



THE GEOMETRIC MEAN LINES OF A CONVEX QUADRILATERAL

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ABSTRACT. The arithmetic-harmonic mean, of two positive numbers, consists of two recursively defined sequences, one being the arithmetic mean and the other the harmonic mean of the previous terms, which both converge to the geometric mean of the initial two numbers. This procedure can be illustrated geometrically by starting with a trapezoid and recursively defining a sequence of decreasing trapezoids, in which the greater base is the midline, while the smaller base is the line that passes through the point of intersection of the diagonals and is parallel to the bases of the previous trapezoid. This sequence of trapezoids shrinks to a line segment, whose length is equal to the geometric mean of the bases of the initial trapezoid. We extend this construction to an arbitrary convex quadrilateral, and define a sequence of decreasing convex quadrilaterals that shrinks to a line segment, that we call a geometric mean line. We estimate the length, of this segment, in terms of the geometric mean of two opposite sides and the angles made by the pairs of opposite sides of the original convex quadrilateral.

1. INTRODUCTION AND MOTIVATIONS

The classic inequality between the harmonic and arithmetic means, of two positive numbers, has a nice visualization using the line that passes through the point of intersection of the diagonals of a trapezoid and is parallel to its bases, and the midline of that trapezoid. A question, that one may ask now, is how the fact that the geometric mean, of two positive numbers, is greater than or equal to the harmonic mean and less than or equal to the arithmetic mean can be visualized in a trapezoid. One way, to realize this, is to construct a sequence of decreasing trapezoids, in which the smaller base and the greater base of the current trapezoid are the harmonic mean and the arithmetic mean of the bases of the previous trapezoid, respectively. This sequence of trapezoids shrinks to a line segment, whose length is exactly the geometric mean of the bases of the initial trapezoid. This construction is a geometric illustration of the fact that the arithmetic-harmonic mean of two positive numbers a and b is their geometric mean, see [1], which means that, if we define $a_0 = a$ and $b_0 = b$, and for all $n \geq 1$:

$$a_n = 2 \left/ \left(\frac{1}{a_{n-1}} + \frac{1}{b_{n-1}} \right) \right. \quad \text{and} \quad b_n = \frac{a_{n-1} + b_{n-1}}{2},$$

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then it is not hard to see that both sequences $\{a_n\}_{n \geq 0}$ and $\{b_n\}_{n \geq 0}$ converge to the geometric mean, $\sqrt{ab}$, of a and b . In fact, the geometric mean is an invariant of the two sequences because we have:

$$\sqrt{a_0 b_0} = \sqrt{a_1 b_1} = \sqrt{a_2 b_2} = \dots$$

Since the line, that passes through the point of intersection of the diagonals of a trapezoid and is parallel to the bases, is the polar of the point of intersection of the non-parallel sides of the trapezoid with respect to the conic given by the union of its two bases, we can extend the construction of the sequence of decreasing trapezoids, to an arbitrary convex quadrilateral, by defining a sequence of decreasing convex quadrilaterals, in which one side of the current convex quadrilateral is the polar of the point of intersection of two opposite sides of the previous quadrilateral with respect to the conic made up of the union of the other two opposite sides, and its opposite side is the line joining the midpoints of two opposite sides of the previous quadrilateral. This sequence of convex quadrilaterals shrinks to a line segment, that we may call a *geometric mean line* of the original convex quadrilateral. The length of this segment is a constant (depending of the angles of the quadrilateral) times the geometric mean of the lengths of two opposite sides of the original convex quadrilateral. We find sharp estimates of this constant.

The problem that we study is the following:

Problem Let $ABCD$ be a convex quadrilateral such that $AD \nparallel BC$. Let $\{I\} := AC \cap BD$. We define a new quadrilateral $A_1 B_1 C_1 D_1$ in the following way:

- If $AB \parallel CD$, then we let A_1 and B_1 be the midpoints of the segments AD and BC , respectively, and C_1 and D_1 the points of intersection of the line, that goes through I and is parallel to AB , with BC and AD , respectively.
- If $AB \nparallel CD$, then let $\{E\} := AB \cap CD$. Let A_1 and B_1 , be the midpoints of AD and BC , respectively, and $\{C_1\} := EI \cap BC$ and $\{D_1\} := EI \cap AD$.

We then repeat this construction, each time constructing a new quadrilateral $A_n B_n C_n D_n$ from a previously constructed $A_{n-1} B_{n-1} C_{n-1} D_{n-1}$, for all $n \geq 2$. For every quadrilateral $MNPQ$, we denote by $[MNPQ]$ the union of the quadrilateral $MNPQ$ and its interior.

- (1) If $ABCD$ is a trapezoid, with $AB \parallel CD$, then find the set:

$$\mathcal{I}_{ABCD} = \bigcap_{n=1}^{\infty} [A_n B_n C_n D_n],$$

and its diameter $\delta(\mathcal{I}_{ABCD})$ in terms of the lengths b_1 and b_2 of AB and CD , respectively, where $\delta(\mathcal{I}_{ABCD}) := \sup\{|RS| \mid R, S \in \mathcal{I}_{ABCD}\}$ ($|RS|$ denotes the length of the segment RS).

- (2) If $AB \nparallel CD$ and $BC \nparallel AD$, then show that for all $n \geq 1$, $A_n B_n \nparallel C_n D_n$, and find

$$\mathcal{I}_{ABCD} := \bigcap_{n=1}^{\infty} [A_n B_n C_n D_n].$$

- (3) Let's suppose that $AB \nparallel CD$ and $AD \nparallel BC$. Let $\{E\} := AB \cap CD$ and $\{F\} := AD \cap BC$. For any two positive numbers α and β , such that $\alpha + \beta < \pi$, let $\mathcal{Q}_{\alpha, \beta}$ denote the set of all convex quadrilaterals $ABCD$ in a plane, such that the measures of the angles $\angle AED$ and $\angle AFB$ are α and β , respectively. Prove that there are two positive constants c_1 and c_2 , such that, for all $ABCD \in \mathcal{Q}_{\alpha, \beta}$, we have:

$$c_1 b_1 b_2 \leq \delta^2(\mathcal{I}_{ABCD}) \leq c_2 b_1 b_2, \quad (1.1)$$

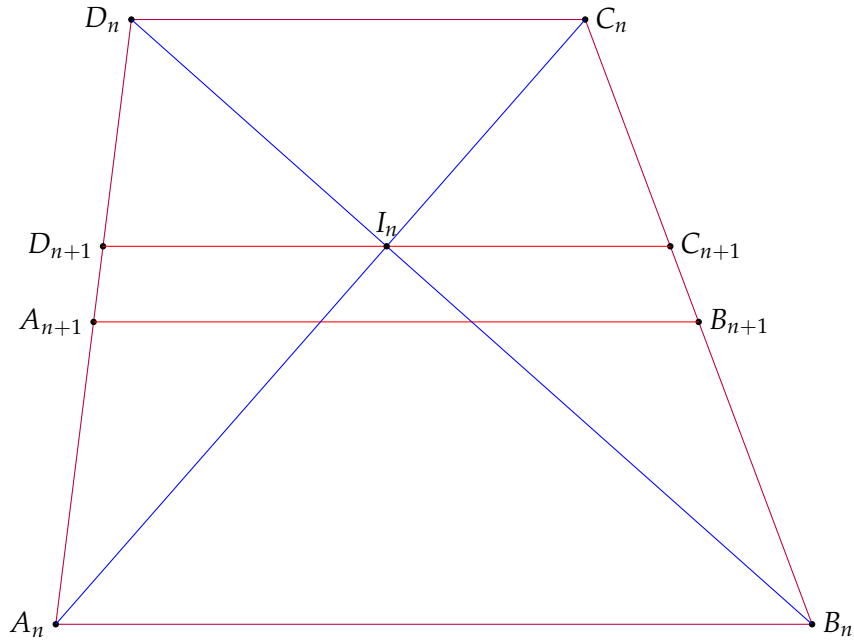


Figure 1. If AB and CD are parallel

where $b_1 := |AB|$ and $b_2 := |CD|$. Find the best possible constants c_1 and c_2 , and if any one of the two inequalities from (1.1) can become an equality, describe all quadrilaterals $ABCD$ for which this happens.

- (4) Describe all pairs (α, β) , of positive numbers, for which $c_1 = c_2$, where c_1 and c_2 , are the two best constants from the previous part.

The paper is structured as follows. In subsection 2.1, we present the geometric mean line of a trapezoid. Finally, in subsection 2.2, we compute the length l_g of a geometric mean line (the other being computed in a similar way) of a convex quadrilateral, and find sharp estimates of its length in terms of the geometric mean of the opposite sides of the quadrilateral.

2. MAIN RESULTS

2.1. The geometric mean line of a trapezoid. Let $ABCD$ be a trapezoid with $AB \parallel CD$. Let $b_1 = |AB|$ and $b_2 = |CD|$. We assume that $|AB| > |CD|$, that means $b_1 > b_2$. We have:

$$\begin{aligned} \frac{|CC_1|}{|C_1B|} &= \frac{|CI|}{|IA|} \\ &= \frac{|CD|}{|AB|} \\ &= \frac{b_2}{b_1} \\ &< 1. \end{aligned}$$

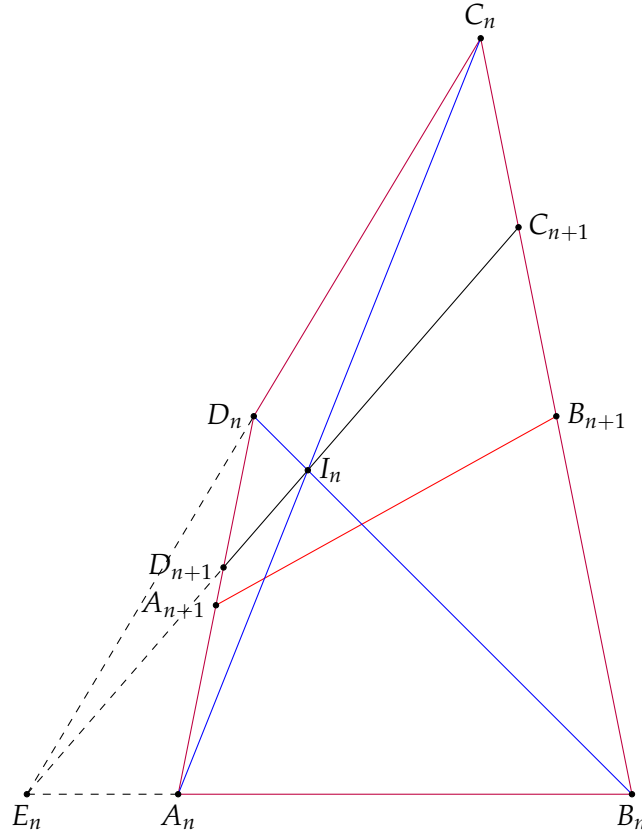


Figure 2. If AB and CD are not parallel

Thus, $|CC_1| < |C_1B|$ and so, $|CC_1| < (1/2)|CB| = |CB_1|$. It follows that C_1 is in between B_1 and C , and D_1 is in between A_1 and D . Let $S(\mathcal{P})$ denote the area of any polygon $\mathcal{P}$. We have:

$$\begin{aligned} S(A_1B_1C_1D_1) &< S(A_1B_1CD) \\ &\leq \frac{1}{2}S(ABCD), \end{aligned}$$

since $S(A_1B_1CD) < S(ABB_1A_1)$. Thus, it follows, that for all $n \geq 1$, we have:

$$S(A_nB_nC_nD_n) < \frac{1}{2^n}S(ABCD).$$

$\mathcal{I}_{ABCD}$ is a convex set, being an intersection of convex sets. We cannot have three non-collinear points M , N , and P in $\mathcal{I}_{ABCD}$, since, otherwise, due to the convexity of $\mathcal{I}_{ABCD}$, the triangle MNP and its interior will be contained in it, which will imply that, for all $n \geq 1$,

$$\begin{aligned} S(MNP) &\leq S(A_nB_nC_nD_n) \\ &< \frac{1}{2^n}S(ABCD), \end{aligned}$$

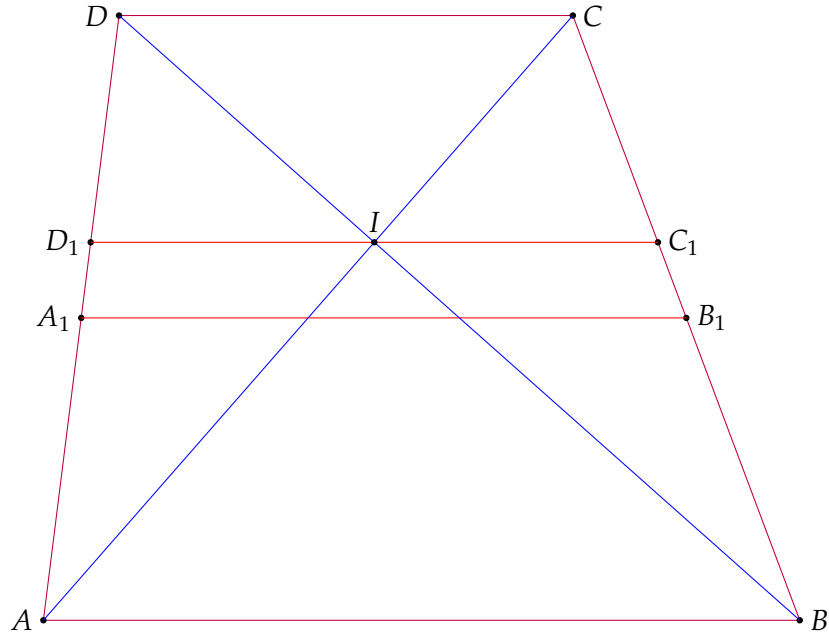


Figure 3. If AB and CD are parallel

which is not possible. Hence any three points in $\mathcal{I}_{ABCD}$ are collinear, and so this set is a segment $[UV]$.

Let us compute first the length $|C_1D_1|$ in terms of b_1 and b_2 . We have:

$$\frac{|IC_1|}{|AB|} = \frac{|CC_1|}{|CB|} \quad (2.1)$$

and

$$\frac{|IC_1|}{|CD|} = \frac{|BC_1|}{|BC|}. \quad (2.2)$$

Adding (2.1) and (2.2), we get:

$$\frac{|IC_1|}{|AB|} + \frac{|IC_1|}{|CD|} = 1.$$

Thus:

$$|IC_1| = \frac{1}{\frac{1}{b_1} + \frac{1}{b_2}}.$$

Similarly:

$$|ID_1| = \frac{1}{\frac{1}{b_1} + \frac{1}{b_2}}.$$

Hence, $|C_1D_1|$ is the harmonic mean of b_1 and b_2 , see also [2], that means:

$$|C_1D_1| = \frac{2b_1b_2}{b_1 + b_2}. \quad (2.3)$$

It is well known, that $|A_1B_1|$ is the arithmetic mean of b_1 and b_2 , that means:

$$|A_1B_1| = \frac{b_1 + b_2}{2}.$$

Let us observe now that the geometric mean, of two positive numbers b_1 and b_2 , is equal to the geometric mean of their arithmetic and harmonic means $(b_1 + b_2)/2$ and $2b_1b_2/(b_1 + b_2)$, due to the fact that:

$$\sqrt{\frac{b_1 + b_2}{2} \cdot \frac{2b_1b_2}{b_1 + b_2}} = \sqrt{b_1b_2}.$$

It follows from here that $|AB| \cdot |CD| = |A_1B_1| \cdot |C_1D_1|$, and thus the product of the bases is an invariant, in the sense that for all $n \geq 1$, $|A_nB_n| \cdot |C_nD_n| = b_1 \cdot b_2$. It is clear that, if $U \in AD$ and $V \in BC$, such that $UV \parallel AB$, then $[UV] \subset A_nB_nC_nD_n$ if and only if $|A_nB_n| \geq |UV| \geq |C_nD_n|$, for all $n \geq 1$. Thus, if we consider the points $U \in AD$ and $V \in BC$, such that $UV \parallel AB$ and $|UV| = \sqrt{b_1b_2}$, then $[UV] \subset [A_nB_nC_nD_n]$, for all $n \geq 1$. Hence, we proved that:

$$\mathcal{I}_{ABCD} = [UV],$$

where $[UV]$ is the segment whose extremities lie on AD and BC , and whose length is the geometric mean of b_1 and b_2 . So:

$$\delta(\mathcal{I}_{ABCD}) = \sqrt{b_1b_2}. \quad (2.4)$$

An interesting known property, of the geometric line $[UV]$, is the fact that it divides the trapezoid $ABCD$ into two similar trapezoids: $UVCD$ and $ABVU$, since it is not hard to see that we have:

$$\frac{|CD|}{|VU|} = \frac{|DU|}{|UA|} = \frac{|UV|}{|AB|} = \frac{|VC|}{|BV|} = \frac{\sqrt{b_2}}{\sqrt{b_1}}$$

and

$$\angle CDU \equiv \angle VUA, \quad \angle DUV \equiv \angle UAB, \quad \angle UVC \equiv \angle ABV.$$

2.2. The geometric mean lines of a convex quadrilateral. Let us suppose now that neither AB and CD , nor AD and BC are parallel. Let $\{E\} = AB \cap CD$ and $\{F\} = AD \cap BC$. Let's assume that C is between B and F , and A is between E and B .

We can see that the points C_1 and F are harmonic conjugated with respect to C and B , and D_1 and F are harmonic conjugated with respect to D and A , in two ways. First way is to consider the (degenerate) conic given by the lines AB and DC , and see that because $\{E\} = AB \cap CD$ and $\{I\} = AC \cap BD$, the line EI is the polar of the point F with respect to this conic. Thus:

$$\frac{|C_1C|}{|C_1B|} = \frac{|FC|}{|FB|} \quad (2.5)$$

and

$$\frac{|D_1D|}{|D_1A|} = \frac{|FD|}{|FA|}. \quad (2.6)$$

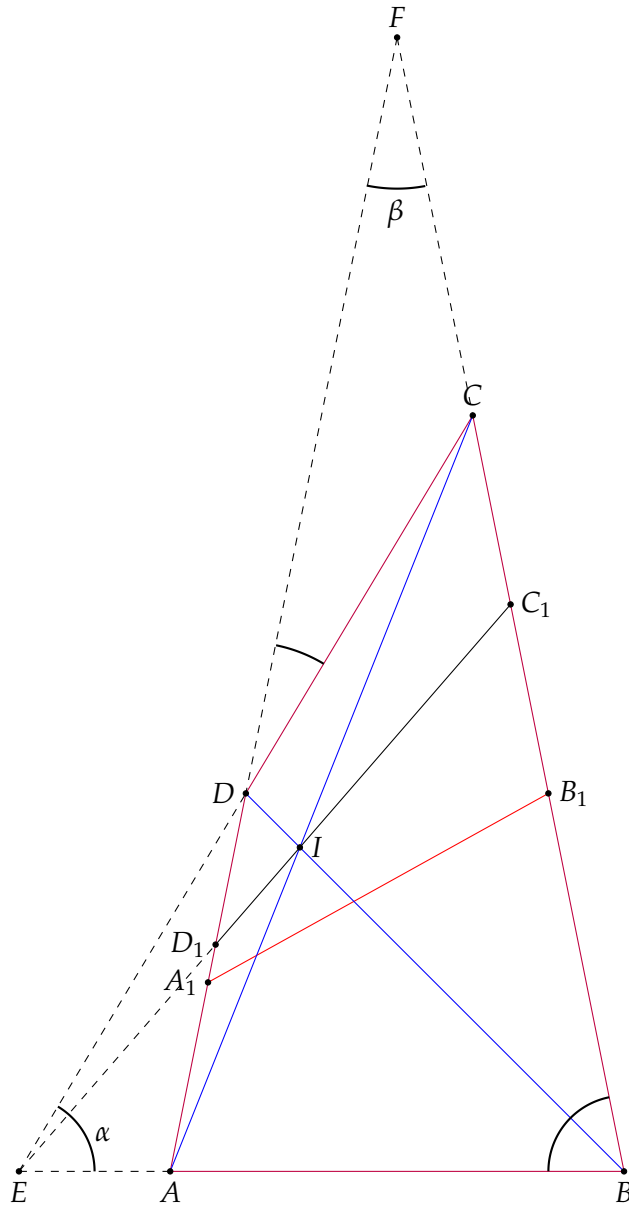


Figure 4. If AB and CD are not parallel

Another way is to apply the theorem of Menelaus in the triangle EBC cut by the transversal (line) FDA , and the theorem of Ceva in the triangle EBC , with respect to the concurrent lines: EC_1 , CA and BD . Thus we get:

$$\frac{|FC|}{|FB|} \cdot \frac{|AB|}{|AE|} \cdot \frac{|DE|}{|DC|} = 1, \quad (2.7)$$

$$\frac{|C_1C|}{|C_1B|} \cdot \frac{|AB|}{|AE|} \cdot \frac{|DE|}{|DC|} = 1. \quad (2.8)$$

We conclude, from (2.7) and (2.8), that $|FC|/|FB| = |C_1C|/|C_1B|$. Similarly, we can see that $|FD|/|FA| = |D_1D|/|D_1A|$.

It follows now that $|FC_1|$ is the harmonic mean of $|FC|$ and $|FB|$, and $|FD_1|$ is the harmonic mean of $|FD|$ and $|FA|$. Since A_1 is the midpoint of AD , and B_1 is the midpoint of BC , we conclude that $|FA_1|$ is the arithmetic mean of $|FD|$ and $|FA|$, and $|FB_1|$ is the arithmetic mean of $|FC|$ and $|FB|$. Thus, $|FC_1| < |FB_1|$ and $|FD_1| < |FA_1|$. Hence, C_1 is between C and B_1 , and D_1 is between D and A_1 .

We cannot have $C_1D_1 \parallel A_1B_1$, since if they were parallel, then it would follow that:

$$\begin{aligned} \frac{|FD_1|}{|FA_1|} &= \frac{|FC_1|}{|FB_1|} \\ \Leftrightarrow \frac{2|FD||FA|}{(|FD| + |FA|)^2} &= \frac{2|FC||FB|}{(|FC| + |FB|)^2} \\ \Leftrightarrow \frac{4|FD||FA|}{(|FD| + |FA|)^2} &= \frac{4|FC||FB|}{(|FC| + |FB|)^2} \\ \Leftrightarrow \frac{y}{(y+1)^2} &= \frac{z}{(z+1)^2}, \end{aligned}$$

where $y := |FD|/|FA| \in (0, 1)$ and $z := |FC|/|FB| \in (0, 1)$. It is easy to see that the equality $y/(y+1)^2 = z/(z+1)^2$ implies either $y = z$ or $yz = 1$. The equality $yz = 1$ is impossible since $0 < yz < 1$, while the equality $y = z$ implies that $AB \parallel CD$ which we assumed not to hold.

Since C is between F and B , and $|CB_1| = |B_1B|$, denoting by h_D and h_A the distances between D and BC , and A and BC , respectively, we have:

$$\begin{aligned} \frac{S(CDB_1)}{S(BAB_1)} &= \frac{h_D}{h_A} \cdot \frac{|CB_1|}{|B_1B|} \\ &= \frac{|FD|}{|FA|} \cdot 1 \\ &< 1. \end{aligned}$$

Thus, we obtain:

$$S(CDB_1) < S(BAB_1). \quad (2.9)$$

Since $DA_1 = A_1A$, we have:

$$S(DB_1A_1) = S(A_1B_1A). \quad (2.10)$$

Adding (2.9) and (2.10), we get:

$$S(A_1B_1CD) < S(ABB_1A_1). \quad (2.11)$$

It follows, from (2.11), that $S(A_1B_1CD) < (1/2)S(ABCD)$. Thus:

$$\begin{aligned} S(A_1B_1C_1D_1) &< S(A_1B_1CD) \\ &< \frac{1}{2}S(ABCD). \end{aligned}$$

We can conclude now, as before, that $\mathcal{I}_{ABCD}$ is a segment.
Since:

$$|FD_1| = \frac{2|FD| \cdot |FA|}{|FD| + |FA|}$$

and

$$|FA_1| = \frac{|FD| + |FA|}{2},$$

we get

$$|FD_1| \cdot |FA_1| = |FD| \cdot |FA|.$$

Thus,

$$|FD| \cdot |FA| = |FD_1| \cdot |FA_1| = |FD_2| \cdot |FA_2| = \dots \quad (2.12)$$

Similarly,

$$|FC| \cdot |FB| = |FC_1| \cdot |FB_1| = |FC_2| \cdot |FB_2| = \dots \quad (2.13)$$

Let $U \in AD$ and $V \in BC$, such that:

$$|FU| = \sqrt{|FD| \cdot |FA|}$$

and

$$|FV| = \sqrt{|FC| \cdot |FB|}.$$

It is now clear that, for all $n \geq 1$, we have:

$$|FD_n| < |FU| < |FA_n|$$

and

$$|FC_n| < |FV| < |FB_n|.$$

Thus, for all $n \geq 1$, $[UV] \subset [A_n B_n C_n D_n]$. Therefore,

$$\mathcal{I}_{ABCD} = [UV].$$

We compute now the length of $[UV]$. Let B denote the measure of the angle $\angle ABC$, and D the measure of the angle $\angle CDF$ (B is an interior angle of the quadrilateral $ABCD$, while D is an exterior angle of $ABCD$). Since $m(\angle BAD) = m(\angle AED) + m(\angle ADE)$, $m(\angle BCD) = m(\angle CFD) + m(\angle CDF)$, $m(\angle ADC) = \pi - m(\angle CDF)$, and $m(\angle BAD) + m(\angle ADC) + m(\angle DCB) + m(\angle CBA) = 2\pi$, we conclude that:

$$\alpha + \beta + B + D = \pi. \quad (2.14)$$

Conversely, it is easy to see that, given any angle B such that $\alpha + \beta + B < \pi$, we can construct a quadrilateral $ABCD$ in $\mathcal{Q}_{\alpha, \beta}$, such that the measure of the angle $\angle ABC$ is B . For example, let ϵ_1 and ϵ_2 be positive numbers such that: $\alpha + \beta + B + \epsilon_1 + \epsilon_2 = \pi$. We construct first a triangle BEF in which the measure of $\angle EBF$ is B , the measure of $\angle BEF$ is $\alpha + \epsilon_1$, and the measure of $\angle BFE$ is $\beta + \epsilon_2$. Then, inside of this triangle, we draw the lines (segments) EC and FA , where $C \in FB$ and $A \in EB$, such that the measure of $\angle BEC$ is α and the measure of $\angle BFA$ is β . Let $\{D\} := EC \cap FA$. The convex quadrilateral $ABCD$ satisfies the required conditions.

We will compute now the length $|UV|$, which represents the diameter of the set $\mathcal{I}_{ABCD}$, in terms of $b_1 = |AB|$, $b_2 = |CD|$, α , β , B , and D .

From the law of sines in the triangle FCD , since $\angle FCD = \alpha + B$, we obtain:

$$|FD| = \frac{b_2 \sin(\alpha + B)}{\sin \beta} \quad (2.15)$$

and

$$|FC| = \frac{b_2 \sin D}{\sin \beta}. \quad (2.16)$$

From the law of sines in the triangle FAB , since $\angle FAB = \alpha + D$, we obtain:

$$|FA| = \frac{b_1 \sin B}{\sin \beta} \quad (2.17)$$

and

$$|FB| = \frac{b_1 \sin(\alpha + D)}{\sin \beta}. \quad (2.18)$$

Since $|FU| = \sqrt{|FD| \cdot |FA|}$ and $|FV| = \sqrt{|FC| \cdot |FB|}$, from the law of cosines in the triangle FUV , we obtain:

$$\begin{aligned} |UV|^2 &= |FU|^2 + |FV|^2 - 2|FU| \cdot |FV| \cdot \cos(\angle UFV) \\ &= |FD| \cdot |FA| + |FC| \cdot |FB| - 2\sqrt{|FD| \cdot |FA| \cdot |FC| \cdot |FB|} \cos \beta \\ &= \frac{b_1 b_2}{\sin^2 \beta} [\sin(\alpha + B) \sin B + \sin(\alpha + D) \sin D \\ &\quad - 2\sqrt{\sin(\alpha + B) \sin B \sin(\alpha + D) \sin D} \cos \beta]. \end{aligned}$$

Using first the formula for changing the product of sines into a difference of cosines, and then the formula for changing a sum of cosines into a product of cosines, we get:

$$\begin{aligned} &|UV|^2 \\ &= \frac{b_1 b_2}{2 \sin^2 \beta} \{ \cos \alpha - \cos(\alpha + 2B) + \cos \alpha - \cos(\alpha + 2D) \\ &\quad - 2\sqrt{\sin B \sin D \sin(\alpha + B) \sin(\alpha + D)} \cos \beta \} \\ &= \frac{b_1 b_2 \cos \alpha}{\sin^2 \beta} - \frac{b_1 b_2}{\sin^2 \beta} \{ \cos(\alpha + B + D) \cos(|B - D|) \\ &\quad - \sqrt{[\cos(|B - D|) - \cos(B + D)] [\cos(|B - D|) - \cos(2\alpha + B + D)]} \cos \beta \} \\ &= \frac{b_1 b_2 \cos \alpha}{\sin^2 \beta} - \frac{b_1 b_2}{\sin^2 \beta} \cos(\pi - \beta) \cos(|B - D|) \\ &\quad - \frac{b_1 b_2}{\sin^2 \beta} \sqrt{[\cos(|B - D|) + \cos(\alpha + \beta)] [\cos(|B - D|) + \cos(\alpha - \beta)]} \cos \beta, \end{aligned}$$

since $B + D = \pi - (\alpha + \beta)$ and $2\alpha + B + D = \pi + (\alpha - \beta)$. Thus, we get:

$$|UV|^2 = \frac{b_1 b_2 \cos \alpha}{\sin^2 \beta} + \frac{b_1 b_2}{\sin^2 \beta} \cos \beta f(|B - D|), \quad (2.19)$$

where:

$$f(x) = \cos x - \sqrt{[\cos x + \cos(\alpha + \beta)][\cos x + \cos(\alpha - \beta)]}. \quad (2.20)$$

The derivative of the function f is:

$$f'(x) = -\sin x + \frac{\cos x \sin x + \cos \alpha \cos \beta \sin x}{\sqrt{[\cos x + \cos(\alpha + \beta)][\cos x + \cos(\alpha - \beta)]}}.$$

Claim. $f'(x) \geq 0$, for all $x \in [0, \pi - \alpha - \beta]$.

Indeed, we have:

$$\begin{aligned} f'(x) &= \frac{\sin x}{\sqrt{[\cos x + \cos(\alpha + \beta)][\cos x + \cos(\alpha - \beta)]}} \\ &\quad \times \left\{ \cos x + \cos \alpha \cos \beta - \sqrt{[\cos x + \cos(\alpha + \beta)][\cos x + \cos(\alpha - \beta)]} \right\} \\ &= \frac{\sin x}{\sqrt{[\cos x + \cos(\alpha + \beta)][\cos x + \cos(\alpha - \beta)]}} \\ &\quad \times \frac{(\cos x + \cos \alpha \cos \beta)^2 - [\cos x + \cos(\alpha + \beta)][\cos x + \cos(\alpha - \beta)]}{\cos x + \cos \alpha \cos \beta + \sqrt{[\cos x + \cos(\alpha + \beta)][\cos x + \cos(\alpha - \beta)]}} \\ &= \frac{\sin x}{\sqrt{[\cos x + \cos(\alpha + \beta)][\cos x + \cos(\alpha - \beta)]}} \\ &\quad \times \frac{\sin^2 \alpha \sin^2 \beta}{\cos x + \cos \alpha \cos \beta + \sqrt{[\cos x + \cos(\alpha + \beta)][\cos x + \cos(\alpha - \beta)]}}. \end{aligned}$$

Since $0 \leq x < \pi - \alpha - \beta < \pi$ and the function $t \mapsto \cos t$ is decreasing on $[0, \pi]$, we have:

$$\begin{aligned} \cos x &> \cos(\pi - \alpha - \beta) \\ &= -\cos(\alpha + \beta) \\ &= -\cos \alpha \cos \beta + \sin \alpha \sin \beta. \end{aligned}$$

Thus:

$$\begin{aligned} \cos x + \cos \alpha \cos \beta &> \sin \alpha \sin \beta \\ &> 0. \end{aligned}$$

It follows now from the last relation that $f'(x) \geq 0$, for all $x \in [0, \pi - \alpha - \beta]$.

In our case $x = |B - D| < B + D = \pi - \alpha - \beta$. Since the function f is increasing on $[0, \pi - \alpha - \beta]$, and

$$|UV|^2 = \frac{b_1 b_2 \cos \alpha}{\sin^2 \beta} + \frac{b_1 b_2}{\sin^2 \beta} \cos \beta f(|B - D|),$$

we conclude that we have two cases.

Case 1. If $0 < \beta \leq \pi/2$, then $\cos \beta \geq 0$, and thus:

$$\frac{\cos \alpha + \cos \beta f(0)}{\sin^2 \beta} b_1 b_2 \leq |UV|^2 \leq \frac{\cos \alpha + \cos \beta f(\pi - \alpha - \beta)}{\sin^2 \beta} b_1 b_2. \quad (2.21)$$

Case 2. If $\pi/2 < \beta < \pi$, then $\cos \beta < 0$, and thus:

$$\frac{\cos \alpha + \cos \beta f(\pi - \alpha - \beta)}{\sin^2 \beta} b_1 b_2 \leq |UV|^2 \leq \frac{\cos \alpha + \cos \beta f(0)}{\sin^2 \beta} b_1 b_2. \quad (2.22)$$

Let us compute $f(0)$ and $f(\pi - \alpha - \beta)$.

$$\begin{aligned} f(0) &= \cos 0 - \sqrt{[\cos 0 + \cos(\alpha + \beta)][\cos 0 + \cos(\alpha - \beta)]} \\ &= 1 - \sqrt{2 \cos^2((\alpha + \beta)/2) \cdot 2 \cos^2((\alpha - \beta)/2)} \\ &= 1 - 2 \cos((\alpha + \beta)/2) \cos((\alpha - \beta)/2) \\ &= 1 - \cos \alpha - \cos \beta. \end{aligned}$$

$$\begin{aligned} f(\pi - \alpha - \beta) &= -\cos(\alpha + \beta) - \sqrt{[-\cos(\alpha + \beta) + \cos(\alpha + \beta)][-\cos(\alpha + \beta) + \cos(\alpha - \beta)]} \\ &= -\cos(\alpha + \beta). \end{aligned}$$

Thus:

$$\begin{aligned} \frac{\cos \alpha + \cos \beta f(0)}{\sin^2 \beta} &= \frac{\cos \alpha + \cos \beta (1 - \cos \alpha - \cos \beta)}{1 - \cos^2 \beta} \\ &= \frac{(1 - \cos \beta)(\cos \alpha + \cos \beta)}{(1 - \cos \beta)(1 + \cos \beta)} \\ &= \frac{\cos \alpha + \cos \beta}{1 + \cos \beta} \end{aligned}$$

and

$$\begin{aligned} \frac{\cos \alpha + \cos \beta f(\pi - \alpha - \beta)}{\sin^2 \beta} &= \frac{\cos \alpha - \cos \beta \cos(\alpha + \beta)}{\sin^2 \beta} \\ &= \frac{\cos((\alpha + \beta) - \beta) - \cos(\alpha + \beta) \cos \beta}{\sin^2 \beta} \\ &= \frac{\sin(\alpha + \beta) \sin \beta}{\sin^2 \beta} \\ &= \frac{\sin(\alpha + \beta)}{\sin \beta}. \end{aligned}$$

Hence, the best constants c_1 and c_2 , such that:

$$c_1 b_1 b_2 \leq |UV|^2 \leq c_2 b_1 b_2$$

are:

$$c_1 = \frac{\cos \alpha + \cos \beta}{1 + \cos \beta}$$

and

$$c_2 = \frac{\sin(\alpha + \beta)}{\sin \beta},$$

if $\beta \leq \pi/2$, and

$$c_1 = \frac{\sin(\alpha + \beta)}{\sin \beta}$$

and

$$c_2 = \frac{\cos \alpha + \cos \beta}{1 + \cos \beta},$$

if $\beta > \pi/2$.

If $\beta \neq \pi/2$ and $\alpha > 0$, then the equality

$$|UV|^2 = \frac{\cos \alpha + \cos \beta}{1 + \cos \beta} b_1 b_2$$

is attained when $x = |B - D| = 0$, which means $B = D$, or equivalently, the quadrilateral $ABCD$ is cyclic (can be inscribed in a circle).

If $\beta \neq \pi/2$ and $\alpha > 0$, then the equality

$$|UV|^2 = \frac{\sin(\alpha + \beta)}{\sin \beta} b_1 b_2$$

is never attained, since $x = |B - D| < \pi - \alpha - \beta$. However, as $D \rightarrow 0$ or $B \rightarrow 0$, since $B + D = \pi - \alpha - \beta$, we get $x \rightarrow \pi - \alpha - \beta$. So, this equality can be attained only asymptotically.

We observe that $|B - D|$ can be intuitively regarded as a way of measuring how far away is the quadrilateral $ABCD$ from being cyclic. So, we may refer to $x = |B - D|$ as a *cyclic defect* of the quadrilateral $ABCD$.

If $\alpha > 0$ and $\beta > 0$, such that $\alpha + \beta < \pi$, then the equality $c_1 = c_2$ can occur if and only if $\cos \beta = 0$, because the function $f(x)$, introduced in the previous step, is strictly increasing. Thus, $c_1 = c_2$ if and only if $\beta = \pi/2$. In this case:

$$|UV|^2 = b_1 b_2 \cos \alpha \tag{2.23}$$

$$= |\vec{AB} \cdot \vec{CD}|, \tag{2.24}$$

where $\vec{v} \cdot \vec{w}$ denotes the scalar (dot) product of the vectors $\vec{v}$ and $\vec{w}$.

Of course, we also have $c_1 = c_2$ (both are equal to 1), in the case $AB \parallel CD$, which means $\alpha = 0$, but this case was excluded since we assumed that both α and β were strictly positive.

Finally, we have also excluded the case when $AD \parallel BC$. In that case, the two segments A_1B_1 and D_1C_1 coincide, since in any trapezoid the line, joining the point of intersection of the nonparallel sides with the point of intersection of the diagonals, passes through the midpoints of the parallel sides. Thus, the sequence of convex quadrilaterals $\{A_n B_n C_n D_n\}_{n \geq 1}$ degenerates into a constant sequence of segments $\{A_n B_n\}_{n \geq 1}$. In this case, $\beta = 0$, and so, if $\alpha > 0$, then the constant $\sin(\alpha + \beta) / \sin \beta$ becomes infinity. Therefore, in this degenerate case, the length of the geometric line cannot be bounded by a constant times the geometric mean of b_1 and b_2 ; however, since the geometric line is also the midline, we can bound it by their arithmetic mean, $(b_1 + b_2)/2$, which can be seen in

the following way.

Let M be the midpoint of the diagonal $[BD]$. We have:

$$\begin{aligned}|A_1B_1| &\leq |A_1M| + |MB_1| \\ &= \frac{1}{2}|AB| + \frac{1}{2}|DC| \\ &= \frac{b_1 + b_2}{2}.\end{aligned}$$

We may call A_1B_1 , C_1D_1 , and UV , the *arithmetic*, *harmonic*, and *geometric lines* of the quadrilateral $ABCD$ corresponding to the opposite sides AB and CD . Of course, there is another arithmetic, harmonic, and geometric triplet corresponding to the other two opposite sides BC and AD .

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