



DUAL OF THE SIX CIRCLES THEOREM

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Abstract. The dual lines/points or circles/points of the Six Circles Theorem is proven. This dual is generalized to n points, distinguishing between whether n is even or odd.

1. INTRODUCTION

The Six Circles Theorem states that given three lines r_1, r_2, r_3 intersecting pair by pair, if the circle C_1 is tangent to r_1 and r_2 , the circle C_2 is tangent to r_2, r_3 and C_1 , the circle C_3 is tangent to r_3, r_1 and C_2 , the circle C_4 is tangent to r_1, r_2 and C_3 , the circle C_5 is tangent to r_2, r_3 and C_4 , the circle C_6 is tangent to r_3, r_1 and C_5 , then the circle C_6 is tangent to the circle C_1 . (See Figure 1)

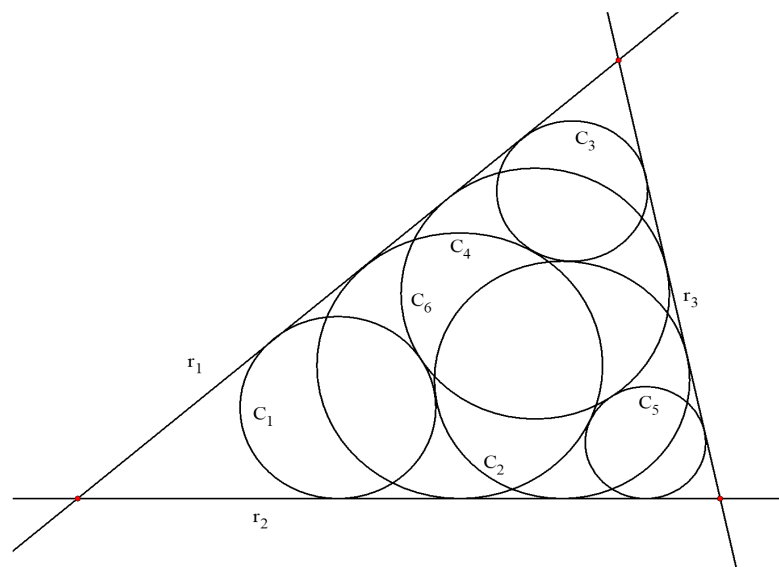


Figure 1. The Six Circles Theorem.

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Key words and phrases. Six Circles Theorem, duality.

Six circles theorem appear in [1] and the first proof in [2]. Other proof can be seen in [3], [4, 401-405].

In [1] it is mentioned that the theorem remains valid if the lines are replaced by circles. In that case, the six circles theorem admit a dual lines/circles theorem that states: given three circles Q_1, Q_2, Q_3 if the circle C_1 is tangent to Q_1 and Q_2 , the circle C_2 is tangent to Q_2, Q_3 and C_1 , the circle C_3 is tangent to Q_3, Q_1 and C_2 , the circle C_4 is tangent to Q_1, Q_2 and C_3 , the circle C_5 is tangent to Q_2, Q_3 and C_4 , the circle C_6 is tangent to Q_3, Q_1 and C_5 , then the circle C_6 is tangent to the circle C_1 . (See Figure 2)

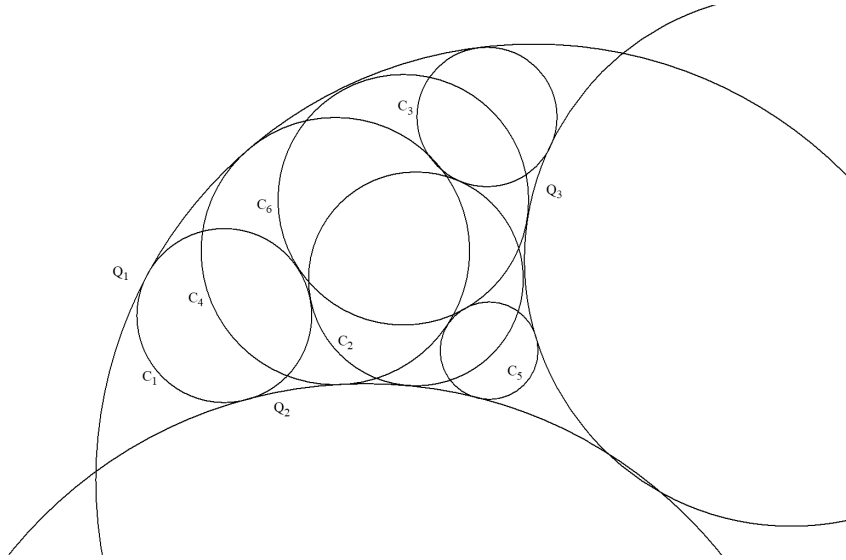


Figure 2. Dual lines/circles Six Circles Theorem.

2. ANOTHER DUAL THEOREM

Nothing is said in [1] about replacing lines with points. In order to formulate a dual lines/points statement that makes sense, it is also necessary to replace ‘circle tangent to two lines’ with ‘circle passing through two points’.

The Dual lines/points Six Circles Theorem states that given three points P_1, P_2, P_3 if the circle C_1 passes through P_1 and P_2 , the circle C_2 passes through P_2, P_3 and is tangent to C_1 , the circle C_3 passes through P_3, P_1 and is tangent to C_2 , the circle C_4 passes through P_1, P_2 and is tangent to C_3 , the circle C_5 passes through P_2, P_3 and is tangent to C_4 , the circle C_6 passes through P_3, P_1 and is tangent to C_5 , then the circle C_6 is tangent to the circle C_1 . (See Figure 3)

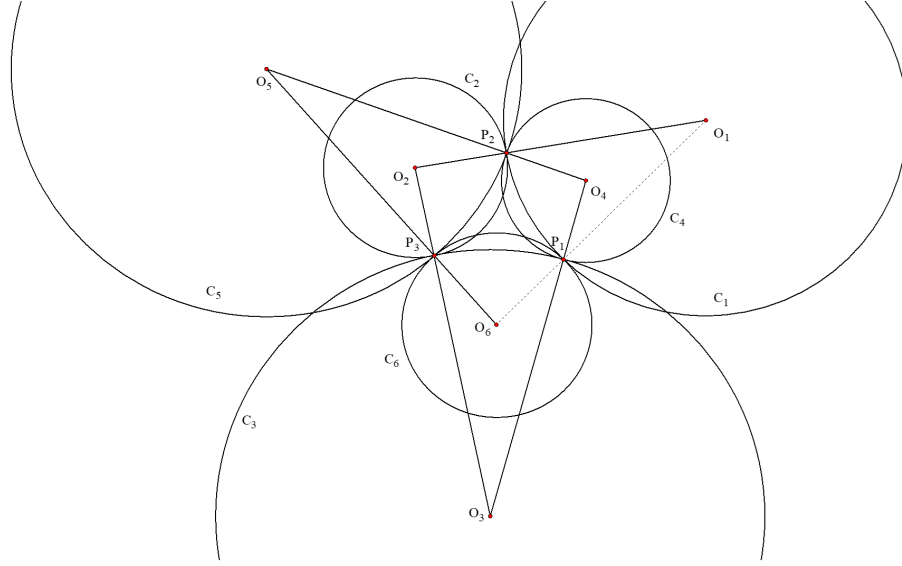


Figure 3. Dual lines/points Six Circles Theorem.

O_1 and O_4 belong to the perpendicular bisector of P_1P_2 so $\angle O_1P_1O_4 = \angle O_1P_2O_4$
 O_2 and O_5 belong to the perpendicular bisector of P_2P_3 so $\angle O_2P_2O_5 = \angle O_1P_3O_5$
 O_3 and O_6 belong to the perpendicular bisector of P_3P_1 so $\angle O_3P_3O_6 = \angle O_3P_1O_6$

$$\angle O_1P_1O_6 = \angle O_1P_1O_4 + \angle O_4P_1O_6 = \angle O_3P_1O_6 + \angle O_4P_1O_6 + \angle O_3P_1O_4 = 2\pi$$

In the case P_1, P_2, P_3 are aligned the proof is analogous.

3. GENERALIZATIONS

In [5] the Six Circles Theorem is generalized by distinguishing according to whether the number of straight lines is even or odd.

A property analogous to that demonstrated for three points in section 2 holds if the number of points is odd. The proof is analogous to the one given previously. If the number of points is even, for a similar property to hold, the points must be cyclic. Let's look at the case of four points.

Given four cyclic points P_1, P_2, P_3, P_4 , if the circle C_1 passes through P_1 and P_2 , the circle C_2 passes through P_2, P_3 and is tangent to C_1 , the circle C_3 passes through P_3, P_4 and is tangent to C_2 , the circle C_4 passes through P_4, P_1 and is tangent to C_3 , then the circle C_4 is tangent to the circle C_1 . The centers of the four tangent circles determine a tangential quadrilateral. (See Figure 4)

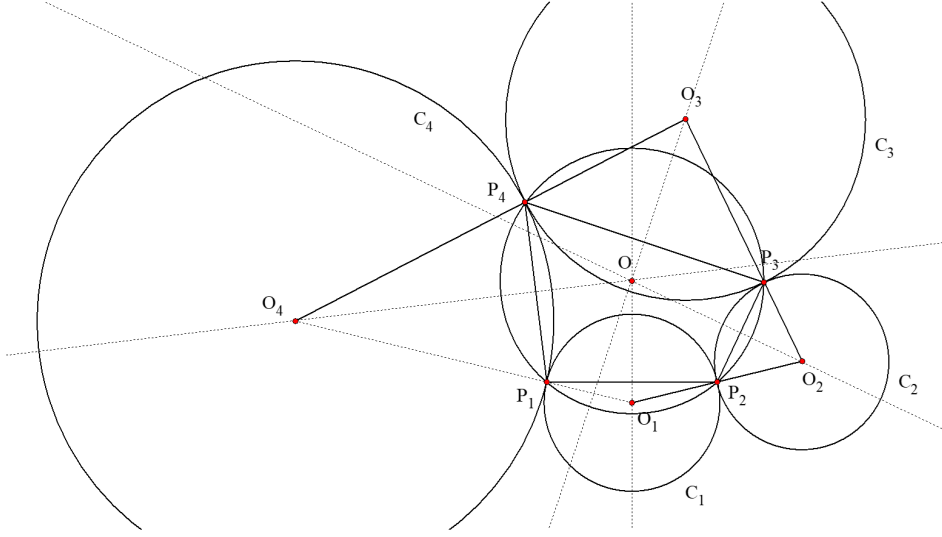


Figure 4. The Dual for four cyclic points.

O_1, O_2, O_3, O_4 are the centers of the circles C_1, C_2, C_3, C_4 respectively. The quadrilateral $P_1P_2P_3P_4$ is cyclic, so

$$\angle P_4P_1P_2 + \angle P_2P_3P_4 = \angle P_1P_2P_3 + \angle P_3P_4P_1$$

Then it is verified

$$(\angle O_4P_1P_4 + \angle P_4P_1P_2 + \angle P_2P_1O_1) + (\angle O_2P_3P_2 + \angle P_2P_3P_4 + \angle P_4P_3O_3) =$$

$$(\angle O_1P_2P_1 + \angle P_1P_2P_3 + \angle P_3P_2O_2) + (\angle O_3P_4P_3 + \angle P_3P_4P_1 + \angle P_1P_4O_4)$$

And therefore

$$(\angle O_4P_1P_4 + \angle P_4P_1P_2 + \angle P_2P_1O_1) + \pi = 2\pi, \text{ so } \angle O_4P_1O_1 = \pi. \text{ The angle bisectors of}$$

$O_1O_2O_3O_4$ are concurrent because they coincide with the perpendicular bisectors of the cyclic quadrilateral $P_1P_2P_3P_4$. $O_1O_2O_3O_4$ is tangential with center O (center of circle containing $P_1P_2P_3P_4$).

Another way to view the above is by considering the radii R_1, R_2, R_3, R_4 of the circles C_1, C_2, C_3, C_4 respectively.

$$O_1O_2O_3O_4 \text{ is tangential} \Leftrightarrow O_1O_2 + O_3O_4 = O_2O_3 + O_4O_1 \Leftrightarrow$$

$$(R_1 + R_2) + (R_3 + R_4) = (R_2 + R_3) + (R_4 + R_1).$$

For cyclic polygons with an even number greater than four, the proofs are analogous.

4. FINAL REFLECTIONS

The presence of three circles (QQQ), three lines (rrr) or three points (PPP) in the previous theorems allows us to ask whether analogous properties hold true in the remaining seven cases of the generalized Apollonius problem: two points and a line (PPr), two lines and a point (rrP), two points and a circle (PPQ), two circles and a point (QQP), two lines and a circle (rrQ), two circles and a line (QQr), a point, a line and a circle (PrQ).

The following diagram allows us to appreciate the dualities circles/points, circles/lines, points/lines that exist among the ten cases (See Diagram 1).

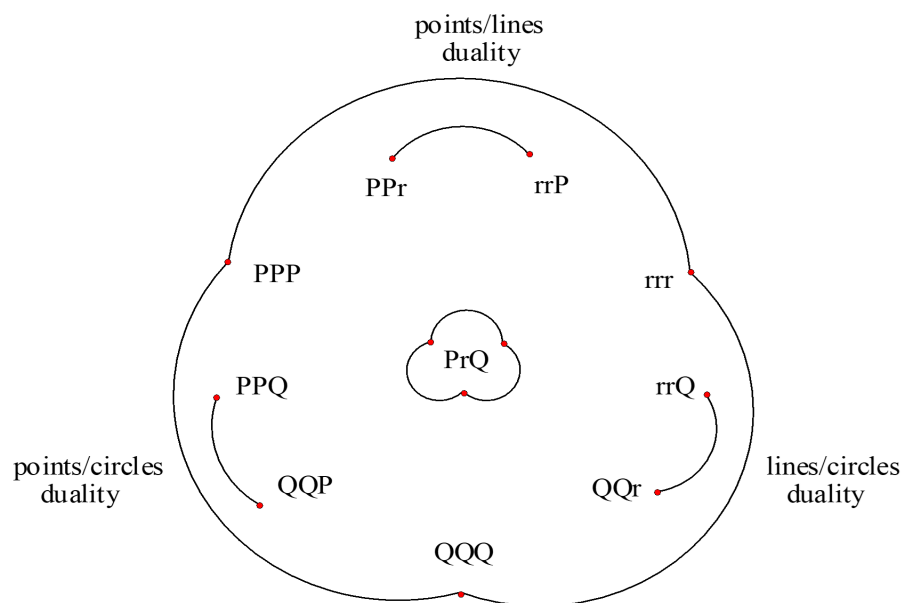


Diagram 1. Dualities among the ten cases of the generalized Apollonius problem.

In the case of duality points/circles PPQ/QQP the property is not fulfilled, nor is it fulfilled in the case of duality points/lines PPr/rrP nor in the case of triple duality PrQ .

References

- [1] C. Evelyn, G. Money-Coutts, J. Tyrrell: The seven circles theorem and other new theorems. Stacey Int., London, 1974.
- [2] J. Tyrrell and M. Powell: A theorem in circle geometry. Bull. London Math. Soc. 3 (1971), 70–74.
- [3] D. Ivanov and S. Tabachnikov: The Six Circles Theorem revisited. arXiv:1312.5260v2 [math.MG] 9 Mar 2014. <https://arxiv.org/pdf/1312.5260>
- [4] D. Ivanov and S. Tabachnikov: Mathematical Omnibus: Thirty Lectures on Classic Mathematics. AMS, 2007
- [5] S. Tabachnikov: Going in Circles: Variations on the Money-Coutts Theorem. Geometriae Dedicata 80: 201–209, 2000.

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