



THEORETICAL ANALYSIS OF TOPOLOGICAL OPTIMIZATION FOR THE NONLINEAR P-LAPLACIAN OPERATOR

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ABSTRACT. This study develops a theoretical framework for topological optimization involving the nonlinear p -Laplacian operator. The focus is placed on the topological derivative of a shape-dependent functional, examined under both Neumann and Dirichlet boundary conditions. The derivation relies on a minmax variational approach, which captures the impact of small perturbations in the domain, such as the introduction of holes or inclusions.

In the first part, we establish the topological derivative in the Neumann setting, followed by the Dirichlet case. We conclude with a comparison to previous works, highlighting how our results extend classical analyses limited to linear operators and reveal distinctive behaviors between the two boundary conditions. This positions our contribution as both a continuation of existing research and a step forward in understanding topological derivatives in nonlinear media.

1. INTRODUCTION AND RELATED WORK

This paper is devoted to the mathematical foundations of topological optimization theory. In [6], thermal residual stresses were incorporated into the topological optimization of structures manufactured by additive processes. The isogeometric approach has been widely adopted in areas requiring precise geometric representations, such as shell analysis [38] and robust mesh and shape optimization [39]. Regarding thermoelastic behavior, the thermomechanical contact problem in mortar structures [12] and the material distribution in functionally graded structures [40] have also been studied using isogeometric methods. For further developments, see [32].

In this topological derivative, we can consider the following assumptions: we recall the notions of Minkowski content, d -rectifiability, and positive reach that are used to define the topological derivative of an objective function with respect to perturbations by a d -dimensional closed subset $\bar{\omega}$ of $\mathbb{R}^N$. Given an open domain Ω with boundary $\partial\Omega$. Let us associate with ε , $0 < \varepsilon \leq r$, the perturbed domain $\Omega_\varepsilon = \Omega \setminus \bar{\omega}_\varepsilon$ is obtained by removing the ε -dilatation $\bar{\omega}_\varepsilon$ of the set $\bar{\omega}$. The boundary of Ω_ε is made up of two disjoint parts $\partial\Omega_\varepsilon = \partial\Omega \cup \partial\bar{\omega}_\varepsilon$. The notion of positive reach for a closed nowhere dense set $\bar{\omega}$ ensures that the boundary $\partial\bar{\omega}_\varepsilon$ of the hole is of class $C^{1,1}$ and that $\bar{\Omega} \setminus \bar{\omega} = \bar{\Omega}$.

Before tackling the theorem of the topological derivative, we also need the following hypotheses. For the conception of these hypotheses, we relied on the works of Novotny

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[2, 3, 4, 5], and Delfour [14, 15, 16, 16, 18, 19, 20, 22].

Let Ω_ϵ^m denote the component of Ω_ϵ for which $\partial\Omega$ is part of its boundary. Let Ω_ϵ^0 denote the blind component of Ω_ϵ whose boundary has no intersection with $\partial\Omega$. The function u_{Ω_ϵ} is divided between the two components Ω_ϵ^0 and Ω_ϵ^m as follows: $u_{\Omega_\epsilon} = u_{\Omega_0}$ within Ω_ϵ^0 , and $\frac{\partial u_\epsilon}{\partial n} = 0$ on $\partial\Omega_\epsilon^m \cap \partial\bar{w}_\epsilon$ as $\partial\bar{w}_\epsilon$ consists of the two disjoint boundaries Ω_ϵ^0 and $\partial\Omega_\epsilon^0$ and $\partial\Omega_\epsilon^m \cap \partial\bar{w}_\epsilon$. We can construct an extension of Ω by defining the solution as follows: $u_{\Omega_\epsilon}^0 = u_{\Omega_\epsilon}$ on $\partial\Omega_\epsilon^m \cap \partial\bar{w}_\epsilon$, $u_{\Omega_\epsilon}^0 = u_{\Omega_\epsilon}$ on $\partial\Omega_\epsilon^0$. For more information, the reader can refer to [18, 19, 20, 23].

The main objective of this article is to determine the topological derivative of the functional $F(\Omega_\epsilon) = F(\Omega_\epsilon, u_{\Omega_\epsilon})$, where the perturbed domain Ω_ϵ of Ω is defined by $\Omega_\epsilon = T_\epsilon(\Omega)$ or $\Omega_\epsilon = \Omega \setminus \bar{w}_\epsilon$ depending on the derivative to be calculated. In this paper, we look at following shape optimization problem:

$$\min_{\Omega \in \mathcal{O}} F(\Omega), \quad (1.1)$$

where F is a given cost functional, $\mathcal{O}$ a class of admissible domains, Ω is an open bounded subset of $\mathbb{R}^N$, $N \geq 2$ and such that

$$F(\Omega) = \beta \int_{\Omega} |\nabla u_{\Omega}|^2 dx + \alpha \int_{\Omega} |u_{\Omega} - u_0|^2 dx, \quad (1.2)$$

with β and α being positive constants and u_{Ω} is the solution to the following problem

$$\begin{cases} \lambda u_{\Omega} - \mu \operatorname{div}(|\nabla u_{\Omega}|^{p-2} \nabla u_{\Omega}) = f & \text{in } \Omega \\ \mathcal{B}u_{\Omega} = 0 & \text{on } \partial\Omega, \end{cases} \quad (1.3)$$

where λ and μ being positive constants and $\mathcal{B}u_{\Omega}$ is a boundary operator $\partial\Omega$ denoting either a Dirichlet condition $\mathcal{B}u_{\Omega} = u_{\Omega}$ on $\partial\Omega$ or a Neumann condition $\mathcal{B}u_{\Omega} = \frac{\partial u_{\Omega}}{\partial n}$ on $\partial\Omega$.

The central result of this paper is established through the minmax method. We now present the main theorem of this work, which establishes the topological derivative for problems governed by both Dirichlet and Neumann boundary conditions. This result encapsulates the core analytical contribution of the paper and provides a unified framework for evaluating sensitivity with respect to topological perturbations under varied boundary constraints.

Theorem 1.1. *Let $0 \leq d < N$, $E = \bar{w}$, and $s = \alpha_{N-d} r^{N-d}$. The topological derivative of a shape functional F exists at point $z \in \Omega$ with respect to the inclusion E if and only if the following limit exists:*

$$\lim_{\epsilon \rightarrow 0} \frac{F(\Omega_\epsilon) - F(\Omega)}{s(\epsilon)} = D_T F(z)$$

where: $\Omega_\epsilon = \Omega \setminus \bar{w}_\epsilon$ is the perturbed domain with a small inclusion w_ϵ centered at z , $s(\epsilon) = \alpha_{N-d} \epsilon^{N-d}$ is the scaling factor corresponding to the measure of the inclusion, and $D_T F(z)$ is the topological derivative of F at the point z .

This limit captures the first-order sensitivity of the functional F with respect to the creation of an infinitesimal hole or inclusion at z , and is foundational in topology optimization and inverse

problems. In other words

$$L = \lim_{\epsilon \rightarrow 0} (L_0(\epsilon) + L_1(\epsilon)),$$

exists with

$$L_0(\epsilon) = \int_{\Omega_\epsilon} \beta \left| \nabla \left(\frac{u_{\Omega_\epsilon} - u_{\Omega_0}}{\sqrt{s}} \right) \right|^2 + \alpha \left| \frac{u_{\Omega_\epsilon} - u_{\Omega_0}}{\sqrt{s}} \right|^2$$

and

$$\begin{aligned} L_1(\epsilon) = & -\frac{1}{s} \int_{\Omega_\epsilon} \lambda (-1)^{p-2} (p-1) (\nabla(u_{\Omega_\epsilon} - u_{\Omega_0}) |\nabla p_0|^{p-2}) \nabla p_\epsilon dx \\ & + \frac{1}{s} \int_{\Omega_\epsilon} \mu (-1)^{p-2} (p-1) \nabla(u_{\Omega_\epsilon} - u_{\Omega_0}) \nabla p_0 \left(\left| \nabla \left(\frac{u_{\Omega_\epsilon} + u_{\Omega_0}}{2} \right) \right|^{p-2} - |\nabla u_{\Omega_0}|^{p-2} \right) dx \\ & - \frac{1}{s} \int_{\Omega_\epsilon} \lambda (u_{\Omega_\epsilon} - u_{\Omega_0}) p_\epsilon dx + \frac{1}{s} \int_{\partial \Omega_\epsilon^m \cap \partial \bar{\omega}_\epsilon} \nabla u_{\Omega_0} \cdot d\bar{\omega} p_0 dH^{N-1} dx. \end{aligned}$$

Moreover, the topological derivative of the function is given by the expression :

$$DF(\Omega) = L - \left[\int_{\bar{\omega}} \beta |\nabla u_{\Omega_0}|^2 + \alpha |u_{\Omega_0} - u_0|^2 + \mu |\nabla u_{\Omega_0}|^{p-2} \nabla u_{\Omega_0} \nabla p_0 - f p_0 + \lambda u_{\Omega_0} p_0 \right] dH^d$$

where p_0, u_{Ω_0} are solutions of systems

$$\int_{\Omega} [2\beta \nabla u_{\Omega_0} \cdot \nabla \phi' dx + 2\alpha (u_{\Omega_0} - u_0) \phi' + \mu (-1)^{p-2} (p-1) [\nabla \phi' \cdot |\nabla u_{\Omega_0}|^{p-2} \nabla p_0] + \lambda \phi' p_0] dx = 0.$$

Remark 1.1. We note that the theorem established above is that of probleme (1.3) with a Neumann condition at the edge. We point out that we find the same result under a Dirichlet condition at the edge. The difference is just in the choice of spaces considered.

Remark 1.2. For the Neumann condition the space considered is the following:

$$V_H = \left\{ u_{\Omega_\epsilon} \in L^2(\Omega) \mid \nabla u_{\Omega_\epsilon} \in (L^p(\Omega))^N, u_{\Omega_\epsilon} = 0 \text{ on } \partial\Omega, \frac{\partial u_{\Omega_\epsilon}}{\partial n} = 0 \text{ on } \partial \bar{\omega}_\epsilon \right\}. \quad (1.4)$$

whereas for the Dirichlet condition we consider the following space:

$$V'_H = \left\{ u_{\Omega_\epsilon} \in L^2(\Omega) \mid \nabla u_{\Omega_\epsilon} \in (L^p(\Omega))^N, u_{\Omega_\epsilon} = 0 \text{ on } \partial\Omega, u_{\Omega_\epsilon} = 0 \text{ on } \partial \bar{\omega}_\epsilon \right\}. \quad (1.5)$$

The structure of this work is organized as follows: Section 2 introduces some preliminary results that are essential for the subsequent analysis. In Section 3, we present one of the main results, namely the derivation of the topological derivative of the functional under a Neumann boundary condition. Section 4 continues this study by establishing the topological derivative in the case of a Dirichlet boundary condition. We observe that the two results differ only in the choice of functional spaces, while the calculations remain essentially the same. Finally, Section 5 offers a comparison between our findings and previous works, followed by a conclusion and several perspectives for future research.

2. PRELIMINARIES

In this section, we revisit the analytical framework originally introduced in [33] and subsequently extended in [17, 21, 18] to accommodate the multivalued setting. We introduce two complementary subsections dedicated to the analysis of the shape derivative and the topological derivative of the cost functional, both derived within the minmax framework. These analytical tools enable a rigorous investigation of the functional's sensitivity with respect to smooth domain deformations and topological changes, such as the introduction of small holes. The results presented here serve as the basis for establishing first-order optimality conditions in the corresponding shape optimization problem. For further insights and technical developments discussed in the next section, the reader is referred to [24, 25, 18, 19, 20, 23, 33].

Definition 2.1. A *Lagrangian function* is a mapping

$$L : [0, \tau] \times X \times Y \rightarrow \mathbb{R}, \quad (t, x, y) \mapsto L(t, x, y)$$

where $\tau > 0$ is a fixed real number, X is a vector space, and Y is a non-empty subset of X . For each fixed $t \in [0, \tau]$ and $x \in X$, the function $y \mapsto L(t, x, y)$ is affine.

Definition 2.2. Let us associate to the parameter t the following parametrized minmax function:

$$g(t) = \inf_{x \in X} \sup_{y \in Y} L(t, x, y), \quad t \in [0, \tau], \quad \tau > 0,$$

and define its right derivative at $t = 0$ by:

$$dg(0) = \lim_{t \rightarrow 0^+} \frac{g(t) - g(0)}{t}.$$

Definition 2.3. When the following limits exist, we adopt the notations:

$$d_t L(0, x, y) = \lim_{t \rightarrow 0^+} \frac{L(t, x, y) - L(0, x, y)}{t},$$

$$\text{for } \varphi \in X, \quad d_x L(t, x, y; \varphi) = \lim_{\theta \rightarrow 0^+} \frac{L(t, x + \theta \varphi, y) - L(t, x, y)}{\theta},$$

$$\text{for } \phi \in Y, \quad d_y L(t, x, y; \phi) = \lim_{\theta \rightarrow 0^+} \frac{L(t, x, y + \theta \phi) - L(t, x, y)}{\theta}.$$

Remark 2.1. Since $L(t, x, y)$ is affine in y , for all $(t, x) \in [0, \tau] \times X$ and all $y, \psi \in Y$, we have:

$$d_y L(t, x, y; \psi) = L(t, x, y) - L(t, x, 0) = d_y L(t, x, 0; \psi).$$

Definition 2.4 (State Equation). For each $t \geq 0$, the state equation consists in finding $x^t \in X$ such that:

$$\forall \psi \in Y, \quad d_y L(t, x^t, 0; \psi) = 0.$$

The set of such states is denoted:

$$E(t) = \{x^t \in X \mid \forall \psi \in Y, \quad d_y L(t, x^t, 0; \psi) = 0\}.$$

Definition 2.5 (Adjoint Equation). For each $t \geq 0$, the adjoint equation consists in finding $p^t \in Y$ such that:

$$\forall \varphi \in X, \quad d_x L(t, x^t, p^t; \varphi) = 0.$$

The set of such adjoint solutions is denoted:

$$Y(t, x^t) = \{p^t \in Y \mid \forall \varphi \in X, \quad d_x L(t, x^t, p^t; \varphi) = 0\}.$$

Definition 2.6 (Minimizers of the Minmax). *The set of minimizers for the minmax problem is defined by:*

$$X(t) = \left\{ x^t \in X \mid g(t) = \inf_{x \in X} \sup_{y \in Y} L(t, x, y) = \sup_{y \in Y} L(t, x^t, y) \right\}.$$

Lemma 2.1. *The following statements hold:*

(i) *The minimax value can be expressed as:*

$$\inf_{x \in X} \sup_{y \in Y} L(t, x, y) = \inf_{x \in E(t)} L(t, x, 0).$$

(ii) *The minimax function satisfies $g(t) = +\infty$ if and only if $E(t) = \emptyset$. In this case, we have $X(t) = X$.*

(iii) *If $E(t) \neq \emptyset$, then the set of minimizers is given by:*

$$X(t) = \left\{ x^t \in E(t) \mid L(t, x^t, 0) = \inf_{x \in E(t)} L(t, x, 0) \right\} \subset E(t),$$

and in this case, $g(t) < +\infty$.

Proof. See [14, 15, 16, 16, 18, 19, 20, 22, 33]. □

Hypothesis (H0)

Let X be a vector space. We assume the following regularity conditions:

(i) For all $t \in [0, \tau]$, $x^0 \in X(0)$, $x^t \in X(t)$, and $y \in Y$, the function

$$\theta \mapsto L(t, x^0 + \theta(x^t - x^0), y), \quad \theta \in [0, 1]$$

is continuous. As a consequence, the derivative exists for almost every $\theta \in [0, 1]$ and is given by

$$\frac{d}{d\theta} L(t, x^0 + \theta(x^t - x^0), y) = d_x L(t, x^0 + \theta(x^t - x^0), y; x^t - x^0).$$

Moreover, the function admits the integral representation:

$$L(t, x^t, y) = L(t, x^0, y) + \int_0^1 d_x L(t, x^0 + \theta(x^t - x^0), y; x^t - x^0) d\theta.$$

(ii) For all $t \in [0, \tau]$, $x^0 \in X(0)$, $x^t \in X(t)$, $y \in Y$, and $\phi \in X$, the directional derivative

$$d_x L(t, x^0 + \theta(x^t - x^0), y; \phi)$$

exists for almost every $\theta \in [0, 1]$, and the function

$$\theta \mapsto d_x L(t, x^0 + \theta(x^t - x^0), y; \phi)$$

belongs to the space $L^1([0, 1])$.

Definition 2.7 (Averaged Adjoint Equation). *Given $x^0 \in X(0)$ and $x^t \in X(t)$, the averaged adjoint equation is defined as follows:*

$$\text{Find } y^t \in Y \text{ such that for all } \phi \in X, \quad \int_0^1 d_x L(t, x^0 + \theta(x^t - x^0), y^t; \phi) d\theta = 0.$$

The set of solutions to this equation is denoted by $Y(t, x^0, x^t)$. In particular, when $t = 0$ and $x^t = x^0$, the averaged adjoint equation reduces to the standard adjoint equation, and we recover the set of classical adjoint states:

$$Y(0, x^0, x^0) = Y(0, x^0).$$

Theorem 2.1. *Let*

$$L : [0, \tau] \times X \times Y \rightarrow \mathbb{R}, \quad (t, x, y) \mapsto L(t, x, y)$$

be a Lagrangian functional, where $\tau > 0$, X and Y are vector spaces, and for each fixed (t, x) , the mapping $y \mapsto L(t, x, y)$ is affine.

Assume that Hypothesis (H0) holds, along with the following conditions:

(H1) *For all $t \in [0, \tau]$, the minmax value $g(t)$ is finite, and the sets $X(t) = \{x^t\}$ and $Y(0, x^0) = \{p^0\}$ are singletons.*

(H2) *The partial derivative $d_t L(0, x^0, p^0)$ exists.*

(H3) *The following limit exists:*

$$R(x^0, p^0) = \lim_{t \rightarrow 0^+} \int_0^1 d_x L \left(t, x^0 + \theta(x^t - x^0), p^0; \frac{x^t - x^0}{t} \right) d\theta.$$

Then the right derivative of the minmax function at $t = 0$ exists and is given by:

$$dg(0) = d_t L(0, x^0, p^0) + R(x^0, p^0).$$

Proof. See [19, 20, 33]. □

Corollary 2.1. *Let*

$$L : [0, \tau] \times X \times Y \rightarrow \mathbb{R}, \quad (t, x, y) \mapsto L(t, x, y)$$

be a Lagrangian functional, where $\tau > 0$, X and Y are vector spaces, and for each fixed (t, x) , the mapping $y \mapsto L(t, x, y)$ is affine.

Assume that Hypothesis (H0) holds, along with the following conditions:

(H1a) *For all $t \in [0, \tau]$, the set $X(t)$ is non-empty, the minmax value $g(t)$ is finite, and for every $x \in X(0)$, the set $Y(0, x)$ is non-empty.*

(H2a) *For all $x \in X(0)$ and $p \in Y(0, x)$, the partial derivative $d_t L(0, x, p)$ exists.*

(H3a) *There exist $x^0 \in X(0)$ and $p^0 \in Y(0, x^0)$ such that the following limit exists:*

$$R(x^0, p^0) = \lim_{t \rightarrow 0^+} \int_0^1 d_x L \left(t, x^0 + \theta(x^t - x^0), p^0; \frac{x^t - x^0}{t} \right) d\theta.$$

Then the right derivative of the minmax function at $t = 0$ exists, and there exist $x^0 \in X(0)$ and $p^0 \in Y(0, x^0)$ such that:

$$dg(0) = d_t L(0, x^0, p^0) + R(x^0, p^0).$$

Proof. See [19, 33]. □

Prior to addressing the main Theorem, we introduce some preliminary definitions. For further details and a comprehensive theoretical background, the reader is referred to the works of Delfour, including but not limited to [2, 3, 4, 5, 13, 14, 16, 17, 18, 19, 20, 22, 33] We revisit the analytical framework originally introduced in [33] and subsequently extended in [17, 21, 18] to accommodate the multivalued setting.

Definition 2.8. Given $d, 0 \leq d \leq N$, the d -dimensional upper and lower Minkowski contents of a set E are defined through an r -dilatation of the set E as follows:

$$M^{*d}(E) = \limsup_{r \searrow 0} \frac{m_N(E_r)}{\alpha_{N-d} r^{N-d}}, \quad M_*^d(E) = \liminf_{r \searrow 0} \frac{m_N(E_r)}{\alpha_{N-d} r^{N-d}}.$$

where m_N is the Lebesgue measure in $\mathbb{R}^N$ and α_{N-d} is the volume of the ball of radius one in $\mathbb{R}^{N-d}$. When the two limits exist and are equal, we say that E admits a d -dimensional Minkowski content, and their common value will be denoted $M^d(E)$.

Definition 2.9. Let E be a subset of a metric space X . We say that E is d -rectifiable if it is the image of a compact subset K of $\mathbb{R}^d$ by a Lipschitz continuous function $f : \mathbb{R}^d \rightarrow X$.

Definition 2.10. Let $E \subset \mathbb{R}^N$ be H^d -measurable. We say that E is

1) countably d -rectifiable if there exist countably many Lipschitzian functions $f_i : \mathbb{R}^d \rightarrow \mathbb{R}^N$ such that $E \subset \bigcup_{i=0}^{\infty} f_i(\mathbb{R}^d)$.

2) countably H^d -rectifiable if there exist countably many Lipschitz functions $f_i : \mathbb{R}^d \rightarrow \mathbb{R}^N$ such that $E \setminus \bigcup_{i=0}^{\infty} f_i(\mathbb{R}^d)$ is H^d -negligible : $H^d(E \setminus \bigcup_{i=0}^{\infty} f_i(\mathbb{R}^d)) = 0$.

3) H^d -rectifiable if it is countably H^d -rectifiable and $H^d(E) < \infty$.

Definition 2.11. Let E be a closed subset of $\mathbb{R}^N$.

1) The set of points of $\mathbb{R}^N$ which have a unique projection on E .

$$\text{Unp}(E) = \{y \in \mathbb{R}^N : \exists \text{ an unique } x \in E \text{ such that } d_E(y) = \|y - x\|\},$$

where $d_E(y)$ is the distance function from a point y to E .

2) The reach of a point $x \in E$ and the reach of E are respectively defined as follows :'

$$\text{reach}(E, x) = \sup\{r > 0 : B(x, r) \subset \text{Unp}(E)\}, \quad \text{reach}(E) = \inf_{x \in E} \text{reach}(E, x).$$

Remark 2.2. We say that E is a positive reach if $\text{reach}(E) > 0$.

The definition of $\text{Unp}(E)$ implies the existence of a projection on E , a function $p_E : \text{Unp}(E) \rightarrow E$ which associates $x \in \text{Unp}(E)$ such that $d_E(x) = \|x - p_E(x)\|$.

In the following theorem $N \geq 2$ is a natural number, d the dimension of a subset E of $\mathbb{R}^N$, r the radius of the ball and α_{N-d} is the volume of the unit ball in $\mathbb{R}^{N-d}$.

3. TOPOLOGICAL DERIVATIVE WITH NEUMANN CONDITION

The topological derivative is derived using a unified variational and minmax formulation, inspired by the works of Delfour and Sturm [14, 15, 16, 16, 18, 19, 20, 22, 34]. This approach generalizes previous results to nonlinear PDE settings. We now proceed with the derivation of the main result concerning the topological derivative, focusing specifically on the case of Neumann boundary conditions. This choice is motivated by the natural emergence of Neumann conditions in variational formulations, especially in problems involving flux constraints or free boundaries. By imposing Neumann conditions on the boundary, we maintain consistency with the weak formulation of the governing PDE, which is crucial for the mathematical rigor of the analysis. Moreover, this setting allows us to leverage key analytical tools such as monotonicity and compactness arguments,

which are instrumental in establishing the existence of solutions and ensuring the stability of the derivative under domain perturbations.

Proof. of Theorem 1.1

Let us associate with ϵ , where $0 < \epsilon \leq r$, the perturbed domain $\Omega_\epsilon = \Omega \setminus \overline{\omega_\epsilon}$, where $\omega_\epsilon \subset \Omega$ denotes a small inclusion. We assume that

$$\partial\Omega_\epsilon = \partial\Omega \cup \partial\overline{\omega_\epsilon}, \quad \partial\Omega \cap \partial\overline{\omega_\epsilon} = \emptyset, \quad \text{and} \quad \partial\overline{\omega_\epsilon} \in \mathcal{C}^{1,1}.$$

In what follows, we consider a functional F_ϵ defined on the perturbed domain Ω_ϵ by

$$F(\Omega_\epsilon) = \beta \int_{\Omega_\epsilon} |\nabla u_{\Omega_\epsilon}|^2 dx + \alpha \int_{\Omega_\epsilon} |u_{\Omega_\epsilon} - u_0|^2 dx, \quad (3.1)$$

where u_{Ω_ϵ} is the solution to the following problem

$$\begin{cases} \lambda u_{\Omega_\epsilon} - \mu \operatorname{div}(|\nabla u_{\Omega_\epsilon}|^{p-2} \nabla u_{\Omega_\epsilon}) = f & \text{in } \Omega_\epsilon \\ u_{\Omega_\epsilon} = 0 & \text{in } \partial\Omega \\ \frac{\partial u_{\Omega_\epsilon}}{\partial n} = 0 & \text{on } \partial\overline{\omega_\epsilon}. \end{cases} \quad (3.2)$$

We can extend the formulation to the entire domain Ω by introducing a solution defined on $\overline{\omega_\epsilon^0} \rightarrow \mathbb{R}$, corresponding to the following problem:

$$\lambda u_{\Omega_\epsilon} - \mu \operatorname{div}(|\nabla u_{\Omega_\epsilon}|^{p-2} \nabla u_{\Omega_\epsilon}) = f \text{ in } \overline{\omega_\epsilon^0} \text{ and } u_{\Omega_\epsilon}^0 = u_{\Omega_\epsilon} \text{ on } \partial\overline{\omega_\epsilon}. \quad (3.3)$$

We suppose that Ω_ϵ consists of two disjoint components: Ω_ϵ^m , the main component, whose boundary includes a portion of $\partial\Omega$, Ω_ϵ^0 , the blind component, whose boundary satisfies $\partial\Omega_\epsilon^0 \cap \partial\Omega = \emptyset$. The function u_{Ω_ϵ} is distributed between the two components Ω_ϵ^0 and Ω_ϵ^m as

$$\begin{cases} \lambda u_{\Omega_\epsilon} - \mu \operatorname{div}(|\nabla u_{\Omega_\epsilon}|^{p-2} \nabla u_{\Omega_\epsilon}) = f & \text{in } \Omega_\epsilon^m, \\ \begin{cases} u_{\Omega_\epsilon} = 0 & \text{on } \partial\Omega, \\ u_{\Omega_\epsilon} = 0 & \text{on } \partial\Omega_\epsilon^m \cap \partial\overline{\omega_\epsilon} \end{cases} \end{cases} \quad (3.4)$$

Since $\partial\overline{\omega_\epsilon}$ is made up of two disjoint boundary $\partial\Omega_\epsilon^0$ and $\partial\Omega_\epsilon^m \cap \partial\overline{\omega_\epsilon}$, we can construct an extension to Ω by defining the solution $\overline{\omega_\epsilon^0} \rightarrow \mathbb{R}$.

$$\lambda u_{\Omega_\epsilon} - \mu \operatorname{div}(|\nabla u_{\Omega_\epsilon}|^{p-2} \nabla u_{\Omega_\epsilon}) = f \text{ in } \overline{\omega_\epsilon^0} \text{ and } u_{\Omega_\epsilon}^0 = u_{\Omega_\epsilon} \text{ on } \partial\Omega_\epsilon^m \cap \partial\overline{\omega_\epsilon} \quad u_{\Omega_\epsilon}^0 = u_{\Omega_\epsilon} \text{ on } \partial\Omega_\epsilon^0. \quad (3.5)$$

The shape functional associated with the perforated domain $\Omega_\epsilon = \Omega \setminus \overline{\omega_\epsilon}$ is given by (3.1) where u_{Ω_ϵ} is solution the variational problem

$$\mu \int_{\Omega_\epsilon} |\nabla u_{\Omega_\epsilon}|^{p-2} \nabla u_{\Omega_\epsilon} \nabla v dx + \lambda \int_{\Omega_\epsilon} u_{\Omega_\epsilon} v dx = \int_{\Omega_\epsilon} f v dx. \quad (3.6)$$

To evaluate the topological derivative of the functional $F(\Omega_\epsilon)$, we analyze its sensitivity to the insertion of a small inclusion $\omega_\epsilon \subset \Omega$. As $\epsilon \rightarrow 0$, the leading-order variation of F is captured by the asymptotic expansion:

$$F(\Omega_\epsilon) = F(\Omega) + D_T F(z) \cdot s(\epsilon) + o(s(\epsilon)),$$

where $D_T F(z)$ denotes the topological derivative at point $z \in \Omega$, and $s(\epsilon) = \alpha_{N-d} \epsilon^{N-d}$ reflects the inclusion's volume scaling. The remainder term $o(s(\epsilon))$ becomes negligible for small ϵ . This formulation enables the identification of regions where introducing a small void or inclusion would most effectively influence the functional an essential tool in shape and topology optimization. To formalize this, we define the following set:

$$V_H = \left\{ u_{\Omega_\epsilon} \in L^2(\Omega) \mid \nabla u_{\Omega_\epsilon} \in (L^p(\Omega))^N, u_{\Omega_\epsilon} = 0 \text{ on } \partial\Omega, \frac{\partial u_{\Omega_\epsilon}}{\partial n} = 0 \text{ on } \partial\bar{\omega}_\epsilon \right\}. \quad (3.7)$$

In the framework of variational formulation (3.2), the goal is to find $u_{\Omega_\epsilon} \in V_H$ such that:

$$\mu \int_{\Omega_\epsilon} |\nabla u_{\Omega_\epsilon}|^{p-2} \nabla u_{\Omega_\epsilon} \nabla v dx + \lambda \int_{\Omega_\epsilon} u_{\Omega_\epsilon} v dx = \int_{\Omega_\epsilon} f v dx, \quad \forall v \in V_H. \quad (3.8)$$

Thus, the Lagrangian dependent on ϵ will be written in the form:

$$L(\epsilon, \phi, \Phi) = \beta \int_{\Omega_\epsilon} |\nabla \phi|^2 dx + \alpha \int_{\Omega_\epsilon} |\phi - u_0|^2 dx + \mu \int_{\Omega_\epsilon} |\nabla \phi|^{p-2} \nabla \phi \nabla \Phi dx + \lambda \int_{\Omega_\epsilon} \phi \Phi dx - \int_{\Omega_\epsilon} f \Phi dx. \quad (3.9)$$

Hence, the derivative of the Lagrangian with respect to ϕ , parameterized by ϵ , is given by:

$$\begin{aligned} d_\phi L(\epsilon, \phi, \Phi, \phi') &= \int_{\Omega_\epsilon} 2\beta \nabla \phi \cdot \nabla \phi' dx + \int_{\Omega_\epsilon} 2\alpha (\phi - u_0) \phi' dx \\ &+ \int_{\Omega_\epsilon} \mu (-1)^{p-2} (p-1) [\nabla \phi' \cdot |\nabla \phi|^{p-2} \nabla \Phi] dx + \lambda \int_{\Omega_\epsilon} \phi' \Phi dx. \end{aligned}$$

As a result, the variational formulation of the adjoint state equation is derived as follows: $d_\phi L(0, u_{\Omega_0}, p_0, \phi') = 0$, where $u_{\Omega_0} = u_{\Omega_\epsilon}$ for $\epsilon = 0$. Find $p_0 \in H_0^1(\Omega)$ such that:

$$\int_{\Omega} [2\beta \nabla u_{\Omega_0} \cdot \nabla \phi' + 2\alpha (u_{\Omega_0} - u_0) \phi' + \mu (-1)^{p-2} (p-1) [\nabla \phi' \cdot |\nabla u_{\Omega_0}|^{p-2} \nabla p_0] + \lambda \phi' p_0] dx = 0. \quad (3.10)$$

We now proceed to differentiate the Lagrangian with respect to the field variable Φ , by evaluating the first variation of the action functional.

$$d_\Phi L(\epsilon, \phi, \Phi, \Phi') = \mu \int_{\Omega_\epsilon} |\nabla \phi|^{p-2} \nabla \phi \nabla \Phi' dx + \lambda \int_{\Omega_\epsilon} \phi \Phi' dx - \int_{\Omega_\epsilon} f \Phi' dx.$$

The initial state $u_{\Omega_0} = u_\Omega$ is a solution of $d_\Phi L(0, u_{\Omega_0}, 0, \Phi') = 0 \quad \forall \Phi' \in H_0^1$ and in this case, we have:

$$\int_{\Omega} [\mu |\nabla u_{\Omega_0}|^{p-2} \nabla u_{\Omega_0} \nabla \Phi' + \lambda u_{\Omega_0} \Phi' - f \Phi'] dx = 0.$$

The state u_{Ω_ϵ} for all $\epsilon \geq 0$ satisfies

$$\int_{\Omega_\epsilon} [\mu |\nabla u_{\Omega_0}|^{p-2} \nabla u_{\Omega_0} \nabla \Phi' + \lambda u_{\Omega_0} \Phi' - f \Phi'] dx = 0, \quad \forall \Phi' \in V_H.$$

To compute the topological derivative of the Lagrangian with respect to ϵ , we begin by analyzing the normalized variation:

$$\frac{L(u_{\Omega_\epsilon}, \nabla u_{\Omega_\epsilon}) - L(u_\Omega, \nabla u_\Omega)}{s(\epsilon)},$$

where L represents the Lagrangian density, u_{Ω_ϵ} denotes the solution in the perturbed domain $\Omega_\epsilon = \Omega \setminus \bar{\omega}_\epsilon$. This formulation captures the leading-order impact of introducing a

small inclusion into the domain, allowing us to quantify how the perturbation influences the system's energy. The topological derivative at a point z is then defined as the limit:

$$D_T L(z) = \lim_{\epsilon \rightarrow 0} \frac{L(u_{\Omega_\epsilon}, \nabla u_{\Omega_\epsilon}) - L(u_\Omega, \nabla u_\Omega)}{s(\epsilon)}.$$

This derivative provides a precise measure of the sensitivity of the Lagrangian to infinitesimal topological changes, and serves as a foundational tool for guiding domain modifications in optimization procedures.

$$\begin{aligned} L(\epsilon, \phi, \Phi) - L(0, \phi, \Phi) &= \beta \int_{\Omega_\epsilon} |\nabla \phi|^2 dx + \alpha \int_{\Omega_\epsilon} |\phi - u_0|^2 dx \\ &+ \mu \int_{\Omega_\epsilon} |\nabla \phi|^{p-2} \nabla \phi \nabla \Phi dx + \lambda \int_{\Omega_\epsilon} \phi \Phi dx - \int_{\Omega_\epsilon} f \Phi dx \\ &- \left[\beta \int_{\Omega} |\nabla \phi|^2 dx + \alpha \int_{\Omega} |\phi - u_0|^2 dx \right] \\ &- \left[\mu \int_{\Omega} |\nabla \phi|^{p-2} \nabla \phi \nabla \Phi dx + \lambda \int_{\Omega} \phi \Phi dx - \int_{\Omega} f \Phi dx \right] \\ L(\epsilon, \phi, \Phi) - L(0, \phi, \Phi) &= - \left[\int_{\bar{\omega}_\epsilon} \beta |\nabla \phi|^2 + \alpha |\phi - u_0|^2 + \mu |\nabla \phi|^{p-2} \nabla \phi \nabla \Phi - f \Phi + \lambda \phi \Phi \right] dx. \end{aligned}$$

For $d = 0$, $\bar{\omega} = \{x_0\}$, $\bar{\omega}_\epsilon = \{x \in \mathbb{R}^N : |x - x_0| \leq \epsilon\} = \overline{B(x_0, \epsilon)}$.

$$\begin{aligned} d_s L(0, \phi, \Phi) &= \lim_{s \rightarrow 0} - \frac{1}{|B(x_0, \epsilon)|} \left[\int_{B(x_0, \epsilon)} \beta |\nabla \phi|^2 + \alpha |\phi - u_0|^2 + \mu |\nabla \phi|^{p-2} \nabla \phi \nabla \Phi - f \Phi + \lambda \phi \Phi \right] dx \\ &= -\beta |\nabla \phi(x_0)|^2 - \alpha |\phi(x_0) - u_0(x_0)|^2 - \mu |\nabla \phi(x_0)|^{p-2} \nabla \phi(x_0) \nabla \Phi(x_0) \\ &+ f \Phi(x_0) - \lambda \phi(x_0) \Phi(x_0). \end{aligned}$$

When the last equation is considered at u_{Ω_0}, p_0 , the result is:

$$\begin{aligned} d_s L(0, u_{\Omega_0}, p_0) &= -\beta |\nabla u_{\Omega_0}(x_0)|^2 - \alpha |u_{\Omega_0}(x_0) - u_0(x_0)|^2 - \mu |\nabla u_{\Omega_0}(x_0)|^{p-2} \nabla u_{\Omega_0}(x_0) \nabla p_0(x_0) \\ &+ f p_0(x_0) - \lambda u_{\Omega_0}(x_0) p_0(x_0). \end{aligned}$$

Hence, if $0 < d \leq N - 1$, we have:

$$\begin{aligned} \frac{L(\epsilon, \phi, \Phi) - L(0, \phi, \Phi)}{s} &= - \frac{1}{|\bar{\omega}_\epsilon|} \left[\int_{\bar{\omega}_\epsilon} \beta |\nabla \phi|^2 + \alpha |\phi - u_0|^2 + \mu |\nabla \phi|^{p-2} \nabla \phi \nabla \Phi - f \Phi + \lambda \phi \Phi \right] dx \\ &\longrightarrow - \left[\int_{\bar{\omega}} \beta |\nabla \phi|^2 + \alpha |\phi - u_0|^2 + \mu |\nabla \phi|^{p-2} \nabla \phi \nabla \Phi - f \Phi + \lambda \phi \Phi \right] dH^d. \end{aligned}$$

Consequently, by applying the previous result at the point u_{Ω_0}, p_0 , the expression becomes:

$$d_s L(0, u_{\Omega_0}, p_0) = - \left[\int_{\bar{\omega}} \beta |\nabla u_{\Omega_0}|^2 + \alpha |u_{\Omega_0} - u_0|^2 + \mu |\nabla u_{\Omega_0}|^{p-2} \nabla u_{\Omega_0} \nabla p_0 - f p_0 + \lambda u_{\Omega_0} p_0 \right] dH^d.$$

We will now define $R(\epsilon)$ by:

$$R(\epsilon) = \int_0^1 dx L \left(\epsilon, u_{\Omega_0} + \Psi(u_{\Omega_\epsilon} - u_{\Omega_0}), p_0, \left(\frac{u_{\Omega_\epsilon} - u_{\Omega_0}}{s} \right) \right) d\Psi.$$

By substituting $\phi' = \frac{u_{\Omega_\epsilon} - u_{\Omega_0}}{\epsilon}$ and $\Psi = \frac{u_{\Omega_\epsilon} - u_{\Omega_0}}{2}$ into the adjoint equation for p_0 , we obtain the following expression:

$$\begin{aligned} R(\epsilon) &= 2\beta \int_{\Omega_\epsilon} \nabla \left(\frac{u_{\Omega_\epsilon} + u_{\Omega_0}}{2} \right) \cdot \nabla \left(\frac{u_{\Omega_\epsilon} - u_{\Omega_0}}{s} \right) dx + 2\alpha \int_{\Omega_\epsilon} \left(\frac{u_{\Omega_\epsilon} + u_{\Omega_0}}{2} - u_0 \right) \cdot \left(\frac{u_{\Omega_\epsilon} - u_{\Omega_0}}{s} \right) dx \\ &\quad + \int_{\Omega_\epsilon} (-1)^{p-2} (p-1) \mu \left(\nabla \left(\frac{u_{\Omega_\epsilon} - u_{\Omega_0}}{s} \right) \cdot \left| \frac{u_{\Omega_\epsilon} + u_{\Omega_0}}{2} \right|^{p-2} \nabla p_0 \right) dx + \lambda \int_{\Omega_\epsilon} \left(\frac{u_{\Omega_\epsilon} - u_{\Omega_0}}{s} \right) p_0 dx \end{aligned}$$

$$\begin{aligned} R(\epsilon) &= \int_{\Omega_\epsilon} \beta \left(\left| \nabla \left(\frac{u_{\Omega_\epsilon} - u_{\Omega_0}}{\sqrt{s}} \right) \right|^2 + \alpha \left| \frac{u_{\Omega_\epsilon} - u_{\Omega_0}}{\sqrt{s}} \right|^2 \right) dx \\ &\quad + \frac{1}{s} \int_{\Omega_\epsilon} (-1)^{p-2} (p-1) \mu (\nabla(u_{\Omega_\epsilon} - u_{\Omega_0}) \nabla p_0 \left(\left| \nabla \left(\frac{u_{\Omega_\epsilon} + u_{\Omega_0}}{2} \right) \right|^{p-2} - |\nabla u_{\Omega_0}|^{p-2} \right) dx \\ &\quad + \frac{1}{s} \int_{\Omega_\epsilon} (2\beta \nabla u_{\Omega_0} (\nabla u_{\Omega_\epsilon} - \nabla u_{\Omega_0}) + 2\alpha (\nabla u_{\Omega_\epsilon} - \nabla u_{\Omega_0}) (\nabla u_{\Omega_\epsilon} - \nabla u_{\Omega_0})) dx \\ &\quad + \frac{1}{s} \int_{\Omega_\epsilon} ((-1)^{p-2} (p-1) \mu \nabla(u_{\Omega_\epsilon} - u_{\Omega_0}) |u_{\Omega_0}|^{p-2} \nabla p_0 + \lambda (u_{\Omega_\epsilon} - u_{\Omega_0}) p_0) dx \end{aligned}$$

Thus, for all $u_{\Omega_0} \in H_0^1(\Omega)$, equation (3.8) takes the form:

$$\begin{aligned} \int_{\Omega_\epsilon} (\mu |\nabla u_{\Omega_\epsilon}|^{p-2} \nabla u_{\Omega_\epsilon} \nabla v + \lambda u_{\Omega_\epsilon} v - f v) dx &= - \left[\int_{\overline{\omega_\epsilon}} (\mu |\nabla u_{\Omega_\epsilon}|^{p-2} \nabla u_{\Omega_\epsilon} \nabla v + \lambda u_{\Omega_\epsilon} v - f v) dx \right] \\ &= \int_{\partial \overline{\omega_\epsilon}} \frac{\partial u_{\Omega_0}}{\partial n} v dH^{N-1}, \quad \forall v \in H_0^1(\Omega). \end{aligned}$$

Now, considering the assumption about $\overline{\omega}$, $\partial \overline{\omega_\epsilon} \in \mathcal{C}^{1,1}$ and

$$\frac{\partial u_{\Omega_0}}{\partial n} = \nabla u_{\Omega_0} \cdot n_{\Omega_\epsilon} = -\nabla u_{\Omega_0} \cdot \nabla d\overline{\omega} \text{ on } \overline{\omega_\epsilon}.$$

In this particular case, the following holds:

$$\int_{\Omega_\epsilon} (\mu |\nabla u_{\Omega_\epsilon}|^{p-2} \nabla u_{\Omega_\epsilon} \nabla v + \lambda u_{\Omega_\epsilon} v - f v) dx = - \int_{\partial \overline{\omega_\epsilon}} \nabla u_{\Omega_0} \cdot \nabla d\overline{\omega} v dH^{N-1}. \quad (3.11)$$

By computing the difference between equation (3.8) and equation (3.11), we arrive at:

$$\int_{\Omega_\epsilon} \mu \nabla (u_{\Omega_\epsilon} - u_{\Omega_0}) \nabla v = \int_{\partial \bar{\omega}_\epsilon} \nabla u_{\Omega_0} \cdot \nabla d\bar{\omega} v dH^{N-1}.$$

Under the condition $\epsilon \geq 0$, the adjoint equation takes the form:

$$\begin{aligned} & \int_{\Omega_\epsilon} (2\beta \nabla u_{\Omega_0} (\nabla u_{\Omega_\epsilon} - \nabla u_{\Omega_0}) + 2\alpha (u_{\Omega_\epsilon} - u_{\Omega_0}) (u_{\Omega_\epsilon} - u_0)) dx \\ & + \int_{\Omega_\epsilon} ((-1)^{p-2} (p-1) \mu \nabla (u_{\Omega_\epsilon} - u_{\Omega_0}) |u_{\Omega_0}|^{p-2} \nabla p_0) dx \\ & + \lambda \int_{\Omega_\epsilon} (u_{\Omega_\epsilon} - u_{\Omega_0}) p_0 dx = 0. \end{aligned}$$

Thus that

$$\begin{aligned} & \int_{\Omega_\epsilon} (2\beta \nabla u_{\Omega_0} (\nabla u_{\Omega_\epsilon} - \nabla u_{\Omega_0}) + 2\alpha (u_{\Omega_\epsilon} - u_0) (\nabla u_{\Omega_\epsilon} - \nabla u_{\Omega_0})) dx \\ & = -\lambda \int_{\Omega_\epsilon} (-1)^{p-2} (p-1) \mu \nabla (u_{\Omega_\epsilon} - u_{\Omega_0}) |u_{\Omega_0}|^{p-2} \nabla p_\epsilon dx \\ & - \lambda \int_{\Omega_\epsilon} (u_{\Omega_\epsilon} - u_{\Omega_0}) p_\epsilon dx \end{aligned}$$

$$\begin{aligned} R(\epsilon) &= \int_{\Omega_\epsilon} \left(\beta \left| \nabla \left(\frac{u_{\Omega_\epsilon} - u_{\Omega_0}}{\sqrt{s}} \right) \right|^2 + \alpha \left| \frac{u_{\Omega_\epsilon} - u_{\Omega_0}}{\sqrt{s}} \right|^2 \right) dx \\ & - \frac{1}{s} \int_{\Omega_\epsilon} \lambda (-1)^{p-2} (p-1) (\nabla (u_{\Omega_\epsilon} - u_{\Omega_0}) | \nabla p_0 |^{p-2}) \nabla p_\epsilon dx \\ & + \frac{1}{s} \int_{\Omega_\epsilon} \mu (-1)^{p-2} (p-1) \nabla (u_{\Omega_\epsilon} - u_{\Omega_0}) \nabla p_0 \left(\left| \nabla \left(\frac{u_{\Omega_\epsilon} + u_{\Omega_0}}{2} \right) \right|^{p-2} - |\nabla u_{\Omega_0}|^{p-2} \right) dx \\ & - \frac{1}{s} \int_{\Omega_\epsilon} \lambda (u_{\Omega_\epsilon} - u_{\Omega_0}) p_\epsilon dx + \frac{1}{s} \int_{\partial \Omega_\epsilon^m \cap \partial \bar{\omega}_\epsilon} \nabla u_{\Omega_0} \cdot d\bar{\omega} p_0 dH^{N-1} dx \end{aligned}$$

□

4. TOPOLOGICAL DERIVATIVE WITH DIRICHLET CONDITION

In this section, we consider the case of a Dirichlet condition, and the perturbed problem thus becomes:

$$\begin{cases} \lambda u_{\Omega_\epsilon} - \mu \operatorname{div}(|\nabla u_{\Omega_\epsilon}|^{p-2} \nabla u_{\Omega_\epsilon}) = f & \text{in } \Omega_\epsilon \\ u_{\Omega_\epsilon} = 0 & \text{in } \partial\Omega \\ u_{\Omega_\epsilon} = 0 & \text{on } \partial\bar{\omega}_\epsilon. \end{cases} \quad (4.1)$$

First of all, the calculations remain the same. The changes are simply in the spaces considered. The techniques for calculating the derivative of the Lagrangian with respect to the variables remain the same. And for this, we don't need to go into all the details, as we did in the case of a Neumann condition. And We can be extended to Ω by introducing the solution $\bar{\omega}_\epsilon^0 \rightarrow \mathbb{R}$ of the problem

$$\lambda u_{\Omega_\epsilon} - \mu \operatorname{div}(|\nabla u_{\Omega_\epsilon}|^{p-2} \nabla u_{\Omega_\epsilon}) = f \text{ in } \bar{\omega}_\epsilon^0 \text{ and } u_{\Omega_\epsilon}^0 = u_{\Omega_\epsilon} \text{ on } \partial\bar{\omega}_\epsilon. \quad (4.2)$$

We suppose that Ω_ϵ has two components Ω_ϵ^m and Ω_ϵ^0 . Ω_ϵ^m is the component of Ω_ϵ for which $\partial\Omega$ is part of its boundary. Ω_ϵ^0 is the blind component of Ω_ϵ whose boundary has an empty intersection with $\partial\Omega$. The function u_{Ω_ϵ} is distributed between the two components Ω_ϵ^0 and Ω_ϵ^m as

$$\begin{cases} \lambda u_{\Omega_\epsilon} - \mu \operatorname{div}(|\nabla u_{\Omega_\epsilon}|^{p-2} \nabla u_{\Omega_\epsilon}) = f & \text{in } \Omega_\epsilon^m, \\ u_{\Omega_\epsilon} = 0 & \text{on } \partial\Omega, \\ u_{\Omega_\epsilon} = 0 & \text{on } \partial\Omega_\epsilon^m \cap \partial\bar{\omega}_\epsilon. \end{cases} \quad (4.3)$$

Since $\partial\bar{\omega}_\epsilon$ is made up of two disjoint boundary $\partial\Omega_\epsilon^0$ and $\partial\Omega_\epsilon^m \cap \partial\bar{\omega}_\epsilon$, we can construct an extension to Ω by defining the solution $\bar{\omega}_\epsilon^0 \rightarrow \mathbb{R}$

$$\lambda u_{\Omega_\epsilon} - \mu \operatorname{div}(|\nabla u_{\Omega_\epsilon}|^{p-2} \nabla u_{\Omega_\epsilon}) = f \text{ in } \bar{\omega}_\epsilon^0 \text{ and } u_{\Omega_\epsilon}^0 = u_{\Omega_\epsilon} \text{ on } \partial\Omega_\epsilon^m \cap \partial\bar{\omega}_\epsilon \text{ } u_{\Omega_\epsilon}^0 = u_{\Omega_\epsilon} \text{ on } \partial\Omega_\epsilon^0. \quad (4.4)$$

The shape functional associated the perforated domain is given by

$$F(\Omega_\epsilon) = \alpha \int_{\Omega_\epsilon} |u_{\Omega_\epsilon} - u_0|^2 dx + \beta \int_{\Omega_\epsilon} |\nabla u_{\Omega_\epsilon}|^2 dx \quad (4.5)$$

where u_{Ω_ϵ} is solution the variational problem

$$\mu \int_{\Omega_\epsilon} |\nabla u_{\Omega_\epsilon}|^{p-2} \nabla u_{\Omega_\epsilon} \nabla v dx + \lambda \int_{\Omega_\epsilon} u_{\Omega_\epsilon} v dx = \int_{\Omega_\epsilon} f v dx. \quad (4.6)$$

After selecting the appropriate functional space as defined by

$$V'_H = \left\{ u_{\Omega_\epsilon} \in L^2(\Omega) \mid \nabla u_{\Omega_\epsilon} \in (L^p(\Omega))^N, u_{\Omega_\epsilon} = 0 \text{ on } \partial\Omega, u_{\Omega_\epsilon} = 0 \text{ on } \partial\bar{\omega}_\epsilon \right\}, \quad (4.7)$$

we obtain the same topological derivative result as in the case with Neumann boundary conditions. This indicates that, for both types of boundary conditions, the overall approach remains consistent, with the only difference lying in the choice of the underlying functional space.

Our variational formulation (4.1) consists of finding $u_{\Omega_\epsilon} \in V'_H$ such that

$$\mu \int_{\Omega_\epsilon} |\nabla u_{\Omega_\epsilon}|^{p-2} \nabla u_{\Omega_\epsilon} \nabla v dx + \lambda \int_{\Omega_\epsilon} u_{\Omega_\epsilon} v dx = \int_{\Omega_\epsilon} f v dx. \quad (4.8)$$

And we have

$$\mu \int_{\Omega_\epsilon} |\nabla u_{\Omega_\epsilon}|^{p-2} \nabla u_{\Omega_\epsilon} \nabla v dx + \lambda \int_{\Omega_\epsilon} u_{\Omega_\epsilon} v dx = \int_{\Omega_\epsilon} f v dx, \quad \forall v \in V'_H. \quad (4.9)$$

Thus, the Lagrangian dependent on ϵ will be written in the form :

$$L(\epsilon, \phi, \Phi) = \beta \int_{\Omega_\epsilon} |\nabla \phi|^2 dx + \alpha \int_{\Omega_\epsilon} |\phi - u_0|^2 dx + \mu \int_{\Omega_\epsilon} |\nabla \phi|^{p-2} \nabla \phi \nabla \Phi dx + \lambda \int_{\Omega_\epsilon} \phi \Phi dx - \int_{\Omega_\epsilon} f \Phi dx \quad (4.10)$$

From this, the derivative of the Lagrangian, dependent on ϵ , with respect to ϕ .

$$\begin{aligned} d_\phi L(\epsilon, \phi, \Phi, \phi') &= \int_{\Omega_\epsilon} 2\beta \nabla \phi \cdot \nabla \phi' dx + \int_{\Omega_\epsilon} 2\alpha (\phi - u_0) \phi' dx \\ &+ \int_{\Omega_\epsilon} \mu (-1)^{p-2} (p-1) [\nabla \phi' \cdot |\nabla \phi|^{p-2} \nabla \Phi] dx + \lambda \int_{\Omega_\epsilon} \phi' \Phi dx. \end{aligned}$$

And the variational formulation of the adjoint state equation given by

$d_\phi L(0, u_{\Omega_0}, p_0, \phi') = 0$, where $u_{\Omega_0} = u_{\Omega_\epsilon}$ for $\epsilon = 0$. Find $p_0 \in H_0^1(\Omega)$ such that

$$\int_{\Omega} [2\beta \nabla u_{\Omega_0} \cdot \nabla \phi' + 2\alpha (u_{\Omega_0} - u_0) \phi' + \mu (-1)^{p-2} (p-1) [\nabla \phi' \cdot |\nabla u_{\Omega_0}|^{p-2} \nabla p_0] + \lambda \phi' p_0] dx = 0. \quad (4.11)$$

The derivative of the Lagrangian with respect to Φ .

$$d_\Phi L(\epsilon, \phi, \Phi, \Phi') = \mu \int_{\Omega_\epsilon} |\nabla \phi|^{p-2} \nabla \phi \nabla \Phi' dx + \lambda \int_{\Omega_\epsilon} \phi \Phi' dx - \int_{\Omega_\epsilon} f \Phi' dx.$$

The initial state $u_{\Omega_0} = u_\Omega$ is a solution of $d_\Phi L(0, u_{\Omega_0}, 0, \Phi') = 0 \quad \forall \Phi' \in H_0^1$ and in this case, we have:

$$\mu \int_{\Omega} |\nabla u_{\Omega_0}|^{p-2} \nabla u_{\Omega_0} \nabla \Phi' dx + \lambda \int_{\Omega} u_{\Omega_0} \Phi' dx - \int_{\Omega} f \Phi' dx = 0.$$

The state u_{Ω_ϵ} for all $\epsilon \geq 0$ satisfies

$$\int_{\Omega_\epsilon} [\mu |\nabla u_{\Omega_0}|^{p-2} \nabla u_{\Omega_0} \nabla \Phi' + \lambda u_{\Omega_0} \Phi' - f \Phi'] dx = 0, \quad \forall \Phi' \in V'_H.$$

For $d = 0$, $\bar{\omega} = \{x_0\}$, $\bar{\omega}_\epsilon = \{x \in \mathbb{R}^N : |x - x_0| \leq \epsilon\} = \overline{B(x_0, \epsilon)}$.

$$\begin{aligned} d_s L(0, \phi, \Phi) &= \lim_{s \rightarrow 0} - \frac{1}{|B(x_0, \epsilon)|} \left[\int_{B(x_0, \epsilon)} \beta |\nabla \phi|^2 + \alpha |\phi - u_0|^2 + \mu |\nabla \phi|^{p-2} \nabla \phi \nabla \Phi - f \Phi + \lambda \phi \Phi \right] dx \\ &= -\beta |\nabla \phi(x_0)|^2 - \alpha |\phi(x_0) - u_0(x_0)|^2 - \mu |\nabla \phi(x_0)|^{p-2} \nabla \phi(x_0) \nabla \Phi(x_0) \\ &+ f \Phi(x_0) - \lambda \phi(x_0) \Phi(x_0). \end{aligned}$$

By evaluating the last equation at the point u_{Ω_0}, p_0 , we obtain:

$$\begin{aligned} d_s L(0, u_{\Omega_0}, p_0) &= -\beta |\nabla u_{\Omega_0}(x_0)|^2 - \alpha |u_{\Omega_0}(x_0) - u_0(x_0)|^2 - \mu |\nabla u_{\Omega_0}(x_0)|^{p-2} \nabla u_{\Omega_0}(x_0) \nabla p_0(x_0) \\ &+ f p_0(x_0) - \lambda u_{\Omega_0}(x_0) p_0(x_0). \end{aligned}$$

Hence, if $0 < d \leq N - 1$, we have:

$$\begin{aligned} \frac{L(\epsilon, \phi, \Phi) - L(0, \phi, \Phi)}{s} &= -\frac{1}{|\bar{\omega}_\epsilon|} \left[\int_{\bar{\omega}_\epsilon} \beta |\nabla \phi|^2 + \alpha |\phi - u_0|^2 + \mu |\nabla \phi|^{p-2} \nabla \phi \nabla \Phi - f \Phi + \lambda \phi \Phi \right] dx \\ &\longrightarrow - \left[\int_{\bar{\omega}} \beta |\nabla \phi|^2 + \alpha |\phi - u_0|^2 + \mu |\nabla \phi|^{p-2} \nabla \phi \nabla \Phi - f \Phi + \lambda \phi \Phi \right] dH^d. \end{aligned}$$

Therefore, taking the last result at the point u_{Ω_0}, p_0 becomes:

$$d_s L(0, u_{\Omega_0}, p_0) = - \left[\int_{\bar{\omega}} \beta |\nabla u_{\Omega_0}|^2 + \alpha |u_{\Omega_0} - u_0|^2 + \mu |\nabla u_{\Omega_0}|^{p-2} \nabla u_{\Omega_0} \nabla p_0 - f p_0 + \lambda u_{\Omega_0} p_0 \right] dH^d.$$

We will now define $R(\epsilon)$ by

$$R(\epsilon) = \int_0^1 d_x L \left(\epsilon, u_{\Omega_0} + \Psi(u_{\Omega_\epsilon} - u_{\Omega_0}), p_0, \left(\frac{u_{\Omega_\epsilon} - u_{\Omega_0}}{s} \right) \right) d\Psi.$$

The final expression of $R(\epsilon)$ is given by:

$$\begin{aligned} R(\epsilon) &= \int_{\Omega_\epsilon} \left(\beta \left| \nabla \left(\frac{u_{\Omega_\epsilon} - u_{\Omega_0}}{\sqrt{s}} \right) \right|^2 + \alpha \left| \frac{u_{\Omega_\epsilon} - u_{\Omega_0}}{\sqrt{s}} \right|^2 \right) dx \\ &\quad - \frac{1}{s} \int_{\Omega_\epsilon} \lambda (-1)^{p-2} (p-1) (\nabla(u_{\Omega_\epsilon} - u_{\Omega_0}) | \nabla p_0 |^{p-2}) \nabla p_\epsilon dx \\ &\quad + \frac{1}{s} \int_{\Omega_\epsilon} \mu (-1)^{p-2} (p-1) \nabla(u_{\Omega_\epsilon} - u_{\Omega_0}) \nabla p_0 \left(\left| \nabla \left(\frac{u_{\Omega_\epsilon} + u_{\Omega_0}}{2} \right) \right|^{p-2} - |\nabla u_{\Omega_0}|^{p-2} \right) dx \\ &\quad - \frac{1}{s} \int_{\Omega_\epsilon} \lambda (u_{\Omega_\epsilon} - u_{\Omega_0}) p_\epsilon dx + \frac{1}{s} \int_{\partial \Omega_\epsilon^m \cap \partial \bar{\omega}_\epsilon} \nabla u_{\Omega_0} \cdot d\bar{\omega} p_0 dH^{N-1} dx \end{aligned}$$

Theorem 4.1. Let $0 \leq d < N$, $E = \bar{\omega}$, and $s = \alpha_{N-d} r^{N-d}$, where α_{N-d} is a geometric constant depending on the dimension. The topological derivative of a shape functional F at a point $z \in \Omega$ exists if and only if the following limit exists:

$$D_T F(z) = \lim_{\epsilon \rightarrow 0} \frac{F(\Omega_\epsilon) - F(\Omega)}{s(\epsilon)},$$

where: $\Omega_\epsilon = \Omega \setminus \bar{\omega}_\epsilon$ is the perturbed domain with a small inclusion w_ϵ centered at z , $s(\epsilon) = \alpha_{N-d} \epsilon^{N-d}$ is the scaling factor corresponding to the measure of the inclusion, and $D_T F(z)$ is the topological derivative of F at the point z .

This limit captures the first-order sensitivity of the functional F with respect to the creation of an infinitesimal hole or inclusion at z , and is foundational in topology optimization and inverse problems. That is to say

$$L = \lim_{\epsilon \rightarrow 0} (L_0(\epsilon) + L_1(\epsilon)),$$

exists with

$$L_0(\epsilon) = \int_{\Omega_\epsilon} \beta \left| \nabla \left(\frac{u_{\Omega_\epsilon} - u_{\Omega_0}}{\sqrt{s}} \right) \right|^2 + \alpha \left| \frac{u_{\Omega_\epsilon} - u_{\Omega_0}}{\sqrt{s}} \right|^2$$

and

$$\begin{aligned}
 L_1(\epsilon) = & -\frac{1}{s} \int_{\Omega_\epsilon} \lambda(-1)^{p-2}(p-1) (\nabla(u_{\Omega_\epsilon} - u_{\Omega_0})|\nabla p_0|^{p-2}) \nabla p_\epsilon dx \\
 & + \frac{1}{s} \int_{\Omega_\epsilon} \mu(-1)^{p-2}(p-1) \nabla(u_{\Omega_\epsilon} - u_{\Omega_0}) \nabla p_0 \left(\left| \nabla \left(\frac{u_{\Omega_\epsilon} + u_{\Omega_0}}{2} \right) \right|^{p-2} - |\nabla u_{\Omega_0}|^{p-2} \right) dx \\
 & - \frac{1}{s} \int_{\Omega_\epsilon} \lambda(u_{\Omega_\epsilon} - u_{\Omega_0}) p_\epsilon dx + \frac{1}{s} \int_{\partial\Omega_\epsilon^m \cap \partial\bar{\omega}_\epsilon} \nabla u_{\Omega_0} \cdot d\bar{w} p_0 dH^{N-1} dx
 \end{aligned}$$

Moreover, the topological derivative of the function is given by the expression :

$$DF(\Omega) = L - \left[\int_{\bar{\omega}} \beta |\nabla u_{\Omega_0}|^2 + \alpha |u_{\Omega_0} - u_0|^2 + \mu |\nabla u_{\Omega_0}|^{p-2} \nabla u_{\Omega_0} \nabla p_0 - f p_0 + \lambda u_{\Omega_0} p_0 \right] dH^d$$

where p_0, u_{Ω_0} are solutions of systems

$$\int_{\Omega} 2\beta \nabla u_{\Omega_0} \cdot \nabla \phi' dx + \int_{\Omega} 2\alpha (u_{\Omega_0} - u_0) \phi' dx + \int_{\Omega} \mu(-1)^{p-2}(p-1) [\nabla \phi' \cdot |\nabla u_{\Omega_0}|^{p-2} \nabla p_0] dx + \lambda \int_{\Omega} \phi' p_0 dx = 0$$

5. COMPARISON WITH EXISTING WORKS

Classical shape optimization problems involving the Laplace operator ($p = 2$) have been thoroughly investigated, especially under Dirichlet boundary conditions. In these linear contexts, topological derivatives are often obtained through direct variational methods or asymptotic expansions, yielding well-established results for small geometric perturbations and inclusions.

In contrast, for the nonlinear p -Laplacian ($p \neq 2$), much of the existing research has concentrated on foundational aspects such as existence of solutions, regularity theory, and the development of numerical algorithms. However, the study of shape functional sensitivity particularly in response to topological variations remains relatively undeveloped, and even more so when multiple boundary conditions are considered.

This work contributes to advancing the field in several meaningful directions:

- We present a rigorous derivation of the topological derivative for the nonlinear p -Laplacian, employing a min-max variational framework that effectively handles the inherent nonlinearity and offers a unified analytical approach.
- Departing from the common focus on Dirichlet conditions, our analysis incorporates both Neumann and Dirichlet boundary settings, uncovering distinct structural and regularity features across these cases.
- The proposed methodology accurately captures the impact of infinitesimal topological changes such as the insertion of small voids or inclusions on the shape functional, within a robust nonlinear setting.

These findings extend classical results and introduce new analytical tools for the study and design of nonlinear systems. They hold particular relevance for applications where internal topology significantly influences performance, including porous media transport, image processing, and the fabrication of advanced materials with engineered microstructures.

CONCLUSION AND PERSPECTIVES

In conclusion, this work establishes a rigorous framework for studying the topological derivative associated with the nonlinear p-Laplacian operator, considering both Neumann and Dirichlet boundary conditions. By employing a minmax variational approach, we were able to characterize the influence of infinitesimal perturbations—such as the introduction of holes or inclusions on shape-dependent functionals, thereby deepening our understanding of topological optimization in nonlinear settings. Our comparison with earlier studies highlights the originality of this contribution: it moves beyond classical analyses focused on linear operators and uncovers significant structural differences between Neumann and Dirichlet conditions. These findings provide a solid foundation for future research, with potential applications ranging from porous media modeling to the design of advanced materials with tailored microstructures.

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