



CONSERVATION OF ALTITUDE-TO-CENTER ANGLES UNDER AN ORTHOCENTRIC INVERSION

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ABSTRACT. Let ABC be a non-right triangle with orthocenter H and circumradius R , and let A^*, B^*, C^* be the images of A, B, C under the inversion centered at H with power $(2R)^2$. We show that the angle a given altitude of ABC makes with the segment from its foot to the circumcenter O reappears exactly, as a numerical identity, as the analogous angle in the derived triangle $A^*B^*C^*$ — measured there not from a circumcenter, but from its incenter (if ABC is acute) or from the appropriate excenter (if ABC is obtuse). This conservation of an angle across two triangles related only by an inversion, and across two different associated triangle centers, is unexpected and is the principal object of this note. The individual closed-form angle expressions used to state and prove it — for the circumcenter, incenter, and excenters in a single triangle — are recorded as supporting lemmas; while elementary, we include full proofs, and note one genuine synthetic explanation among them (Lemma 2 and the first case of Lemma 3 share a common Apollonius-circle argument).

1. NOTATIONS

Let ABC be a triangle with angles A, B, C , opposite sides $a = BC, b = CA, c = AB$, circumradius R , orthocenter H , circumcenter O , incenter I , and excenters I_A, I_B, I_C opposite A, B, C respectively. We assume throughout that ABC is not right-angled.

This restriction is needed already at the level of Lemma 1, independent of the application in §5: if some angle of ABC equals 90° , the corresponding instance has $\cos A = 0$ in the denominator, and the degeneracy is genuine rather than a mere artifact of the formula — the circumcenter then lies on the side opposite the right angle, so O and the relevant altitude foot both lie on that side, forcing the angle θ_A to be exactly 90° outright, rather than merely undefined. No corresponding restriction is needed for the equilateral case: every instance of Lemmas 1–3 remains perfectly well defined there, simply equal to 0 (since O, I, H coincide), and Theorem 6 (§4) correspondingly reduces to the true, if unilluminating, identity $0 = 0$.

Let A', B', C' denote the feet of the altitudes from A, B, C . For a triangle PQR (with its own circumcenter, incenter, and excenters) and a vertex P , we write $\theta_P(PQR), \gamma_P(PQR), \alpha_P(PQR)$ for the (undirected) angle that the altitude from P makes, at its foot, with the segment from that foot to the circumcenter, incenter, and P -excenter of PQR respectively

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— so that Lemmas 1–3 below, although stated for a single triangle ABC , apply verbatim to any triangle, including the derived triangle $A^*B^*C^*$ introduced in §3.

2. THREE ANGLE FORMULAS (PRELIMINARIES)

The identity that is the object of this note (Theorem 6, §4) compares an angle computed from one triangle's circumcenter with an angle computed from a different triangle's incenter or excenter. To state it, we first record closed forms for all three angles in a single triangle. These are elementary consequences of standard formulas for the circumcenter, incenter, and excenters, and may already be known in some form; we include full derivations only so that the note is self-contained — the substantive content of the paper begins in §4.

Lemma 2.1. $\tan \theta_A(ABC) = \frac{|\sin(B - C)|}{\cos A}$, and symmetrically for B, C .

Proof. Let M be the midpoint of BC . Since $OM \perp BC$ and $AA' \perp BC$, we have $OM \parallel AA'$, so $\tan \theta_A = A'M/OM$. With $OM = R \cos A$, $BM = a/2$, $BA' = c \cos B$, and $a = 2R \sin A$, $c = 2R \sin C$ (Law of Sines), $A'M = R(\sin A - 2 \sin C \cos B)/1 = R \sin(B - C)$ upon expanding $\sin A = \sin(B + C)$; dividing by OM gives the formula (up to the sign conventions of the case $C > B$ or $A > 90^\circ$, absorbed into the absolute value). $\square$

Lemma 2.2. $\tan \gamma_A(ABC) = \frac{|\sin((B - C)/2)|}{\sin(A/2)}$, and symmetrically for B, C .

Proof. Let I' be the touch point of the incircle on BC . Since $II' \parallel AA'$, $\tan \gamma_A = A'I'/r$. Using $s = (a + b + c)/2$, $BI' = BI \cdot \cos(B/2)$ and $BA' = c \cdot \cos B$, assume first $B \geq C$ (so A' lies between B and I'):

$$\begin{aligned} \tan \gamma_A &= A'I'/II' = (BI \cdot \cos(B/2) - c \cdot \cos B)/r = \cot(B/2) - (c/r) \cdot \cos B \\ &= \cot(B/2) - (2s/c) \cdot \cot B = \cot(B/2) - \cot B - \frac{\cos B}{\sin A} - \frac{\sin C \cdot \cot B}{\sin A} \\ &= \frac{1}{\sin B} - \frac{\cos B}{\sin A} - \frac{\sin C \cdot \cot B}{\sin A} = \frac{\sin A - \sin B \cos B - \cos B \sin C}{\sin A \sin B} \\ &= \frac{\sin(B + C) - \sin B \cos B - \cos B \sin C}{\sin A \sin B} = \frac{\cos C - \cos B}{\sin A} = \frac{\sin((B - C)/2)}{\sin(A/2)}, \end{aligned}$$

with the case $C > B$ contributing the symmetric sign, absorbed into the stated absolute value. $\square$

Lemma 2.3. $\tan \alpha_A(ABC) = \frac{|\sin((B - C)/2)|}{\sin(A/2)}$;

$$\tan \alpha_B = \frac{\cos((A - C)/2)}{\cos(B/2)} \quad \text{and} \quad \tan \alpha_C = \frac{\cos((A - B)/2)}{\cos(C/2)},$$

where $\alpha_A, \alpha_B, \alpha_C$ are measured from the A -excenter I_A .

Proof (sketch). For α_A : with $L = AI \cap BC$, one shows A, L lie on the Apollonius circle for foci I, I_A with AL as diameter; since $\angle AA'L = 90^\circ$, A' lies on the same circle, forcing BC to bisect $\angle IA'I_A$, whence $\alpha_A = \gamma_A$.

For α_B (and symmetrically α_C): let K be the touch point of the A -excircle on AC . Since $I_A K \perp AC$ and $BB' \perp AC$, the angle α_B at B' equals $\angle B'I_A K$ at K , so $\tan \alpha_B = B'K/I_A K$.

Using the standard tangent length $AK = s$ (so $I_A K = s \cdot \tan(A/2)$, from the right triangle $AI_A K$) and $B'K = AK - AB' = s - c \cdot \cos A$:

$$\tan \alpha_B = \frac{B'K}{I_A K} = \frac{s - c \cdot \cos A}{s \cdot \tan(A/2)} = \frac{(s - c \cdot \cos A) \cdot (1 + \cos A)}{s \cdot \sin A}$$

using the half-angle identity $1/\tan(A/2) = (1 + \cos A)/\sin A$. Writing $c \cdot \cos A = (b^2 + c^2 - a^2)/(2b)$ (Law of Cosines) and simplifying:

$$\begin{aligned} \tan \alpha_B &= \frac{1}{\sin A} \cdot \left(1 + \frac{a - c}{b} \cdot \cos A\right) = \frac{\sin B + (\sin A - \sin C) \cdot \cos A}{\sin B \cdot \sin A} \\ &= \frac{\cos A + \cos C}{\sin B} = \frac{2 \cdot \cos((A + C)/2) \cdot \cos((A - C)/2)}{2 \cdot \sin(B/2) \cdot \cos(B/2)} = \frac{\cos((A - C)/2)}{\cos(B/2)}. \square \end{aligned}$$

As Lemma 2 and the first case of Lemma 3 give the same formula, we record this explicitly, since it is the only one of the three angle coincidences below with a synthetic proof rather than a shared algebraic form:

Corollary 2.1. $\gamma_A(ABC) = \alpha_A(ABC)$ for every non-right triangle ABC : the incenter and the A -excenter subtend equal angles at the foot of the A -altitude. No comparable synthetic link is currently known between α_B, α_C and any incenter-based angle — these are different animals: their formulas arise from a structurally different argument (the A -excircle's tangency point, not a harmonic range), not a variant of the Apollonius-circle proof above. We simply note the algebraic commonality: the α_B, α_C formulas mirror Lemma 2's, with the half-angle sines of the incenter case replaced by cosines.

3. THE ORTHOCENTRIC INVERSION

Fix a non-right triangle ABC with orthocenter H and circumradius R , and let τ be the inversion centered at H with power $(2R)^2$. Write $A^* = \tau(A)$, $B^* = \tau(B)$, $C^* = \tau(C)$, and let $A^*B^*C^*$ denote the resulting (generically non-degenerate) triangle.

This inversion is natural for the following reason. The circumcircles of HBC, HCA, HAB (the Johnson circles of H) each have radius exactly R and pass through H , so H is equidistant, at distance R , from each of their centers; consequently, the circle Ω of radius $2R$ centered at H is tangent to all three. One finds that τ carries A, B, C to the vertices of the triangle tangent to Ω , and carries Ω itself onto the incircle of $A^*B^*C^*$ when ABC is acute, or onto the appropriate excircle when ABC is obtuse — in either case:

Proposition 3.1 (Angle and center identification). *If ABC is acute, H is the incenter of $A^*B^*C^*$, and the angle of $A^*B^*C^*$ at A^* equals $180^\circ - 2A$ (symmetrically at B^*, C^*). If ABC is obtuse at A , H is the A^* -excenter of $A^*B^*C^*$, and the angle of $A^*B^*C^*$ at A^* equals $2A - 180^\circ$ (with the angles at B^*, C^* equal to $2B$ and $2C$, respectively).*

Proof (sketch). Since Ω is tangent to all three sides of $A^*B^*C^*$ (§3) and is centered at H , H is automatically the incenter of $A^*B^*C^*$ when it lies inside the triangle (the acute case), or the relevant excenter when it lies outside (the obtuse case) — this needs no separate argument beyond identifying which side of the triangle H falls on. It remains to find the angle at A^* . Let c_2, c_3 be the centers of ω_2, ω_3 . The two sides of $A^*B^*C^*$ meeting at A^* are the images under τ of ω_2, ω_3 , hence tangent to Ω at the points where rays Hc_2, Hc_3 meet Ω ; the angle between two tangent lines to a circle equals 180° minus the angle their tangent points subtend at the center, so the angle at A^* equals $180^\circ - \angle c_2 H c_3$.

Since H and A are the two intersection points of ω_2 and ω_3 (both circles pass through H and A , by construction), the segment HA is a common chord, and a short inscribed-angle argument in each circle (using that ω_2, ω_3 have radius R and pass through C, B respectively) gives $\angle c_2 H c_3 = 2A$. Hence the angle at A^* equals $180^\circ - 2A$ when this is positive (A acute), or its supplement-turned-reflex reading $2A - 180^\circ$ identifies the A^* -excenter configuration when A itself is obtuse. The same argument at B, C gives the remaining angles. $\square$

The present note's contribution begins with what this identification lets us conclude about angles.

4. THE CONSERVATION THEOREM

We can now state the principal result of this note.

Theorem 4.1 (Conservation). *With notation as above,*

$$\theta_A(ABC) = \gamma_{A^*}(A^*B^*C^*) \quad \text{if } ABC \text{ is acute,}$$

$$\theta_X(ABC) = \alpha_{X^*}(A^*B^*C^*) \quad \text{for each } X = A, B, C, \text{ if } ABC \text{ is obtuse,}$$

and symmetrically at B, B^* and C, C^* in the acute case. That is: the angle θ_X , defined in ABC using its circumcenter O , is numerically identical to the angle γ or α defined in the different triangle $A^*B^*C^*$ using its incenter or excenter (which happens to be H).

Proof. Suppose ABC is acute. By Proposition 3.1, the angles of $A^*B^*C^*$ are $A^* = 180^\circ - 2A$, $B^* = 180^\circ - 2B$, $C^* = 180^\circ - 2C$ (writing A^* for both the vertex and, by standard abuse, the angle of $A^*B^*C^*$ there). Applying Lemma 2 to the triangle $A^*B^*C^*$ at its own vertex A^* :

$$\tan \gamma_{A^*}(A^*B^*C^*) = |\sin((B^* - C^*)/2)| / \sin(A^*/2).$$

Substituting $B^* - C^* = (180^\circ - 2B) - (180^\circ - 2C) = 2(C - B)$ and $A^*/2 = 90^\circ - A$, this becomes:

$$|\sin(C - B)| / \sin(90^\circ - A) = |\sin(B - C)| / \cos A,$$

which is exactly $\tan \theta_A(ABC)$ by Lemma 1. The same substitution at B^* and C^* gives the remaining two identities.

Now suppose ABC is obtuse at A . By Proposition 3.1, the angles of $A^*B^*C^*$ are $A^* = 2A - 180^\circ$, $B^* = 2B$, $C^* = 2C$. Applying Lemma 3 (all three formulas, since H is a single excenter — the A^* -excenter — of $A^*B^*C^*$, governing angles at all three of A^*, B^*, C^*):

$$\begin{aligned} \tan \alpha_{A^*} &= |\sin((B^* - C^*)/2)| / \sin(A^*/2) = |\sin(B - C)| / \sin(A - 90^\circ) = |\sin(B - C)| / \cos A \\ &= \tan \theta_A(ABC), \end{aligned}$$

$$\begin{aligned} \tan \alpha_{B^*} &= \cos((A^* - C^*)/2) / \cos(B^*/2) = \cos(A - 180^\circ - C) / \cos B = |\sin(A - C)| / \cos B \\ &= \tan \theta_B(ABC), \end{aligned}$$

and symmetrically for α_{C^*} . In each case, the substitution of Proposition 3.1's angle values into Lemma 2 or Lemma 3 reduces the incenter- or excenter-angle of $A^*B^*C^*$ identically to the circumcenter-angle of ABC , which is the assertion of the theorem. $\square$

Illustration. Figures 1 and 2 show Theorem 6 for one acute and one obtuse example, with ABC (black) and its image $A^*B^*C^*$ (gray) drawn together in the same plane. The angles $\theta_A, \theta_B, \theta_C$ are marked at the feet of ABC (measured toward O) in red, blue, and green respectively, and the correspondingly-colored angles γ or α are marked at the altitude feet of $A^*B^*C^*$ (measured toward H). Each colored angle is, by Theorem 6, numerically equal to its same-colored counterpart, despite belonging to different triangles and different associated centers. Also shown: the three Johnson circles $\omega_1, \omega_2, \omega_3$ (dotted), the circumcircle of ABC (solid), and Ω itself (solid purple) — the incircle of $A^*B^*C^*$ in the acute case, or the A^* -excircle in the obtuse case, where light gray lines extend A^*B^* and A^*C^* beyond B^* and C^* to show the tangency explicitly.

Acute case: ABC (black), image $A^*B^*C^*$ (gray).
Dotted: Johnson circles. Solid: circumcircle of ABC ; purple = Ω (incircle of $A^*B^*C^*$).

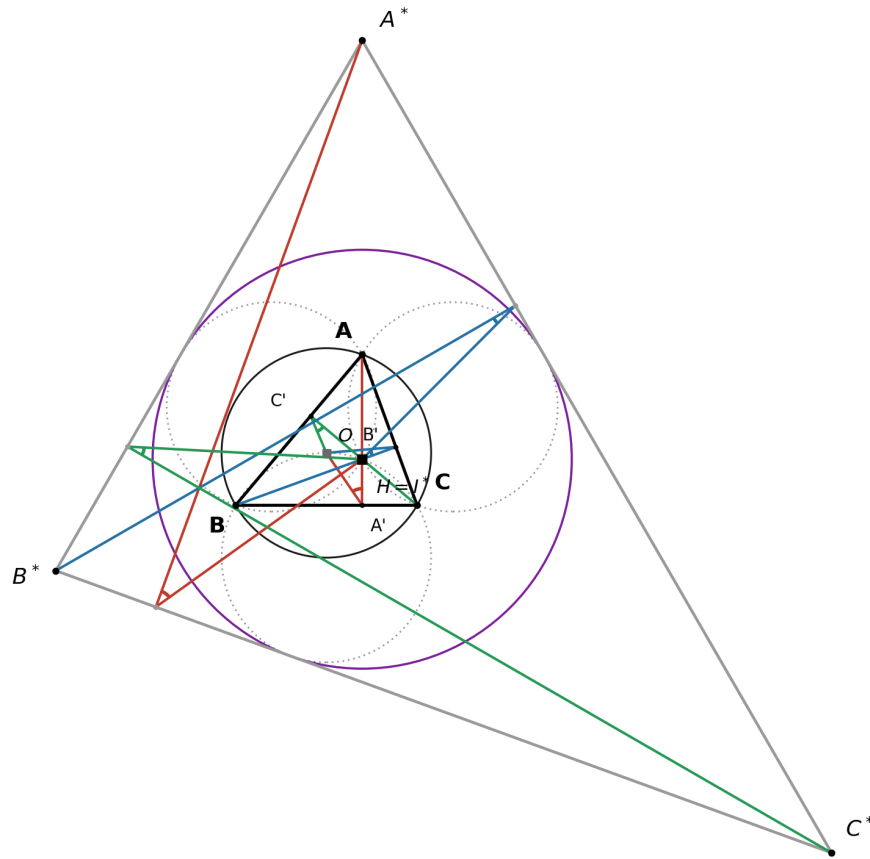


Figure 1. Acute case. $\theta_A, \theta_B, \theta_C$ in ABC (toward O) match $\gamma_{A^*}, \gamma_{B^*}, \gamma_{C^*}$ in $A^*B^*C^*$ (toward $H = I^*$).

Obtuse case (at A): ABC (black), image $A^*B^*C^*$ (gray).
Dotted: Johnson circles. Purple = Ω (A^* -excircle); light lines show tangency to A^*B^* , A^*C^* .

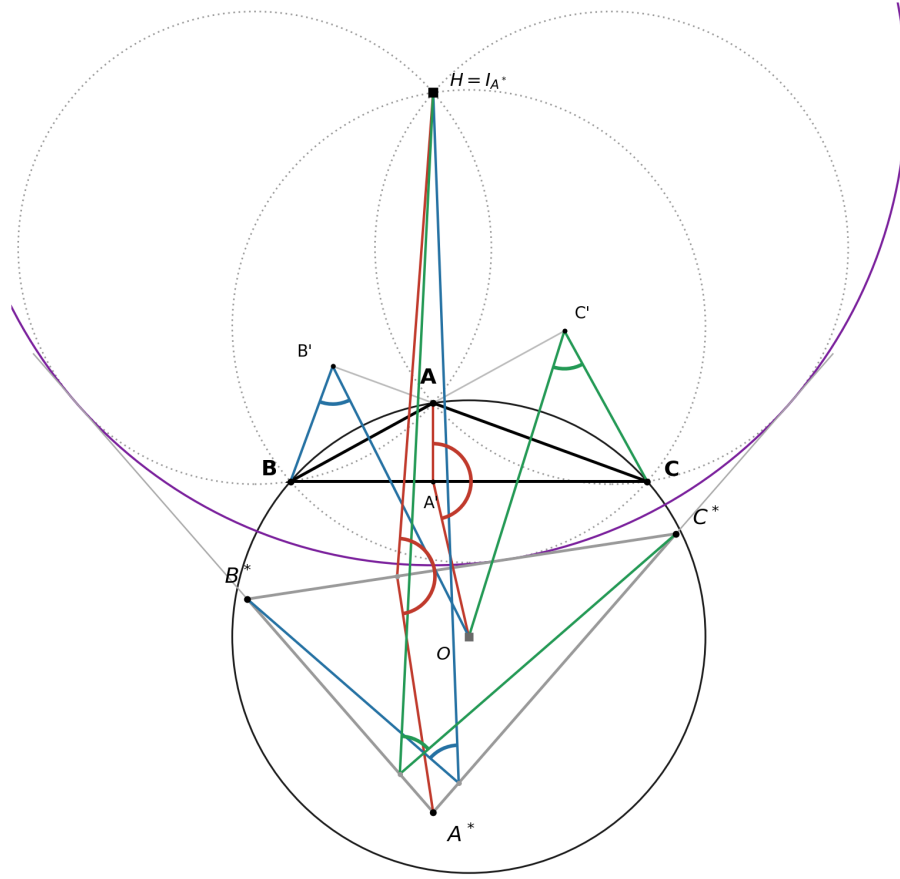


Figure 2. Obtuse case (at A). $\theta_A, \theta_B, \theta_C$ in ABC match $\alpha_{A^*}, \alpha_{B^*}, \alpha_{C^*}$ in $A^*B^*C^*$ (toward H = the A^* -excenter).

5. APPLICATION: FULL STATEMENT AND PROOF

We close by giving a complete example of Theorem 6 in use — the theorem it was originally developed to prove.

Theorem 5.1. *Let ABC be a non-equilateral triangle with circumcenter O and orthocenter H , distinct from A, B, C . Let $\omega_1, \omega_2, \omega_3$ be the circumcircles of HBC, HCA, HAB respectively, and let Ω be the circle containing $\omega_1, \omega_2, \omega_3$ and tangent to each. Let A' be the foot of the altitude from A ; the line through H parallel to OA' meets ω_1 again at a point D , and the line through D parallel to BC meets ω_1 again at a point M . Define N and P analogously with respect to B, ω_2 and C, ω_3 . Then:*

- (a) *the circle (MNP) is tangent to Ω ;*
- (b) *the circles $(HAM), (HBN)$ and (HCP) have a second common point.*

Proof. By §3, $\omega_1, \omega_2, \omega_3$ have radius R and pass through H (they are the Johnson circles of H), Ω is the circle of radius $2R$ centered at H , and τ carries A, B, C to the vertices of

the triangle $A^*B^*C^*$ tangent to Ω , with H equal to the incenter of $A^*B^*C^*$ (if ABC acute) or an excenter (if ABC obtuse).

Step 1: locating M via an isogonal argument. Let O_1 be the center of ω_1 . Since H lies on the altitude AA' , the ray $A'H$ coincides with the ray $A'A$, so the angle at A' between $A'O$ and $A'H$ is exactly θ_A . As HD is parallel to OA' by construction, the corresponding angle at H between HA' and HD also equals θ_A .

Now, HA' (perpendicular to BC , since H lies on the altitude from A) and HO_1 (a circumradius of triangle HBC from its vertex H) are isogonal with respect to angle BHC — the standard fact that in any triangle, the altitude and the circumradius from a given vertex are symmetric about that vertex's angle bisector. The point M , obtained from D by the further “parallel to BC ” step, is exactly the isogonal partner of D with respect to H 's angle in triangle HBC . In particular:

$$\angle O_1HM = \theta_A(ABC).$$

Step 2: applying the inversion. The ray HO_1 is fixed setwise by τ (any ray from the center of inversion maps to itself); and ω_1 , passing through H , maps to the line ℓ perpendicular to HO_1 that forms one side of the triangle $A^*B^*C^*$ (§3). Since M lies on ω_1 , and its image M^* lies on the same ray HM as M (a positive-power inversion preserves rays from the center), the angle $\angle O_1HM^*$ equals $\angle O_1HM = \theta_A(ABC)$ as well — angles at the center of inversion are simply not moved by it.

Because HO_1 is perpendicular to ℓ , the direction HO_1 is exactly the direction of the altitude from A^* onto $\ell = B^*C^*$ in triangle $A^*B^*C^*$. So M^* , lying on ℓ , subtends angle $\theta_A(ABC)$ with H as measured from that altitude direction — that is, M^* is the point of B^*C^* at which the line to H makes angle $\theta_A(ABC)$ with the A^* -altitude.

By Theorem 6, $\theta_A(ABC)$ equals $\gamma_{A^*}(A^*B^*C^*)$ (if ABC acute) or $\alpha_{A^*}(A^*B^*C^*)$ (if ABC obtuse) — precisely the angle that the A^* -altitude foot, by definition, makes with the line from that foot to H (the incenter or A^* -excenter of $A^*B^*C^*$). Since the A^* -altitude foot is the unique point of B^*C^* with this property, we conclude M^* is exactly the foot of the altitude from A^* in $A^*B^*C^*$.

By the identical argument at B, C , the points N^*, P^* (images of N, P) are the feet of the altitudes from B^*, C^* in $A^*B^*C^*$.

The condition of H distinct from A, B and C is important here, and it translates into ABC being non-right-angled: for a right-angled triangle, two of the circles $\omega_1, \omega_2, \omega_3$ would need different definitions (since H coincides with the right-angle vertex), and, more importantly, the triangle $A^*B^*C^*$ would degenerate.

Step 3: conclusion via Feuerbach, for statement (a). Since M^*, N^*, P^* are the three altitude feet of $A^*B^*C^*$, the circle through them is its nine-point circle. By Feuerbach's theorem, the nine-point circle of any triangle is tangent to its incircle and to each of its excircles; in particular it is tangent to whichever of these Ω corresponds to (the incircle if ABC is acute, the appropriate excircle if ABC is obtuse, by §3).

Finally, Ω (radius $2R$, centered at H) is fixed as a set by τ , since the inversion's power $(2R)^2$ exactly matches Ω 's own radius. As inversion preserves tangency, applying τ to the tangency of the nine-point circle of $A^*B^*C^*$ with Ω — and using that τ carries M^*, N^*, P^* back to M, N, P (inversion being an involution), hence the nine-point circle back to circle (MNP) — gives that circle (MNP) is tangent to Ω , proving (a). Obviously, in the case of an equilateral triangle ABC , the two circles (MNP) and Ω would coincide

rather than merely touch — which is exactly why ABC is assumed non-equilateral in this theorem.

Step 4: conclusion via concurrence of altitudes, for statement (b). Since circle (H, A, M) passes through the inversion center H , it maps under τ to a line — specifically, the line through $\tau(A) = A^*$ and $\tau(M) = M^*$. By Step 2, M^* is the foot of the altitude from A^* in $A^*B^*C^*$, so this line is exactly the A^* -altitude of $A^*B^*C^*$. By the identical argument, circle (H, B, N) maps to the B^* -altitude, and circle (H, C, P) maps to the C^* -altitude of $A^*B^*C^*$.

Since the three altitudes of any triangle are concurrent — at its orthocenter — the three image lines share a common point, namely the orthocenter of $A^*B^*C^*$. As τ is an involution, applying it once more carries this common point back to a point shared by all three original circles (H, A, M) , (H, B, N) , (H, C, P) — together with H itself, which they already share by construction, since each passes through H by definition — giving the required second common point and proving (b). $\square$

6. CONCLUSIONS

The force of Theorem 6 is that it links two angles defined by two different triangle centers (a circumcenter on one side, an incenter or excenter on the other) in two different triangles, related only by a single inversion — and it does so with no residual dependence on the inversion at all: both sides of each identity are expressed purely in terms of the angles of ABC . One should not expect this from the formulas alone: Lemma 1 and Lemma 2 have no a priori relationship (one is built from O , the other from I , of the same triangle, and are simply different functions of A, B, C); it is only because Proposition 3.1 supplies the exact angle-doubling relation $A^* = 180^\circ - 2A$ (or $2A - 180^\circ$) that substituting into Lemma 2 or 3 happens to reconstruct Lemma 1's expression precisely, rather than some unrelated function of A, B, C .

This is, to our knowledge, the main point of interest in this circle of results: three otherwise-unremarkable angle formulas, applied to two triangles related by a natural but unusual inversion, reproduce one another exactly. We record the individual formulas (§2) because Theorem 6's proof needs them stated precisely, but the conservation phenomenon of Theorem 6 itself — not the formulas in isolation — is the result we consider novel and worth recording.

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