



CHARACTERIZATION OF LOCAL GRADIENT VECTOR FIELDS ON CONTACT-SYMPLECTIC AND CONTACT PAIRS

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ABSTRACT. In this paper we shown that gradient vector fields and, more generally, local gradient vector fields on contact-symplectic pairs and contact pairs can be characterized in terms of Lagrangian submanifolds of the tangent bundle with respect to an induced symplectic structure.

1. INTRODUCTION AND MOTIVATIONS

In the context of a $2n$ -dimensional symplectic manifold (M, ω) , a Lagrangian submanifold is defined as an n -dimensional submanifold $N \subset M$ such that the restriction of the symplectic form ω to N vanishes identically (see [1, 10] for comprehensive details). Given such a manifold, the musical isomorphism $\flat_\omega : TM \rightarrow T^*M$, defined by the interior product $\flat_\omega(X) = i_X\omega$, constitutes a vector bundle isomorphism over the identity. Let $\Omega_0 = -d\theta$ represent the canonical symplectic form on the cotangent bundle T^*M , where θ denotes the Liouville (or canonical) 1-form. Utilizing the isomorphism $\flat_\omega$, one can pull back the canonical symplectic structure to the tangent bundle via $\Omega = \flat_\omega^*\Omega_0$. Consequently, Ω defines a symplectic structure on TM , and by construction, $\flat_\omega$ serves as a symplectomorphism between (TM, Ω) and (T^*M, Ω_0) . Existing literature (see[3, 4]) establishes that for any smooth function $H \in C^\infty(M)$, the image of the Hamiltonian vector field X_H under the map $\flat_\omega$ corresponds to a Lagrangian submanifold of (TM, Ω) . While the converse is not universally valid, insofar as a vector field X on a symplectic manifold whose image determines a Lagrangian section of the tangent bundle is not necessarily Hamiltonian, it remains true if X is at least a locally Hamiltonian vector field (see[4]). Building upon these symplectic foundations, Cantrijn, de León, and Lacomba characterized local gradient vector fields on cosymplectic manifolds by identifying them with Lagrangian submanifolds of (TM, Ω) (see[8]). Parallel to this, contact-symplectic pairs and contact pairs were introduced and investigated by Bande in [6, 15]. A contact-symplectic pair equips a manifold with two distinct yet interrelated geometric structures: a contact form η and a symplectic form ω . This dual framework facilitates the simultaneous analysis of dynamics inherent to both contact and symplectic geometries, proving

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particularly useful for systems bridging Hamiltonian (conservative) and contact (non-conservative) mechanics. Similarly, the concept of a Contact Pair allows for the investigation of dynamical systems governed by two independent types of non-integrability or distinct time directions, encoded by the forms α and η . The study of such pairs opens new avenues for generalizing reduction theory and the concepts of Hamiltonian and gradient vector fields to contexts where dynamics are constrained by two independent Pfaffian structures. In recent contributions, Mbodji, Dathe, and Diango (see [17, 7] for more details) formalized the notions of Hamiltonian and gradient vector fields within the frameworks of contact-symplectic pairs and contact pairs. Drawing inspiration from the classical symplectic and cosymplectic cases, it is natural to investigate the characterization of gradient and local gradient vector fields in these specific settings. In this paper, we provide a characterization of gradient vector fields and, more broadly, local gradient vector fields, on contact-symplectic pairs and contact pairs by relating them to Lagrangian submanifolds of the tangent bundle fibers with respect to the induced symplectic structure.

2. MAIN RESULTS

2.1. Contact-symplectic pairs. This subsection provides a review of contact-symplectic pair structures and the definition of their associated gradient vector fields. For more information see [6].

Definition 2.1. ([6]) *A contact-symplectic pair of type (k, h) on a manifold M of odd dimension $(2k + 2h + 1)$ is a pair (η, ω) , where η is a 1-form and ω is a closed 2-form such that:*

$$\eta \wedge (d\eta)^h \wedge \omega^k \text{ is a volume form on } M, \text{ and } (d\eta)^{h+1} = 0, \omega^{k+1} = 0.$$

The forms η and ω are so necessarily constant class $2h + 1$ and $2k$. For $k = 0$, we find the contact forms, and for $h = 0$ the cosymplectic structures. To a such structure are naturally associated two distributions: the distribution of vector fields which annul η and $d\eta$, and the distribution of vector fields which annul ω . This distributions are completely integrable because the forms η and ω are constant class $2h + 1$ and $2k$. They determine the characteristic foliations of η and ω , which we will note $\mathfrak{F}$ and $\mathfrak{G}$.

The characteristic foliation of ω is of codimension $2k$ and her leaves are contact manifolds of contact structure associated η . The characteristic foliation of η is of codimension $2h + 1$ and her leaves are symplectic manifold of symplectic form associated ω .

The foliations $\mathfrak{F}$ and $\mathfrak{G}$ are hollowing out transversal and additional. A vector field Z , known as the Reeb vector field, is carried by manifolds with contact-symplectic pairs and is uniquely identified by:

$$\eta(Z) = 1 \text{ and } i_Z(d\eta^h \wedge \omega^k) = 0.$$

Contrary to riemannian manifolds, the contact-symplectic pairs of type (k, h) have not local invariants. That is to do to theorem which follows, called the Darboux theorem. It establishes an unique local model of contact-symplectic pairs of type (k, h) .

Theorem 2.1. [6] *Let (η, ω) be a contact-symplectic pair of type (k, h) on $M^{2k+2h+1}$ and p a point of $M^{2k+2h+1}$. Then, it exists a local coordinates system*

$(U, x_1, \dots, x_k, y_1, \dots, y_k, p_1, \dots, p_h, q_1, \dots, q_h, z)$ around of p , such that

$$\eta|_U = \sum_{i=1}^h p_i dq_i + dz, \omega|_U = \sum_{i=1}^k dx_i \wedge dy_i, Z|_U = \partial_z$$

The coordinates system $(U, x_1, \dots, x_k, y_1, \dots, y_k, p_1, \dots, p_h, q_1, \dots, q_h, z)$ is called coordinates system of Darboux or canonical coordinates. Consider the bundle homomorphism $\flat_{(\eta, \omega)} : \chi(M) \longrightarrow \Omega^1(M)$ defined by:

$$\flat_{(\eta, \omega)}(X) = \eta(X) \cdot \eta + i_X(\omega + d\eta), \forall X \in \chi(M).$$

The following proposition is proven in [17].

Proposition 2.1. ([17]) *The map $\flat_{(\eta, \omega)}$ is an isomorphism of $C^\infty(M, \mathbb{R})$ -module.*

Thanks to this isomorphism $\flat_{(\eta, \omega)}$, Mbodji and Dathe can associate at every function $H \in C^\infty(M, \mathbb{R})$ an unique vector field $\text{grad}H \in \chi(M)$, called gradient of H .

Definition 2.2. ([17]) *Let $(M^{2k+2h+1}, \eta, \omega)$ be a contact-symplectic manifold of type (k, h) and $H \in C^\infty(M, \mathbb{R})$ a function. Is called gradient vector field of H the single vector field $\text{grad}H$ defined by*

$$\text{grad}H = \flat_{(\eta, \omega)}^{-1}(dH)$$

. Likewise

$$\begin{cases} i_{\text{grad}H}(\omega + d\eta) = dH - Z(H)\eta \\ \eta(\text{grad}H) = Z(H) \end{cases}$$

In terms of canonical coordinates, the gradient vector field $\text{grad}H$ reads

$$\text{grad}H|_U = \frac{\partial H}{\partial x_i} \partial_{x_i} - \frac{\partial H}{\partial y_i} \partial_{y_i} + \left(\frac{\partial H}{\partial q_j} - p_i \frac{\partial H}{\partial z} \right) \partial p_j - \frac{\partial H}{\partial p_j} \partial q_j + \left(\frac{\partial H}{\partial z} + p_j \frac{\partial H}{\partial p_j} \right) \partial z$$

(see [17] for more details).

2.2. Contact pairs. This subsection provides a review of contact-symplectic pair structures and the definition of their associated gradient vector fields.

Definition 2.3. *A contact pair of type (k, h) on a manifold $M^{2k+2h+2}$ of dimension $(2k+2h+2)$ is a pair of Pfaffian forms (η, α) such that:*

$$\eta \wedge (d\eta)^k \wedge \alpha \wedge d\alpha^h \text{ is a volume form on } M^{2k+2h+2}, \text{ and } d\eta^{k+1} = 0, d\alpha^{h+1} = 0.$$

The forms η and α are so necessarily constant class $2k+1$ and $2h+1$. To a such structure are naturally associated two distributions: the distribution of vector fields which annul η and $d\eta$, and the distribution of vector fields which annul α and $d\alpha$. This distributions are completely integrable because the forms η and ω are constant class $2k+1$ and $2h+1$. They determine the characteristic foliations of η and α , which we will note $\mathfrak{F}$ and $\mathfrak{G}$.

The characteristic foliation of η is of codimension $2h+1$ and her leaves are contact manifolds of contact structure associated α . The characteristic foliation of η is of codimension $2k+1$ and her leaves are contact manifold of contact form associated η .

The foliations $\mathfrak{F}$ and $\mathfrak{G}$ are hollowing out transversal and additional. A vectors fields Z_η, Z_α known as the Reeb vectors fields, is carried by manifolds with contact-symplectic pairs and is uniquely identified by:

$$\eta(Z_\eta) = 1 \text{ and } i_{Z_\eta}(\alpha \wedge d\alpha^h \wedge d\eta^k) = 0.$$

and

$$\alpha(Z_\alpha) = 1 \text{ and } i_{Z_\alpha}(\eta \wedge d\eta^k \wedge d\alpha^h) = 0.$$

The Reeb vector field Z_η, Z_α of contact pair (η, α) commute and satisfy the following condition:

$$\eta(Z_\alpha) = 0, i_{Z_\alpha} d\eta = i_{Z_\alpha} d\eta = 0, \alpha(Z_\eta) = 0, i_{Z_\eta} d\alpha = i_{Z_\eta} d\alpha = 0,$$

So, Z_η (resp. Z_α) is tangent to characteristic foliation of η (resp. α) and it is the Reeb vector field (in the classical sense) of contact form induced by η (resp. α) on each leaf of their foliation. Contrary to riemannian manifolds, the contact pairs have not local invariants. That is to do to theorem which follows, called the Darboux theorem. It establishes an unique local model of contact pairs.

Theorem 2.2. [15] *Let (η, α) be a contact pair on $M^{2k+2h+2}$ and p a point of $M^{2k+2h+2}$. Then, it exists a local coordinates system*

$(U, x_1, \dots, x_k, y_1, \dots, y_k, z_1, p_1, \dots, p_h, q_1, \dots, q_h, z_2)$ around of p , such that

$$\eta|_U = \sum_{i=1}^k x_i dy_i + dz_1, \alpha|_U = \sum_{i=1}^k p_i dq_i + dz_2, Z_\eta|_U = \partial_{z_1}, Z_\alpha|_U = \partial_{z_2}$$

The coordinates system $(U, x_1, \dots, x_k, y_1, \dots, y_k, z_1, p_1, \dots, p_h, q_1, \dots, q_h, z_2)$ is called coordinates system of Darboux or canonical coordinates. Consider the bundle homomorphism $\flat_{(\eta, \alpha)} : \chi(M) \longrightarrow \Omega^1(M)$ defined by:

$$\flat_{(\eta, \alpha)}(X) = \eta(X).\eta + \alpha(X).\alpha + i_X(d\eta + d\alpha), \forall X \in \chi(M).$$

The following proposition is proven in [15].

Proposition 2.2. *The map $\flat_{(\eta, \alpha)}$ is an isomorphism of $C^\infty(M^{2k+2h+2}, \mathbb{R})$.*

Thanks to this isomorphism, MBODJI, DIANGO and DATHE associated at every function

$H \in C^\infty(M^{2k+2h+2}, \mathbb{R})$ an unique vectors field $gradH \in \chi(M)$, called gradient of H , which is defined by

Definition 2.4. *Let $(M^{2k+2h+2}, \eta, \alpha)$ be a contact pair and $H \in C^\infty(M^{2k+2h+2}, \mathbb{R})$ a function. Is called gradient vectors field of H the single vectors field $gradH$ defined by*

$$gradH = \flat_{(\eta, \alpha)}^{-1}(dH)$$

. Likewise

$$\begin{cases} i_{gradH}(\omega + d\eta) = dH - Z_\eta(H)\eta - Z_\alpha(H)\alpha \\ \eta(gradH) = Z_\eta(H), \alpha(gradH) = Z_\alpha(H). \end{cases}$$

In terms of canonical coordinates , the gradient vector field $gradH$ reads

$$gradH|_U = (\frac{\partial H}{\partial y_i} - x_i \frac{\partial H}{\partial z_1})\partial_{x_i} - \frac{\partial H}{\partial x_i}\partial_{y_i} + (\frac{\partial H}{\partial z_1} + x_i \frac{\partial H}{\partial x_i})\partial_{z_1} + (\frac{\partial H}{\partial q_i} - p_i \frac{\partial H}{\partial z_2})\partial_{p_i} - \frac{\partial H}{\partial p_i}\partial_{q_i} + (\frac{\partial H}{\partial z_2} + p_i \frac{\partial H}{\partial p_i})\partial_{z_2}.$$

2.3. Lagrangian submanifold.

Definition 2.5. *Let (M, ω) be a symplectic manifold of dimension $2n$. A submanifold $N \subset M$ is called a Lagrangian submanifold if it satisfies the following conditions:*

- (1) *The restriction of the symplectic form ω to N is zero.*
- (2) *$\dim(N) = n$.*

Among the most fundamental examples of Lagrangian submanifolds are those arising from the geometry of cotangent bundles. The following proposition establishes a profound link between closed differential forms on a manifold and Lagrangian sections of its cotangent bundle, a result that will play a pivotal role throughout the remainder of this paper.

Proposition 2.3. [20] *Let M be a differential manifold, and let T^*M be its cotangent bundle equipped with its canonical symplectic 2-form $\Omega_0 = -d\theta$, where θ is the Liouville 1-form. Let β be a differential 1-form on M . The image of β is a Lagrangian submanifold of (T^*M, Ω_0) if and only if β is closed.*

Let M be a differential manifold. Following Tulczyjew, one can define two derivations, i_T and d_T from $\Omega(M)$ into $\Omega(TM)$ respectively (see [3]). Given a p -form α on $M^{2k+2h+1}$, $i_T\alpha$ is a $(p-1)$ -form on TM defined by

$$i_T\alpha(v)(\omega_1, \dots, \omega_{p-1}) = \alpha(x)(v, u_1, \dots, u_{p-1})$$

where $v \in T_x M$, $\omega_i \in T_v(TM)$ and $u_i = T\varphi(\omega_i)$ for $i = 1, \dots, p-1$. In particular, $i_T f = 0$ for function and if α is a 1-form, $i_T\alpha(v) = \langle v, \alpha(\varphi(v)) \rangle$ where, as usual, $\langle, \rangle$ denotes the natural pairing between vectors and covectors. If $\alpha \in \Omega^1(M)$ is locally represented by $\alpha = \alpha_i dq_i$, then, in natural bundle coordinates (q_i, v_i) on TM ,

$$i_T\alpha = \alpha_i v_i. \quad (2.1)$$

The derivative operator $d_T : \Omega(M) \rightarrow \Omega(TM)$ is defined by

$$d_T = i_T d + di_T.$$

Let now α be an p -form on $M^{2k+2h+1}$, then its complete lift to TM is the p -form α^c which is fully characterized by the property

$$\alpha^c(X_1^c, \dots, X_p^c) = (\alpha(X_1, \dots, X_p))^c$$

for any $X_i \in \chi(M)$ (see [2]). One can then easily verify, using local coordinate expressions for instance, that

$$\alpha^c = d_T\alpha. \quad (2.2)$$

2.4. Gradient vector fields and Lagrangian submanifolds on contact-symplectic pairs.

Let (M, η, ω) be a contact-symplectic pair of type (k, h) . We denote by φ and ψ the naturel projection of TM and T^*M on M , respectively, and we recall that θ stands for the canonical Liouville 1-form on T^*M . Observe that the naturel projections φ and ψ are defined by:

$$\forall (m, v) \in TM, \forall (m, \beta) \in T^*M, \varphi(m, v) = m, \psi(m, \beta) = m,$$

and the Liouville 1-form θ is defined by:

$$\forall (m, \beta) \in T^*M, \forall u \in T_{(m, \beta)} T^*M, \theta_{(m, \beta)}(u) = \langle \beta, T_{(m, \beta)}\psi(u) \rangle.$$

Let $\flat_{(\eta, \omega)} : TM \rightarrow T^*M$ be the isomorphisme of fiber defined by:

$$\forall (m, v) \in TM, \flat_{(\eta, \omega)}(m, v) = (m, i_v(\omega_m + d\eta_m) + \eta_m(v)\eta_m)$$

and ξ, χ the naturel projection of $T(TM)$ and $T(T^*M)$ on TM and T^*M , respectively, defined by:

$$\forall (m, v, u) \in T(TM), \forall (m, \beta, \mu) \in T(T^*M), \xi(m, v, u) = (m, v), \chi(m, \beta, \mu) = (m, \beta)$$

We have the following relations

$$\psi \circ \flat_{(\eta, \omega)} = \varphi, \chi \circ T\flat_{(\eta, \omega)} = \flat_{(\eta, \omega)} \circ \xi.$$

Proposition 2.4. *Let (M, η, ω) be a contact-symplectic pair of type (k, h) . So we have*

$$\flat_{(\eta, \omega)}^* \theta = i_T(\omega + d\eta) + (i_T\eta)\varphi^*\eta$$

Proof. $\forall (m, v) \in TM$, and $\forall u \in T_{(m, \beta)} TM, \theta_{(m, \beta)}$, where $\beta = i_v(\omega_m + d\eta_m) + \eta_m(v)\eta_m$. We have

$$\begin{aligned}
 \flat_{(\eta, \omega)}^* \theta_{(m, v)}(u) &= \theta_{(m, \beta)}(T_{(m, v)} \flat_{(\eta, \omega)}(u)) \\
 &= \langle \beta, T_{(m, \beta)} \psi(T_{(m, v)} \flat_{(\eta, \omega)}(u)) \rangle \\
 &= \langle \beta, T_{(m, v)} \psi \circ \flat_{(\eta, \omega)}(u) \rangle \\
 &= \langle \beta, T_{(m, v)} \varphi(u) \rangle \\
 &= i_v(\omega_m + d\eta_m)(T_{(m, v)} \varphi(u)) + \eta_m(v)\eta_m(T_{(m, v)} \varphi(u)) \\
 &= i_T(\omega + d\eta)_{(m, v)}(u) + \eta_m(v)\varphi^* \eta_{(m, v)}(u)
 \end{aligned}$$

Consequently we obtain, $\flat_{(\eta, \omega)}^* \theta = i_T(\omega + d\eta) + (i_T \eta) \varphi^* \eta$. $\square$

Let now Ω_0 be the symplectic form on the cotangent bundle T^*M defined by: $\Omega_0 = -d\theta$. Since $\flat_{(\eta, \omega)}$ is an isomorphism, then the form $\Omega = \flat_{(\eta, \omega)}^* \Omega_0$ is a symplectic form on the tangent bundle TM . We obtain the following corollary

Corollary 2.1. *Let (M, η, ω) be a contact-symplectic pair of type (k, h) . The symplectic form Ω on the tangent bundle TM is given by*

$$\Omega = -(d_T \omega + d_T d\eta + d_T \eta \wedge \varphi^* \eta - i_T d\eta \wedge \varphi^* \eta + i_T \eta \wedge \varphi^* d\eta). \quad (2.3)$$

Proof. We have

$$\begin{aligned}
 \Omega &= -d\flat_{(\eta, \omega)}^* \theta \\
 &= -[di_T \omega + di_T d\eta + d(i_T \eta) \varphi^* \eta] \\
 &= -[di_T \omega + di_T d\eta + di_T \eta \wedge \varphi^* \eta + (i_T \eta) \varphi^* d\eta].
 \end{aligned}$$

Now we have $d_T \omega = i_T d\omega + di_T \omega$, $d_T d\eta = i_T dd\eta + di_T d\eta$ and $d_T \eta = i_T d\eta + di_T \eta$. Since ω is closed and $d \circ d = 0$ then $d\omega = 0$ and $dd\eta = 0$. Therefore we obtain

$$\Omega = -(d_T \omega + d_T d\eta + d_T \eta \wedge \varphi^* \eta - i_T d\eta \wedge \varphi^* \eta + i_T \eta \wedge \varphi^* d\eta)$$

$\square$

Remark 2.1. (1) Taking into account the relation (2.2), the relation (2.3) can still be rewritten as

$$\Omega = -(\omega^c + d\eta^c + \eta^c \wedge \varphi^* \eta - i_T d\eta \wedge \varphi^* \eta + i_T \eta \wedge \varphi^* d\eta).$$

(2) Let (x_i, y_i, p_i, q_i, z) be a Darboux coordinate system of M and $(x_i, y_i, p_i, q_i, z, l_i, u_i, v_i, w_i, t)$ the coordinate system of the corresponding tangent bundle TM . According to the Darboux's theorem and the relation (2.1), we obtain $\omega = \sum_{i=1}^k dx_i \wedge dy_i$, $\eta = \sum_{i=1}^h p_i dq_i +$

dz and $i_T dx_i = l_i, i_T dy_i = u_i, i_T dp_i = v_i, i_T dq_i = w_i, i_T dz = t$. So we have,

$$\begin{aligned}
 d_T \omega &= di_T \omega \\
 &= d((i_T dx_i) \wedge dy_i - (i_T dy_i) \wedge dx_i) \\
 &= d(l_i dy_i - u_i dx_i) \\
 &= dl_i \wedge dy_i - du_i \wedge dx_i \\
 d_T d\eta &= di_T d\eta \\
 &= d((i_T dp_i) \wedge dq_i - (i_T dq_i) \wedge dp_i) \\
 &= dv_i \wedge dq_i - dw_i \wedge dp_i \\
 di_T \eta \wedge \varphi^* \eta &= w_i p_j dp_i \wedge dq_j + w_i dp_i \wedge dz + p_i p_j dw_i \wedge dq_j + p_i dw_i \wedge dz + p_j dt \wedge dq_j + dt \wedge dz \\
 i_T \eta \wedge \varphi^* d\eta &= p_i w_i dp_j \wedge dq_j + td p_j \wedge dq_j
 \end{aligned}$$

Consequently we obtain,

$$\begin{aligned}
 \Omega = - \Big(&dl_i \wedge dy_i - du_i \wedge dx_i + dv_i \wedge dq_i - dw_i \wedge dp_i \\
 &+ w_i p_j dp_i \wedge dq_j + w_i dp_i \wedge dz + p_i p_j dw_i \wedge dq_j + p_i dw_i \wedge dz \\
 &+ p_j dt \wedge dq_j + dt \wedge dz + p_i w_i dp_j \wedge dq_j + td p_j \wedge dq_j \Big)
 \end{aligned}$$

Proposition 2.5. *Let (M, η, ω) be a contact-symplectic pair of type (k, h) . Then for each $\varphi \in C^\infty(M)$, $\text{im}(\text{grad } \varphi)$ is a Lagrangian submanifold of (TM, Ω) .*

Proof. Let $\varphi \in C^\infty(M)$. The exterior derivative $d\varphi$ is a closed 1-form on M . According to Proposition 2.3, its image $\text{im}(d\varphi)$ is a Lagrangian submanifold of (T^*M, Ω_0) . By construction, the musical isomorphism $\flat_{(\eta, \omega)}$ is a symplectomorphism from (TM, Ω) to (T^*M, Ω_0) . It follows that its inverse image $\flat_{(\eta, \omega)}^{-1}(\text{im}(d\varphi))$ is a Lagrangian submanifold of (TM, Ω) .

Since the gradient of φ is defined via the relation $\text{grad } \varphi = \flat_{(\eta, \omega)}^{-1}(d\varphi)$, we have the equality of their images:

$$\text{im}(\text{grad } \varphi) = \flat_{(\eta, \omega)}^{-1}(\text{im}(d\varphi))$$

Consequently, $\text{im}(\text{grad } \varphi)$ is a Lagrangian submanifold of (TM, Ω) . $\square$

The converse of Proposition 2.5 does not hold in general; that is, if a vector field X on a contact-symplectic pair manifold determines a Lagrangian section of the tangent bundle, it is not necessarily a gradient vector field. Drawing again from the analogous situation in the purely symplectic case, we are naturally led to consider the notion of a *local gradient vector field*.

Definition 2.6. *A vector field X on a contact-symplectic pair (M, η, ω) is called a local gradient vector field if its musical image $\flat_{(\eta, \omega)}(X)$ is a closed 1-form, which means:*

$$d(i_X(\omega + d\eta) + \eta(X)\eta) = 0. \quad (2.4)$$

Alternatively, since both ω and $d\eta$ are closed, a direct computation using Cartan's magic formula yields the equivalent condition in terms of the Lie derivative:

$$L_X(\omega + d\eta) = \eta \wedge L_X \eta + (i_X d\eta) \wedge \eta - \eta(X)d\eta. \quad (2.5)$$

We are now ready to state the main result of this section, from which Proposition 2.5 can be recovered as an immediate corollary.

Theorem 2.3. *Let (M, η, ω) be a contact-symplectic pair. A vector field X on M is a local gradient vector field if and only if $\text{im}(X)$ is a Lagrangian submanifold of (TM, Ω) .*

Proof. Let X be a vector field on a contact-symplectic pair manifold (M, η, ω) . By definition, X is a local gradient vector field if and only if

$$d(i_X(\omega + d\eta) + \eta(X)\eta) = 0. \quad (2.6)$$

This condition states precisely that the 1-form $b_{(\eta, \omega)}(X) = i_X(\omega + d\eta) + \eta(X)\eta$ is closed. Recall the classical property that the image of a 1-form on M is a Lagrangian submanifold of (T^*M, Ω_0) if and only if the 1-form is closed (see [20]). Thus, the image $\text{im}(b_{(\eta, \omega)}(X))$ is a Lagrangian submanifold of (T^*M, Ω_0) . Since $b_{(\eta, \omega)} : (TM, \Omega) \rightarrow (T^*M, \Omega_0)$ is a symplectomorphism, its inverse preserves the Lagrangian property. Therefore, the pull-back of this image, which satisfies

$$b_{(\eta, \omega)}^{-1}(\text{im}(b_{(\eta, \omega)}(X))) = \text{im}(X), \quad (2.7)$$

is a Lagrangian submanifold of (TM, Ω) . $\square$

This theorem is the contacto-symplectic analogue of the characterization of local Hamiltonian vector fields on symplectic manifolds (see [4]).

2.5. Gradient vector fields and Lagrangian submanifolds on contact-pairs. Let (M, η, α) be a contact pair of type (k, h) . We denote by φ and ψ the naturel projection of TM and T^*M on M , respectively, and we recall that θ stands for the canonical Liouville 1-form on T^*M . Observe that the naturel projections φ and ψ are defined by:

$$\forall (m, v) \in TM, \forall (m, \beta) \in T^*M, \varphi(m, v) = m, \psi(m, \beta) = m,$$

and the Liouville 1-form θ is defined by:

$$\forall (m, \beta) \in T^*M, \forall u \in T_{(m, \beta)}T^*M, \theta_{(m, \beta)}(u) = \langle \beta, T_{(m, \beta)}\psi(u) \rangle.$$

Let $b_{(\eta, \omega)} : TM \rightarrow T^*M$ be the isomorphisme of fiber defined by:

$$\forall (m, v) \in TM, b_{(\eta, \omega)}(m, v) = (m, i_v(d\alpha_m + d\eta_m) + \eta_m(v)\eta_m + \alpha_m(v)\alpha_m)$$

and ξ, χ the naturel projection of $T(TM)$ and $T(T^*M)$ on TM and T^*M , respectively, defined by:

$$\forall (m, v, u) \in T(TM), \forall (m, \beta, \mu) \in T(T^*M), \xi(m, v, u) = (m, v), \chi(m, \beta, \mu) = (m, \beta)$$

We have the following relations

$$\psi \circ b_{(\eta, \alpha)} = \varphi, \chi \circ T b_{(\eta, \alpha)} = b_{(\eta, \alpha)} \circ \xi.$$

Proposition 2.6. *Let (M, η, α) be a contact pair of type (k, h) . So we have*

$$b_{(\eta, \alpha)}^* \theta = i_T(d\alpha + d\eta) + (i_T \eta) \varphi^* \eta + (i_T \alpha) \varphi^* \alpha$$

Proof. $\forall (m, v) \in TM$, and $\forall u \in T_{(m, \beta)} TM, \theta_{(m, \beta)}$, where $\beta = i_v(d\alpha_m + d\eta_m) + \eta_m(v)\eta_m + \alpha_m(v)\alpha_m$.

We have

$$\begin{aligned}
 \flat_{(\eta, \alpha)}^* \theta_{(m, v)}(u) &= \theta_{(m, \beta)}(T_{(m, v)} \flat_{(\eta, \alpha)}(u)) \\
 &= \langle \beta, T_{(m, \beta)} \psi(T_{(m, v)} \flat_{(\eta, \alpha)}(u)) \rangle \\
 &= \langle \beta, T_{(m, v)} \psi \circ \flat_{(\eta, \alpha)}(u) \rangle \\
 &= \langle \beta, T_{(m, v)} \varphi(u) \rangle \\
 &= i_v(d\alpha_m + d\eta_m)(T_{(m, v)} \varphi(u)) + \eta_m(v)\eta_m(T_{(m, v)} \varphi(u)) + \alpha_m(v)\alpha_m(T_{(m, v)} \varphi(u)) \\
 &= i_T(d\alpha + d\eta)_{(m, v)}(u) + \eta_m(v)\varphi^* \eta_{(m, v)}(u) + \alpha_m(v)\varphi^* \alpha_{(m, v)}(u)
 \end{aligned}$$

Consequently we obtain, $\flat_{(\eta, \alpha)}^* \theta = i_T(d\alpha + d\eta) + (i_T \eta)\varphi^* \eta + (i_T \alpha)\varphi^* \alpha$. $\square$

Let now Ω be the symplectic form on the cotangent bundle T^*M defined by: $\Omega_0 = -d\theta$. Since $\flat_{(\eta, \alpha)}$ is an isomorphism, then the form $\Omega = \flat_{(\eta, \alpha)}^* \Omega_0$ is a symplectic form on the tangent bundle TM . We obtain the following corollary

Corollary 2.2. *Let (M, η, α) be a contact pair of type (k, h) . The symplectic form Ω on the tangent bundle TM is given by*

$$\Omega = -(d_T d\alpha + d_T d\eta + d_T \eta \wedge \varphi^* \eta - i_T d\eta \wedge \varphi^* \eta + i_T \eta \wedge \varphi^* d\eta + d_T \alpha \wedge \varphi^* \alpha - i_T d\alpha \wedge \varphi^* \alpha + i_T \alpha \wedge \varphi^* d\alpha). \quad (2.8)$$

Proof. We have

$$\begin{aligned}
 \Omega &= -d\flat_{(\eta, \alpha)}^* \theta \\
 &= -d(i_T(d\alpha + d\eta) + (i_T \eta)\varphi^* \eta + (i_T \alpha)\varphi^* \alpha) \\
 &= -(di_T d\alpha + di_T d\eta + (di_T \eta) \wedge \varphi^* \eta + (i_T \eta)\varphi^* d\eta + (di_T \alpha) \wedge \varphi^* \alpha + (i_T \alpha)\varphi^* d\alpha).
 \end{aligned}$$

Now we have $d_T d\alpha = i_T d d\alpha + di_T d\alpha$, $d_T d\eta = i_T d d\eta + di_T d\eta$, $d_T \eta = i_T d\eta + di_T \eta$ and $d_T \alpha = i_T d\alpha + di_T \alpha$. Since $d \circ d = 0$ then $dd\alpha = 0$ and $dd\eta = 0$. Therefore we obtain

$$\begin{aligned}
 \Omega &= -(d_T d\alpha + d_T d\eta + d_T \eta \wedge \varphi^* \eta - i_T d\eta \wedge \varphi^* \eta \\
 &\quad + i_T \eta \wedge \varphi^* d\eta + d_T \alpha \wedge \varphi^* \alpha - i_T d\alpha \wedge \varphi^* \alpha + i_T \alpha \wedge \varphi^* d\alpha)
 \end{aligned} \quad (2.9)$$

$\square$

Remark 2.2. (1) Taking into account the relation (2.2), the relation (2.8) can still be rewritten as

$$\Omega = -(d\alpha^c + d\eta^c + \eta^c \wedge \varphi^* \eta - i_T d\eta \wedge \varphi^* \eta + i_T \eta \wedge \varphi^* d\eta + \alpha^c \wedge \varphi^* \alpha - i_T d\alpha \wedge \varphi^* \alpha + i_T \alpha \wedge \varphi^* d\alpha).$$

(2) Let $(x_i, y_i, p_i, q_i, z_1, z_2)$ be a Darboux coordinate system of M and $(x_i, y_i, p_i, q_i, z_1, z_2, l_i, u_i, v_i, w_i, t_1, t_2)$ the coordinate system of the corresponding tangent bundle TM . According to the Darboux's theorem and the relation (2.1), we obtain $\alpha = \sum_{i=1}^k x_i dy_i + dz_1$, $\eta = \sum_{i=1}^h p_i dq_i + dz_2$ and $i_T dx_i = l_i$, $i_T dy_i = u_i$, $i_T dp_i = v_i$, $i_T dq_i = w_i$, $i_T dz_1 = t_1$, $i_T dz_2 = t_2$. So we

have,

$$\begin{aligned}
 d_T d\alpha &= di_T d\alpha \\
 &= d((i_T dx_i) \wedge dy_i - (i_T dy_i) \wedge dx_i) \\
 &= d(l_i dy_i - u_i dx_i) \\
 &= dl_i \wedge dy_i - du_i \wedge dx_i \\
 d_T d\eta &= di_T d\eta \\
 &= d((i_T dp_i) \wedge dq_i - (i_T dq_i) \wedge dp_i) \\
 &= dv_i \wedge dq_i - dp_i \wedge dw_i \\
 d_T \eta \wedge \varphi^* \eta &= (i_T d\eta + di_T \eta) \wedge \varphi^* \eta \\
 &= p_i p_j dw_i \wedge dq_j + p_i dw_i \wedge dz_2 + v_i dq_i \wedge dz_2 \\
 i_T d\eta \wedge \varphi^* \eta &= v_i dq_i \wedge dz_2 - p_j w_i dp_i \wedge dq_j - w_i dp_i \wedge dz_2 \\
 i_T \eta \wedge \varphi^* d\eta &= p_j w_j dp_i \wedge dq_i + t_2 dp_i \wedge dq_i \\
 \alpha^c \wedge \varphi^* \alpha &= x_i x_j du_i \wedge dy_j + x_i du_i \wedge dz_1 + l_i dy_i \wedge dz_1 \\
 i_T d\alpha \wedge \varphi^* \alpha &= l_i dy_i \wedge dz_1 - p_j u_i dx_i \wedge dy_j - u_i dx_i \wedge dz_1 \\
 i_T \alpha \wedge \varphi^* d\alpha &= x_j u_j dx_i \wedge dy_i + t_1 dx_i \wedge dy_i
 \end{aligned}$$

Consequently we obtain,

$$\begin{aligned}
 \Omega = & -(dl_i \wedge dy_i - du_i \wedge dx_i + dv_i \wedge dq_i - dp_i \wedge dw_i + v_i dq_i \wedge dz_2 - p_j w_i dp_i \wedge dq_j - w_i dp_i \wedge dz_2 - v_i dq_i \wedge dz_2 \\
 & + w_i dp_i \wedge dz_2 - l_i dy_i \wedge dz_1 + p_j u_i dx_i \wedge dy_j + u_i dx_i \wedge dz_1 + x_j u_j dx_i \wedge dy_i + t_1 dx_i \wedge dy_i).
 \end{aligned} \tag{2.10}$$

Proposition 2.7. *Let (M, α, η) be a contact pair. Then for each $\varphi \in C^\infty(M)$, $\text{im}(\text{grad } \varphi)$ is a Lagrangian submanifold of (TM, Ω) .*

Proof. Let $\varphi \in C^\infty(M)$. The exterior derivative $d\varphi$ is a closed 1-form on M . According to Proposition 2.3, its image $\text{im}(d\varphi)$ is a Lagrangian submanifold of (T^*M, Ω_0) .

By construction, the musical isomorphism $\flat_{(\eta, \alpha)}$ is a symplectomorphism from (TM, Ω) to (T^*M, Ω_0) . It follows that its inverse image $\flat_{(\eta, \alpha)}^{-1}(\text{im}(d\varphi))$ is a Lagrangian submanifold of (TM, Ω) .

Since the gradient of φ is defined via the relation $\text{grad } \varphi = \flat_{(\eta, \alpha)}^{-1}(d\varphi)$, we have the equality of their images:

$$\text{im}(\text{grad } \varphi) = \flat_{(\eta, \omega)}^{-1}(\text{im}(d\varphi))$$

Consequently, $\text{im}(\text{grad } \varphi)$ is a Lagrangian submanifold of (TM, Ω) . $\square$

The converse of Proposition 2.7 does not hold in general; that is, if a vector field X on a contact pair manifold determines a Lagrangian section of the tangent bundle, it is not necessarily a gradient vector field. Drawing again from the analogous situation in the purely symplectic case, we are naturally led to consider the notion of a *local gradient vector field*.

Definition 2.7. *A vector field X on a contact pair manifold (M, η, α) is called a local gradient vector field if its musical image $\flat_{(\eta, \alpha)}(X)$ is a closed 1-form, which satisfies the condition:*

$$d(i_X(d\alpha + d\eta) + \eta(X)\eta + \alpha(X)\alpha) = 0. \tag{2.11}$$

Alternatively, since $d\alpha$ and $d\eta$ are both closed, a direct application of Cartan's magic formula yields:

$$\mathcal{L}_X(d\alpha + d\eta) = \eta \wedge \mathcal{L}_X\eta + (i_X d\eta) \wedge \eta - \eta(X)d\eta + \alpha \wedge \mathcal{L}_X\alpha + (i_X d\alpha) \wedge \alpha - \alpha(X)d\alpha. \quad (2.12)$$

We now arrive at the main result of this section, from which proposition 2.7 can in fact be derived as a corollary.

Theorem 2.4. *Let (M, η, ω) be a contact-symplectic pair. A vector field X on M is a local gradient vector field if and only if $\text{im}(X)$ is a Lagrangian submanifold of (TM, Ω) .*

Proof. Let X be a vector field on a contact pair manifold (M, η, α) . By definition, X is a local gradient vector field if and only if

$$d(i_X(d\alpha + d\eta) + \eta(X)\eta + \alpha(X)\alpha) = 0. \quad (2.13)$$

This is equivalent to stating that the 1-form $\flat_{(\eta, \alpha)}(X) = i_X(d\alpha + d\eta) + \eta(X)\eta + \alpha(X)\alpha$ is closed. According to the classical property that the image of a 1-form on M is a Lagrangian submanifold of (T^*M, Ω_0) if and only if the 1-form is closed (see [20]), the image $\text{im}(\flat_{(\eta, \alpha)}(X))$ is a Lagrangian submanifold of (T^*M, Ω_0) . Since the musical bundle map $\flat_{(\eta, \alpha)} : (TM, \Omega) \rightarrow (T^*M, \Omega_0)$ is a symplectomorphism, its inverse preserves the Lagrangian property. Therefore, the pullback of this sub-manifold, which satisfies

$$\flat_{(\eta, \alpha)}^{-1}(\text{im}(\flat_{(\eta, \alpha)}(X))) = \text{im}(X), \quad (2.14)$$

is a Lagrangian submanifold of the tangent bundle (TM, Ω) . $\square$

This theorem is the contact pair analogue of the characterization of local Hamiltonian vector fields on symplectic manifolds (see.[4])

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