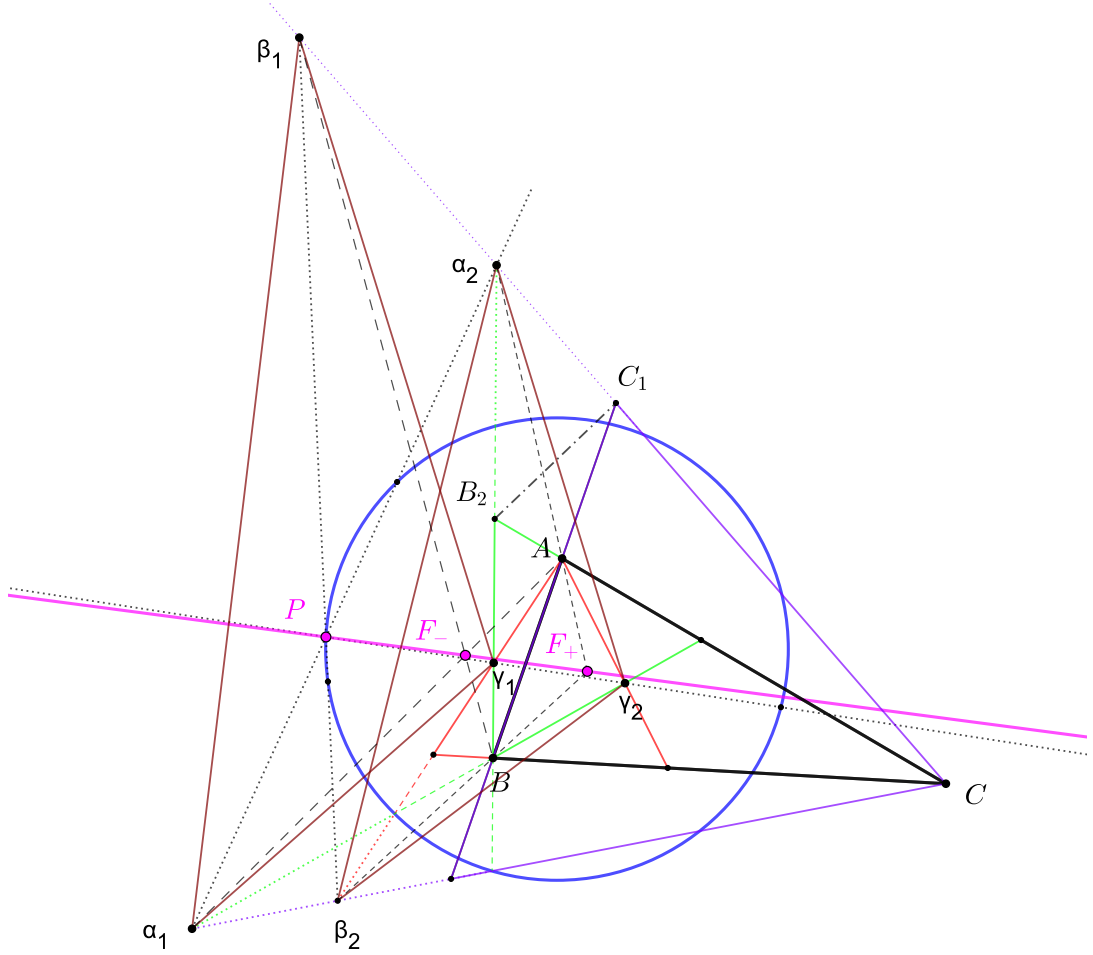


Fig. 8 P lies on the Fermat axis F_+F_-

Proposition 5.2. *The points P, F_+, F_- are collinear.*

Proof. Let us show that P is on the Fermat axis of the triangle ABC . For this, we consider the triangles $\alpha_1\beta_1\gamma_1, \alpha_2\beta_2\gamma_2$ and, taking into account (4.1) and (4.2), we find that $\beta_1\gamma_2 \cap \beta_2\gamma_1 = A$, $\gamma_1\alpha_2 \cap \gamma_2\alpha_1 = B$, and $\alpha_1\beta_2 \cap \alpha_2\beta_1 = C$. Using [9], #166, it follows that triangles $\alpha_1\beta_1\gamma_1, \alpha_2\beta_2\gamma_2$, and ABC are in perspective two by two and have the same axis of perspective. Under these conditions, the theorem from [6], #378 can be applied, and we get that their centers of perspective are collinear. Taking into account the previous sentence, this means that the points P, F_+, F_- are collinear. $\square$

The equilateral triangles $AB'C'', BC'A''$ and $CA'B''$ (cf. Proposition 2.2) are formed by the side lines of triangle ABC and the lines $B'C'', C'A'', A'B''$, respectively; dual, the lines $B''C', C''A', A''B'$ determine the equilateral triangles $AB''C', BC''A', CA''B'$, respectively. We denote $A_+^*B_+^*C_+^*$, the triangle formed by the lines $B'C'', C'A'', A'B''$ and $A_-^*B_-^*C_-^*$, the dual triangle

formed by the lines $B''C'$, $C''A'$, $A''B'$ (i.e. $A_+^* = C'A'' \cap A'B''$, $A_-^* = C''A' \cap A''B'$ and the other vertices are similarly defined) (Fig. 9).

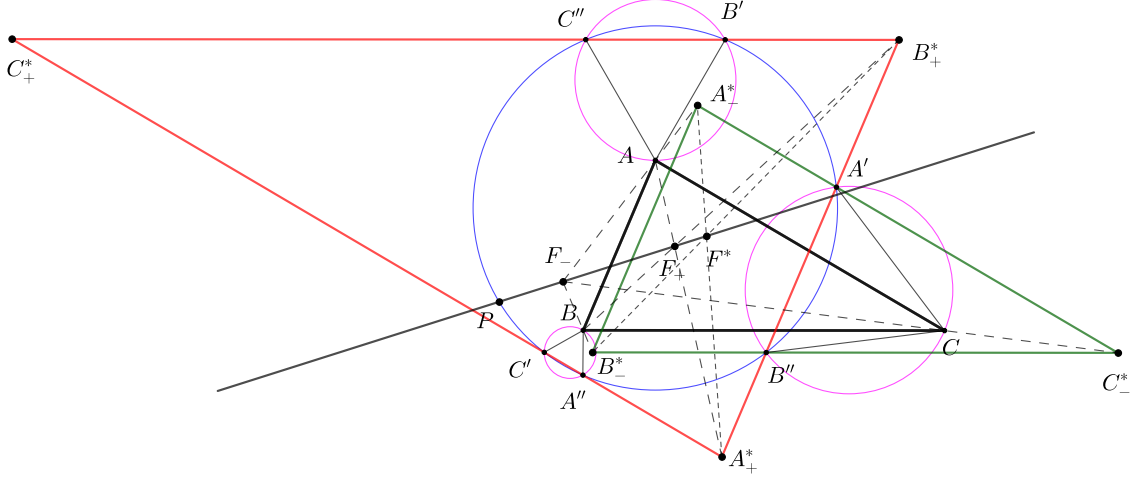


Fig. 9 Triangles $A_+^*B_+^*C_+^*$ and $A_-^*B_-^*C_-^*$ and the axis F_+F_-

Now, our attention is focused on the triplet $(ABC, A_+^*B_+^*C_+^*, A_-^*B_-^*C_-^*)$. By Corollary 2.1, we have $BC \parallel B'C'' \parallel B''C'$ and similar relations to them. Consequently, $BC \parallel B_+^*C_+^* \parallel B_-^*C_-^*$, and, therefore, the corresponding sides of triangles ABC , $A_+^*B_+^*C_+^*$, $A_-^*B_-^*C_-^*$ are parallel. Then, these triangles are homothetic two by two.

Proposition 5.3. *We have: (i) triangles ABC and $A_+^*B_+^*C_+^*$ are homothetic, with F_+ as center of homothety and the ratio given by*

$$k_+ = -\frac{\sqrt{3}}{24\Delta} \left[3(a^2 + b^2 + c^2) + 4\sqrt{3}\Delta \right] = -\frac{1}{2S} \left(\sqrt{3}S_\omega + S \right); \quad (5.1)$$

*(ii) triangles ABC and $A_-^*B_-^*C_-^*$ are homothetic, with F_- as center of homothety and the ratio given by*

$$k_- = \frac{\sqrt{3}}{24\Delta} \left[3(a^2 + b^2 + c^2) - 4\sqrt{3}\Delta \right] = \frac{1}{2S} \left(\sqrt{3}S_\omega - S \right); \quad (5.2)$$

*(iii) the center of homothety of triangles $A_+^*B_+^*C_+^*$ and $A_-^*B_-^*C_-^*$, namely F^* , lies on the Fermat axis F_+F_- and his ratio of homothety is given by*

$$k^* = -\frac{3(a^2 + b^2 + c^2) - 4\sqrt{3}\Delta}{3(a^2 + b^2 + c^2) + 4\sqrt{3}\Delta} = -\frac{\sqrt{3}S_\omega - S}{\sqrt{3}S_\omega + S}. \quad (5.3)$$

Proof. (i) Let us show that AA_+^* , BB_+^* , CC_+^* are concurrent in F_+ (Fig. 9). We will only show that $F_+ \in AA_+^*$. Indeed, AF_+ is the radical axis of the circles (BC_2A_1) and (CA_2B_1) ([2], Proposition 5.1). On the other hand, the lines $A_+^*C_+^*$ and $A_+^*B_+^*$ are the radical axes of the pairs of circles $(\mathcal{C}(H, R), (BC_2A_1))$ and $(\mathcal{C}(H, R), (CA_2B_1))$, respectively. Therefore, A_+^* is the radical center of these three circles; in particular, A_+^* lies on the radical axis of the circles (BC_2A_1) and (CA_2B_1) , that is $F_+ \in AA_+^*$.

Now, we will express the ratio $\frac{A_+^*B_+^*}{AB} \left(= \frac{c_+^*}{c} \right)$ by the sidelengths of the given triangle. We have

$c_+^* = B_+^*A' + A'B'' + B''A_+^*$ (Fig. 9). Applying the sine formula to triangle $A'B_+^*B'$ (angles: $\frac{\pi}{3}$, B , $\frac{2\pi}{3} -$

B), we find: $B_+^* A' = \frac{c}{2} \left(\sqrt{3} \frac{a^2 - b^2 + c^2}{4\Delta} + 1 \right)$. In the same way, by choosing the triangle $A_+^* B'' A''$ (angles: $A, \frac{\pi}{3}, \frac{2\pi}{3} - A$), we get: $B'' A_+^* = \frac{c}{2} \left(\sqrt{3} \frac{-a^2 + b^2 + c^2}{4\Delta} + 1 \right)$. Finally, in the equilateral triangle $A' B'' C$ we have $A' B'' = B'' C$, and applying the sine formula to triangle $A_2 B'' C$ (angles: $\frac{\pi}{3}, \pi - B, B - \frac{\pi}{3}$) we find $A' B'' = \sin \frac{\pi}{3} \frac{A_2 C}{\sin B} = \frac{b}{\sin B} \cdot \sin \left(\frac{\pi}{3} - C \right) = \frac{c}{2} \left(\sqrt{3} \frac{a^2 + b^2 - c^2}{4\Delta} - 1 \right)$. Consequently, $c_+^* = \frac{c}{2} \left(\sqrt{3} \frac{a^2 + b^2 + c^2}{4\Delta} + 1 \right)$, where we get (5.1).

(ii) With the same arguments and similar calculations as in (i).

(iii) The collinearity of the points F^*, F_+, F_- can be deduced from general theoretical results.

First proof. We note that the homothety that carries $A_+^* B_+^* C_+^*$ into $A_-^* B_-^* C_-^*$ is the product of the homothety that carries $A_+^* B_+^* C_+^*$ into ABC by the homothety that carries ABC into $A_-^* B_-^* C_-^*$. Then, it is sufficient to take into account the following properties: 1) the inverse of a homothety has the same center as the given; 2) the center of the product of two homotheties is collinear with the centers of the factors ([12], p. 29).

Second proof. We use the fact that two homothetic triangles are in perspective, and that the center of perspective coincides with their center of homothety and the axis of perspective is the line at infinity.

It is clear that the triangles $ABC, A_+^* B_+^* C_+^*, A_-^* B_-^* C_-^*$ are in perspective two by two. From the parallelism of the corresponding sides of these triangles it follows that the line at infinity is the common axis of perspective. According to the Theorem in [9], #168 or in [6], #378, their centers of perspective are collinear. So, F^*, F_+, F_- are collinear.

It remains to show the formula (5.3). But $k^* = -\frac{k_-}{k_+}$, and taking into account (5.1) and (5.2), we obtain the required formula. $\square$

Corollary 5.1. *The position of the point F^* on the axis $F_+ F_-$ is given by*

$$\frac{\overline{F^* F_-}}{\overline{F^* F_+}} = \frac{\sqrt{3}(S_\omega^2 - S^2) + 2SS_\omega}{\sqrt{3}(S_\omega^2 - S^2) - 2SS_\omega}. \quad (5.4)$$

Proof. The product of homothety that carries $A_-^* B_-^* C_-^*$ into $A_+^* B_+^* C_+^*$ with the homothety that carries $A_+^* B_+^* C_+^*$ into ABC is the homothety that carries $A_-^* B_-^* C_-^*$ into ABC . Consequently, relative to the centers of these homotheties we have the relation ([12], p. 29)

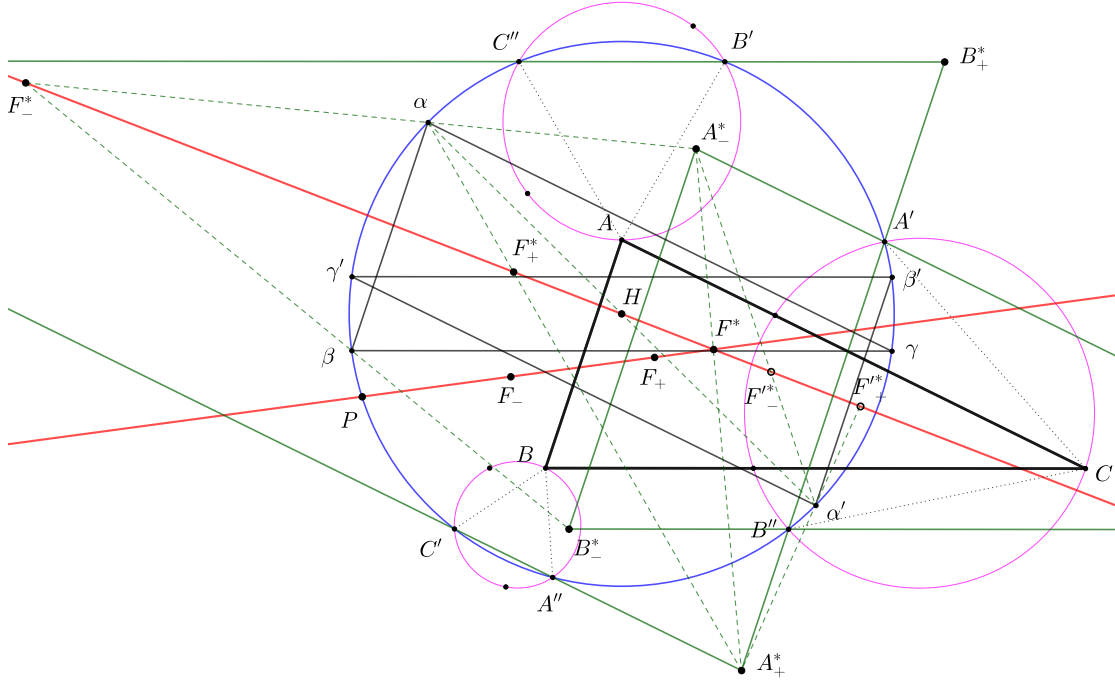
$$\overline{F_- F^*} = \frac{1 - k_+}{1 - (k_-)^{-1} k_+} \overline{F_- F_+},$$

from where

$$\frac{\overline{F^* F_-}}{\overline{F^* F_+}} = \frac{k_- (1 - k_+)}{k_+ (1 - k_-)},$$

and using (5.1) and (5.2), we immediately find the equation (5.4). $\square$

Let's now consider triangles $\alpha\beta\gamma, \alpha'\beta'\gamma', A_+^* B_+^* C_+^*$, and $A_-^* B_-^* C_-^*$. We know that the corresponding sides of these triangles are parallel; so, these triangles are homothetic two by two. The center of homothety of the triangles $A_+^* B_+^* C_+^*$, and $A_-^* B_-^* C_-^*$ was denoted with F^* . With F_+^* and F_-^* we will denote the centers of homothety of pairs $(\alpha\beta\gamma, A_+^* B_+^* C_+^*)$ and $(\alpha\beta\gamma, A_-^* B_-^* C_-^*)$, respectively. Regarding the pairs $(\alpha'\beta'\gamma', A_+^* B_+^* C_+^*)$ and $(\alpha'\beta'\gamma', A_-^* B_-^* C_-^*)$ the centers of homothety will be denoted $F_+^{*'} and $F_-^{*'}$, respectively (Fig. 10).$


Fig. 10 Centers of homothety on the line h

Proposition 5.4. *The centers of homothety F^* , H , F_+ , F_- , F'_+ , F'_- are collinear.*

Proof. It is clear that H is the center of homothety of the antipodal triangles $\alpha\beta\gamma$ and $\alpha'\beta'\gamma'$. We denote the line HF^* with h and show that F^* , H , F_+ , F_- , F'_+ , F'_- $\in h$. Indeed, it is sufficient to show that by choosing three of these centers, they are collinear. But, this fact follows by applying the theorem on the three centres of homothety ([12], p. 29). $\square$

Remark 5.1. *Finally, the triangles $\alpha\beta\gamma$, $\alpha'\beta'\gamma'$, $A_+^*B_+^*C_+^*$, $A_-^*B_-^*C_-^*$, having parallel corresponding sides, are homothetic two by two and the corresponding homothetic centers of homothety lie on the lines F_+F_- and h .*

Remark 5.2. *The points in each of the systems 1) $P, \alpha, \alpha_1, \alpha_2$, 2) $P, \beta, \beta_1, \beta_2$, 3) $P, \gamma, \gamma_1, \gamma_2$, are collinear. Consequently, P is the center of perspective of triangles $\alpha\beta\gamma$, $\alpha_1\beta_1\gamma_1$, $\alpha_2\beta_2\gamma_2$, and their three axes of perspective are concurrent.*

6. BARYCENTRIC COORDINATES OF POINTS P AND F^* . EQUATIONS OF THE CIRCLE $\mathcal{C}(H, R)$ AND THE LINE HF^*

To the intended aim, we recall some things known or easy to show. Thus, the barycentric coordinates of the vertices of triangles AA_1A_2 , BB_1B_2 , CC_1C_2 (other than A, B, C) are given by

$$\begin{aligned} A_1 &= \left(0 : b \sin \left(C + \frac{\pi}{3}\right) : -c \sin \left(B - \frac{\pi}{3}\right)\right), \quad A_2 = \left(0 : -b \sin \left(C - \frac{\pi}{3}\right) : c \sin \left(B + \frac{\pi}{3}\right)\right), \\ B_1 &= \left(-a \sin \left(C - \frac{\pi}{3}\right) : 0 : c \sin \left(A + \frac{\pi}{3}\right)\right), \quad B_2 = \left(a \sin \left(C + \frac{\pi}{3}\right) : 0 : -c \sin \left(A - \frac{\pi}{3}\right)\right), \\ C_1 &= \left(a \sin \left(B + \frac{\pi}{3}\right) : -b \sin \left(A - \frac{\pi}{3}\right) : 0\right), \quad C_2 = \left(-a \sin \left(B - \frac{\pi}{3}\right) : b \sin \left(A + \frac{\pi}{3}\right) : 0\right) \end{aligned}$$

([2], p. 23). Using Conway's notations ([13], p. 33), they are rewritten in the form

$$\begin{aligned} A_1 &= (0 : \sqrt{3}S_C + S : \sqrt{3}S_B - S), & A_2 &= (0 : \sqrt{3}S_C - S : \sqrt{3}S_B + S), \\ B_1 &= (\sqrt{3}S_C - S : 0 : \sqrt{3}S_A + S), & B_2 &= (\sqrt{3}S_C + S : 0 : \sqrt{3}S_A - S), \\ C_1 &= (\sqrt{3}S_B + S : \sqrt{3}S_A - S : 0), & C_2 &= (\sqrt{3}S_B - S : \sqrt{3}S_A + S : 0). \end{aligned}$$

Moreover, to avoid long expressions we will use the following notations:

$$\begin{aligned} \sigma_A &= \sqrt{3}S_A + S, & \sigma_B &= \sqrt{3}S_B + S, & \sigma_C &= \sqrt{3}S_C + S; \\ \tau_A &= \sqrt{3}S_A - S, & \tau_B &= \sqrt{3}S_B - S, & \tau_C &= \sqrt{3}S_C - S. \end{aligned} \quad (6.1)$$

Then,

$$\begin{aligned} A_1 &= (0 : \sigma_C : \tau_B), & A_2 &= (0 : \tau_C : \sigma_B), \\ B_1 &= (\tau_C : 0 : \sigma_A), & B_2 &= (\sigma_C : 0 : \tau_A), \\ C_1 &= (\sigma_B : \tau_A : 0), & C_2 &= (\tau_B : \sigma_A : 0). \end{aligned}$$

As a result, the extended sides of the triangles AA_1A_2 , BB_1B_2 , CC_1C_2 (other than BC , CA , AB) have the following equations in barycentric coordinates:

$$\begin{aligned} (AA_1) \quad \tau_B y - \sigma_C z &= 0, & (AA_2) \quad \sigma_B y - \tau_C z &= 0, \\ (BB_1) \quad -\sigma_A x + \tau_C z &= 0, & (BB_2) \quad -\tau_A x + \sigma_C z &= 0, \\ (CC_1) \quad \tau_A x - \sigma_B y &= 0, & (CC_2) \quad \sigma_A x - \tau_B y &= 0. \end{aligned}$$

Now, we find the barycentric coordinates of the points $\alpha_1, \beta_1, \gamma_1; \alpha_2, \beta_2, \gamma_2$. Taking into account that $\alpha_1 = BB_1 \cap CC_2$, $\alpha_2 = BB_2 \cap CC_1$ etc. (Fig. 6 or 8), by a simple calculation we have

$$\begin{aligned} \alpha_1 &= (\tau_B \tau_C : \tau_C \sigma_A : \sigma_A \tau_B), & \alpha_2 &= (\sigma_B \sigma_C : \sigma_C \tau_A : \tau_A \sigma_B), \\ \beta_1 &= (\sigma_B \tau_C : \tau_C \tau_A : \tau_A \sigma_B), & \beta_2 &= (\tau_B \sigma_C : \sigma_C \sigma_A : \sigma_A \tau_B), \\ \gamma_1 &= (\tau_B \sigma_C : \sigma_C \tau_A : \tau_A \tau_B), & \gamma_2 &= (\sigma_B \tau_C : \tau_C \sigma_A : \sigma_A \sigma_B). \end{aligned}$$

Also, through an elementary calculation we find the equations of the lines $\alpha_1\alpha_2$, $\beta_1\beta_2$, $\gamma_1\gamma_2$:

$$(\alpha_1\alpha_2) \quad \sigma_A \tau_A (S_B - S_C)x - \sigma_B \tau_B (S_C + S_A)y + \sigma_C \tau_C (S_A + S_B)z = 0 \quad (6.2)$$

$$(\beta_1\beta_2) \quad \sigma_A \tau_A (S_B + S_C)x + \sigma_B \tau_B (S_C - S_A)y - \sigma_C \tau_C (S_A + S_B)z = 0 \quad (6.3)$$

$$(\gamma_1\gamma_2) \quad -\sigma_A \tau_A (S_B + S_C)x + \sigma_B \tau_B (S_C + S_A)y + \sigma_C \tau_C (S_A - S_B)z = 0. \quad (6.4)$$

We mention that, according to Proposition 4.1, these are the equations of the radical axes of the circles (AA_1A_2) , (BB_1B_2) , (CC_1C_2) .

On the other hand, we find the barycentric coordinates of points A' , B' , C' ; A'' , B'' , C'' taking into account that $A' = BB_1 \cap CC_1$, $A'' = BB_2 \cap CC_2$ etc. (Fig. 4). We get:

$$A' = (\sigma_B \tau_C : \tau_C \tau_A : \sigma_A \sigma_B), \quad A'' = (\tau_B \sigma_C : \sigma_C \sigma_A : \tau_A \tau_B), \quad (6.5)$$

$$B' = (\sigma_B \sigma_C : \sigma_C \tau_A : \tau_A \tau_B), \quad B'' = (\tau_B \tau_C : \tau_C \sigma_A : \sigma_A \sigma_B), \quad (6.6)$$

$$C' = (\tau_B \tau_C : \sigma_C \sigma_A : \sigma_A \tau_B), \quad C'' = (\sigma_B \sigma_C : \tau_C \tau_A : \tau_A \sigma_B). \quad (6.7)$$

Let's find out the equation of circle $\mathcal{C}(H, R)$ in barycentric coordinates. Indeed, the powers of the vertices A , B , C of the reference triangle with respect to this circle are (Fig. 3):

$$\begin{aligned} p_A(\mathcal{C}(H, R)) &= AH^2 - R^2 = 4R^2 \cos^2 A - R^2 = 3R^2 - a^2, \\ p_B(\mathcal{C}(H, R)) &= 3R^2 - b^2, \end{aligned}$$

$$p_C(\mathcal{C}(H, R)) = 3R^2 - c^2.$$

Therefore, the equation of the circle is

$$\sum_{cyclic} a^2 yz - (x + y + z) \sum_{cyclic} (3R^2 - a^2) x = 0. \quad (6.8)$$

But, we have:

$$\begin{aligned} 3R^2 - a^2 &= 3 \frac{a^2 b^2 c^2}{4S^2} - a^2 = \frac{a^2}{4S^2} (3b^2 c^2 - 4S^2) \\ &= \frac{1}{4S^2} (S_B + S_C) [3(S_C + S_A)(S_A + S_B) - 4S^2] \\ &= \frac{1}{4S^2} (S_B + S_C) [3S_A^2 + 3(S_{BC} + S_{CA} + S_{AB}) - 4S^2] \\ &= \frac{1}{4S^2} (S_B + S_C) (3S_A^2 - S^2) = \frac{1}{4S^2} (S_B + S_C) \sigma_A \tau_A, \end{aligned}$$

and the equation of circle $\mathcal{C}(H, R)$ is written in the form

$$4S^2 \cdot \sum_{cyclic} (S_B + S_C) yz - (x + y + z) \sum_{cyclic} (S_B + S_C) (3S_A^2 - S^2) x = 0. \quad (6.9)$$

The preparations for the intended purposes are concluded. We first propose to determine the barycentric coordinates of the point P .

Proposition 6.1. *The barycentric coordinates of the point P are*

$$P = (S_A(3S_B^2 - S^2)(3S_C^2 - S^2) : S_B(3S_C^2 - S^2)(3S_A^2 - S^2) : S_C(3S_A^2 - S^2)(3S_B^2 - S^2)) \quad (6.10)$$

or

$$P = (S_A \sigma_B \sigma_C \tau_B \tau_C : S_B \sigma_C \sigma_A \tau_C \tau_A : S_C \sigma_A \sigma_B \tau_A \tau_B) \quad (6.11)$$

or still

$$P = ((-a^2 + b^2 + c^2) f(b, c, a) f(c, a, b), \dots : \dots), \quad (6.12)$$

where $f(a, b, c) = a^2 + b^2 + c^2 + bc - 2ca - 2ab$.

Proof. Since $P = \alpha_1 \alpha_2 \cap \beta_1 \beta_2 \cap \gamma_1 \gamma_2$, necessarily the barycentric coordinates of P verify the system of equations (6.2)-(6.4). It is easy to show that the solution of this system (up to one factor) is the triplet (6.10). The conclusion follows. $\square$

Remark 6.1. *Other expressions for the barycentric coordinates of the point P :*

$$\begin{aligned} 1) P &= \left(\frac{S_A}{3S_A^2 - S^2} : \dots : \dots \right), \\ 2) P &= \left(\frac{a \cos A}{3 - 4 \sin^2 A} : \dots : \dots \right), \\ 3) P &= \left(\frac{\sin 2A}{1 + \cos 2A} : \dots : \dots \right). \end{aligned}$$

Corollary 6.1. (i) *The points P, F_+, F_- are collinear.*

(ii) *The point P is on the circle $\mathcal{C}(H, R)$ (Fig. 8).*

Proof. (i) The barycentric coordinates of the Fermat points can be written in the form ([13], pp. 36 and 37)

$$F_+ = \left((\sqrt{3}S_B + S) (\sqrt{3}S_C + S) : \dots : \dots \right),$$

$$F_- = \left((\sqrt{3}S_B - S) : (\sqrt{3}S_C - S) : \dots : \dots \right) \quad (6.13)$$

or

$$F_+ = (\sigma_B \sigma_C : \sigma_C \sigma_A : \sigma_A \sigma_B), \quad F_- = (\tau_B \tau_C : \tau_C \tau_A : \tau_A \tau_B). \quad (6.14)$$

Thus, the collinearity of the three points requires only the calculation of a determinant. We omit this calculation.

(ii) By a routine calculation, it is shown that the coordinates of the point P , verify the equation (6.9). So, $P \in \mathcal{C}(H, R)$. $\square$

Remark 6.2. *The statements (i) and (ii) of the preceding sentence have been previously proved by synthetic arguments; they are precisely Proposition 5.2 and Proposition 4.3.*

Now, we will find out the barycentric coordinates of point F^* and establish with analytical arguments some of its properties. As with point P , we will need some preparation beforehand. Let's first find the equations of lines $B'C''$, $C'A''$, $A'B''$; $B''C'$, $C''A'$, $A''B'$. Taking into account (6.5)-(6.7), we will easily find that

$$\begin{aligned} (B'C'') \quad 4S^2x - \sigma_B \sigma_C (x + y + z) &= 0, & (B''C') \quad 4S^2x - \tau_B \tau_C (x + y + z) &= 0, \\ (C'A'') \quad 4S^2y - \sigma_C \sigma_A (x + y + z) &= 0, & (C''A') \quad 4S^2y - \tau_C \tau_A (x + y + z) &= 0, \\ (A'B'') \quad 4S^2z - \sigma_A \sigma_B (x + y + z) &= 0, & (A''B') \quad 4S^2z - \tau_A \tau_B (x + y + z) &= 0. \end{aligned}$$

Since $A_+^* = C'A'' \cap A'B''$, $A_-^* = C''A' \cap A''B'$ etc., by a routine calculation we obtain the coordinates of the vertices of triangles $A_+^*B_+^*C_+^*$, and $A_-^*B_-^*C_-^*$:

$$\begin{aligned} A_+^* &= (4S^2 - \sigma_C \sigma_A - \sigma_A \sigma_B : \sigma_C \sigma_A : \sigma_A \sigma_B), & A_-^* &= (4S^2 - \tau_C \tau_A - \tau_A \tau_B : \tau_C \tau_A : \tau_A \tau_B), \\ B_+^* &= (\sigma_B \sigma_C : 4S^2 - \sigma_A \sigma_B - \sigma_B \sigma_C : \sigma_A \sigma_B), & B_-^* &= (\tau_B \tau_C : 4S^2 - \tau_A \tau_B - \tau_B \tau_C : \tau_A \tau_B), \\ C_+^* &= (\sigma_B \sigma_C : \sigma_C \sigma_A : 4S^2 - \sigma_B \sigma_C - \sigma_C \sigma_A), & C_-^* &= (\tau_B \tau_C : \tau_C \tau_A : 4S^2 - \tau_B \tau_C - \tau_C \tau_A). \end{aligned}$$

The equations of lines $A_+^*A_-^*$, $B_+^*B_-^*$, $C_+^*C_-^*$ are

$$\begin{aligned} (A_+^*A_-^*) \quad (b^2 - c^2) \sigma_A \tau_A (x + y + z) - 4c^2 S^2 y + 4b^2 S^2 z &= 0, \\ (B_+^*B_-^*) \quad 4c^2 S^2 x + (c^2 - a^2) \sigma_B \tau_B (x + y + z) - 4a^2 S^2 z &= 0, \\ (C_+^*C_-^*) \quad -4b^2 S^2 x + 4a^2 S^2 y + (a^2 - b^2) \sigma_C \tau_C (x + y + z) &= 0. \end{aligned}$$

Proposition 6.2. *The barycentric coordinates of the point F^* are*

$$F^* = ((S_B + S_C) (3S_B S_C + S_A^2) + 4S_A S_B S_C : \dots : \dots) \quad (6.15)$$

or

$$F^* = (\varphi(a, b, c) : \varphi(b, c, a) : \varphi(c, a, b)), \quad (6.16)$$

where $\varphi(a, b, c) = a^6 - b^6 - c^6 + b^4 c^2 + b^2 c^4 + 2a^2 b^2 c^2$.

Remark 6.3. *The collinearity of the points F_+ , F_- , F^* , stated in Proposition 5.3, (iii) can be easily demonstrated using the barycentric coordinates of these points (the relations (6.13) and (6.15)).*

Remark 6.4. *Formula (5.4) is the key to a second way of finding the barycentric coordinates of the point F^* .*

Indeed, it is well known that, if the points $F_+ (\alpha_+ : \beta_+ : \gamma_+)$ and $F_- (\alpha_- : \beta_- : \gamma_-)$ check the condition $\alpha_+ + \beta_+ + \gamma_+ = \alpha_- + \beta_- + \gamma_-$, and if $k = \frac{F^* F_-}{F^* F_+}$, then we have $F^* = \left(\frac{\alpha_- - k \alpha_+}{1 - k} : \dots : \dots \right)$.

The sums of the coordinates of the Fermat points given by (6.13) or (6.14) are

$$\begin{aligned} q_+ &= \sum_{cyclic} \sigma_B \sigma_C = \sum_{cyclic} (\sqrt{3} S_B + S) (\sqrt{3} S_C + S) = \sum_{cyclic} (3 S_{BC} + S^2 + \sqrt{3} a^2 S) \\ &= 6 S^2 + 2 \sqrt{3} S_\omega S = 2 \sqrt{3} S (\sqrt{3} S + S_\omega), \\ q_- &= 2 \sqrt{3} S (\sqrt{3} S - S_\omega). \end{aligned}$$

Therefore, we will consider $F_+ (q_- \sigma_B \sigma_C : \dots : \dots)$, $F_- (q_+ \tau_B \tau_C : \dots : \dots)$. By Corollary 5.1,

$$k = \frac{\sqrt{3} (S_\omega^2 - S^2) + 2 S S_\omega}{\sqrt{3} (S_\omega^2 - S^2) - 2 S S_\omega}.$$

Then, the barycentric coordinates of the point F^* are

$$\alpha^* = \frac{\alpha_- - k \alpha_+}{1 - k} = \frac{1}{1 - k} (q_+ \tau_B \tau_C - k q_- \sigma_B \sigma_C) \text{ etc.}$$

Performing the calculations, we find the same expressions for coordinates of F^* up to a constant factor.

Proposition 6.3. *The equation of the line h (i.e. HF^*) is*

$$S_A (S_B^2 - S_C^2) \alpha + S_B (S_C^2 - S_A^2) \beta + S_C (S_A^2 - S_B^2) \gamma = 0 \quad (6.17)$$

or

$$a^2 (b^2 - c^2) (-a^2 + b^2 + c^2) \alpha + b^2 (c^2 - a^2) (a^2 - b^2 + c^2) \beta + c^2 (a^2 - b^2) (a^2 + b^2 - c^2) \gamma = 0. \quad (6.18)$$

Proof. We start with $H = (S_{BC} : S_{CA} : S_{AB})$ and $F^* = (S^2 S_A + 3 S_\omega S_B S_C : \dots : \dots)$. By a simple calculation the equation (6.17) is obtained. $\square$

Remark 6.5. *Points P and F^* are polynomial centers ([7], p. 46) located on the Fermat axis.*

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GH. ASACHI TECHNICAL UNIVERSITY OF IAȘI
CAROL I BLV., NO 11A, 700506 IAȘI
IAȘI, ROMÂNIA
Email address: tbirsan111@gmail.com