



A REFINEMENT OF THE TANGENT ERDŐS-MORDELL INEQUALITY

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Abstract. Let P be an interior point of a triangle ABC . Denote by d_a, d_b, d_c the distances from P to the sides BC, CA, AB , respectively, and by t_a, t_b, t_c the distances from P to the tangents at A, B, C to the circumcircle of ABC . The tangent Erdős-Mordell inequality gives

$$2(d_a + d_b + d_c) \leq t_a + t_b + t_c \leq PA + PB + PC.$$

We give a symmetric refinement of its first inequality. Set $S = t_a + t_b + t_c$ and

$$\Delta = \frac{1}{2}(|t_b + t_c - t_a| + |t_c + t_a - t_b| + |t_a + t_b - t_c| - S).$$

Then

$$2(d_a + d_b + d_c) \leq \sqrt{S^2 - \Delta^2} \leq S \leq PA + PB + PC.$$

The new bound is strict when one of t_a, t_b, t_c is greater than the sum of the other two. Its proof follows from a weighted form of the tangent inequality.

1. Introduction

Let ABC be a triangle, with the standard notation

$$a = BC, b = CA, c = AB.$$

Let P be an interior point of ABC , and write

$$d_a = \text{dist}(P, BC), d_b = \text{dist}(P, CA), d_c = \text{dist}(P, AB).$$

The classical Erdős-Mordell inequality states that

$$PA + PB + PC \geq 2(d_a + d_b + d_c).$$

This problem was posed by Erdős, and a solution was later given by Mordell and Barrow; see [2, 3].

Let Γ be the circumcircle of ABC . Denote by ℓ_a, ℓ_b, ℓ_c the tangents to Γ at A, B, C , respectively, and put

$$t_a = \text{dist}(P, \ell_a), t_b = \text{dist}(P, \ell_b), t_c = \text{dist}(P, \ell_c).$$

Dao, Nguyen, and Pham [1] proved the strengthened chain

$$2(d_a + d_b + d_c) \leq t_a + t_b + t_c \leq PA + PB + PC. \quad (1.1)$$

We refer to the first inequality in (1.1) as the tangent Erdős-Mordell inequality.

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Our aim is to refine this first inequality. Set

$$S = t_a + t_b + t_c \quad (1.2)$$

and

$$\Delta = \frac{1}{2} (|t_b + t_c - t_a| + |t_c + t_a - t_b| + |t_a + t_b - t_c| - S). \quad (1.3)$$

The following equivalent form will be useful:

$$\Delta = \max\{0, t_a - t_b - t_c, t_b - t_c - t_a, t_c - t_a - t_b\}. \quad (1.4)$$

To see this, note first that at most one of the last three quantities in (1.4) can be positive. If t_a, t_b, t_c satisfy the triangle inequalities, all three expressions inside the absolute values in (1.3) are nonnegative and their sum is S , so $\Delta = 0$. If, for instance, $t_a > t_b + t_c$, then only $t_b + t_c - t_a$ is negative, and (1.3) gives $\Delta = t_a - t_b - t_c$. The other two cases are the same after a cyclic relabeling. Hence (1.3) and (1.4) are equivalent.

Since $t_a, t_b, t_c > 0$, formula (1.4) gives $0 \leq \Delta < S$. Thus $\sqrt{S^2 - \Delta^2}$ is well defined.

Theorem 1.1. With the notation above,

$$2(d_a + d_b + d_c) \leq \sqrt{S^2 - \Delta^2} \leq S \leq PA + PB + PC. \quad (1.5)$$

Moreover,

$$\sqrt{S^2 - \Delta^2} = \begin{cases} t_a + t_b + t_c, & \text{if } t_a, t_b, t_c \text{ satisfy the triangle inequalities,} \\ 2\sqrt{t_a(t_b + t_c)}, & \text{if } t_a > t_b + t_c, \end{cases} \quad (1.6)$$

together with the two analogous cyclic cases.

At the boundary $t_a = t_b + t_c$, the first line of (1.6) applies, and both displayed expressions have the same value $2t_a$.

The strict case in Theorem 1.1 does occur. For example, take

$$A = (0, 0), B = (4, 0), C = (1, 2), P = (3.93, 0.02).$$

Then P lies inside ABC , and numerical calculation gives

$$t_a \approx 3.90213, t_b \approx 0.07194, t_c \approx 3.17281,$$

so $t_a > t_b + t_c$. In this example,

$$2(d_a + d_b + d_c) \approx 7.09669 < \sqrt{S^2 - \Delta^2} \approx 7.11658 < S \approx 7.14688.$$

2. Proof

We begin with the relation between the tangent distances and the distances from P to the sides.

Lemma 2.1. One has

$$t_a = \frac{c}{a}d_b + \frac{b}{a}d_c, t_b = \frac{c}{b}d_a + \frac{a}{b}d_c, t_c = \frac{b}{c}d_a + \frac{a}{c}d_b. \quad (2.1)$$

Consequently,

$$S = \left(\frac{b}{c} + \frac{c}{b}\right)d_a + \left(\frac{c}{a} + \frac{a}{c}\right)d_b + \left(\frac{a}{b} + \frac{b}{a}\right)d_c. \quad (2.2)$$

Proof. Let K and R be the area and the circumradius of ABC . Introduce the areal coordinates

$$\lambda_a = \frac{[PBC]}{K}, \lambda_b = \frac{[PCA]}{K}, \lambda_c = \frac{[PAB]}{K}.$$

Then $\lambda_a + \lambda_b + \lambda_c = 1$, and

$$\lambda_a = \frac{ad_a}{2K}, \lambda_b = \frac{bd_b}{2K}, \lambda_c = \frac{cd_c}{2K}. \quad (2.3)$$

Consider the tangent ℓ_a at A . By the tangent chord theorem and the sine rule, the distances from B and C to ℓ_a are

$$c \sin C = \frac{c^2}{2R}, \quad b \sin B = \frac{b^2}{2R}.$$

The signed distance to an oriented line is an affine function. The points B, C, P lie in the same open half plane bounded by ℓ_a , while $A \in \ell_a$. Choose the signed distance to be positive on that half plane. On B, C, P it then agrees with the ordinary distance, while its value at A is zero. Since $P = \lambda_a A + \lambda_b B + \lambda_c C$, it follows that

$$t_a = \lambda_b \frac{c^2}{2R} + \lambda_c \frac{b^2}{2R}.$$

Using (2.3) and $4RK = abc$, we obtain $t_a = (c/a)d_b + (b/a)d_c$. The other two formulas follow cyclically, and their sum is (2.2). $\square$

From (2.2) and $x + x^{-1} \geq 2$ for $x > 0$, we immediately obtain

$$S \geq 2(d_a + d_b + d_c). \quad (2.4)$$

The next lemma contains the weighted argument that gives the improvement.

Lemma 2.2. If $t_a \geq t_b + t_c$, then

$$(d_a + d_b + d_c)^2 \leq t_a(t_b + t_c). \quad (2.5)$$

The two cyclic analogues also hold.

Proof. Assume $t_a \geq t_b + t_c$ and set

$$\lambda = \sqrt{\frac{t_b + t_c}{t_a}}.$$

Then $0 < \lambda \leq 1$. By Lemma 2.1,

$$\lambda t_a + \frac{t_b + t_c}{\lambda} = \frac{1}{\lambda} \left(\frac{c}{b} + \frac{b}{c} \right) d_a + \left(\lambda \frac{c}{a} + \frac{1}{\lambda} \frac{a}{c} \right) d_b + \left(\lambda \frac{b}{a} + \frac{1}{\lambda} \frac{a}{b} \right) d_c. \quad (2.6)$$

The coefficient of d_a satisfies

$$\frac{1}{\lambda} \left(\frac{c}{b} + \frac{b}{c} \right) \geq \frac{2}{\lambda} \geq 2.$$

For the other two coefficients, AM–GM gives

$$\lambda \frac{c}{a} + \frac{1}{\lambda} \frac{a}{c} \geq 2, \quad \lambda \frac{b}{a} + \frac{1}{\lambda} \frac{a}{b} \geq 2.$$

Since $d_a, d_b, d_c > 0$, (2.6) yields

$$2(d_a + d_b + d_c) \leq \lambda t_a + \frac{t_b + t_c}{\lambda}. \quad (2.7)$$

For the chosen value of λ , the right hand side of (2.7) is $2\sqrt{t_a(t_b + t_c)}$. Dividing by 2 and squaring proves (2.5). $\square$

We can now prove Theorem 1.1. By (1.4), there are two possibilities.

If t_a, t_b, t_c satisfy the triangle inequalities, then $\Delta = 0$, and hence $\sqrt{S^2 - \Delta^2} = S$. The first inequality in (1.5) is then exactly (2.4).

Otherwise, after a cyclic relabeling, suppose that $t_a > t_b + t_c$. In this case $\Delta = t_a - t_b - t_c$, so

$$S^2 - \Delta^2 = (t_a + t_b + t_c)^2 - (t_a - t_b - t_c)^2 = 4t_a(t_b + t_c).$$

Lemma 2.2 therefore gives

$$2(d_a + d_b + d_c) \leq 2\sqrt{t_a(t_b + t_c)} = \sqrt{S^2 - \Delta^2}.$$

This proves the first inequality in (1.5). The second follows from $0 \leq \Delta < S$.

Finally, $A \in \ell_a$, so the distance from P to ℓ_a cannot exceed the distance from P to the particular point A on that line. Thus $t_a \leq PA$. Similarly, $t_b \leq PB$ and $t_c \leq PC$. Hence

$$S \leq PA + PB + PC,$$

which completes the proof.

Remark 2.1. The equality cases can be read directly from the proof. In the strict dominant case, say $t_a > t_b + t_c$, one has $\lambda < 1$. The coefficient of d_a in (2.6) is then strictly greater than 2, and $d_a > 0$. Hence

$$2(d_a + d_b + d_c) < \sqrt{S^2 - \Delta^2}.$$

Thus equality in the first inequality of (1.5) is possible only when $\Delta = 0$. In that case equality means $S = 2(d_a + d_b + d_c)$. From (2.2), and because $d_a, d_b, d_c > 0$, this happens if and only if $a = b = c$. Conversely, for an equilateral triangle, equality in the first inequality holds for every interior point P .

Equality in $\sqrt{S^2 - \Delta^2} \leq S$ holds if and only if $\Delta = 0$, equivalently, if and only if t_a, t_b, t_c satisfy the triangle inequalities.

Equality in $S \leq PA + PB + PC$ holds if and only if P is the circumcenter O . Indeed, equality in $t_a \leq PA$ means that A is the foot of the perpendicular from P to ℓ_a . Since $OA \perp \ell_a$, this is equivalent to P, A, O being collinear. Simultaneous equality for A, B, C forces $P = O$, and the converse is immediate. Because P is required to be interior, this can occur precisely when ABC is acute.

Consequently, equality throughout (1.5) holds if and only if ABC is equilateral and P is its center.

3. Conclusion

The refinement comes from a weighted version of the tangent inequality. If $t_a \geq t_b + t_c$, then the proof of Lemma 2.2 actually gives, for every $0 < \lambda \leq 1$,

$$2(d_a + d_b + d_c) \leq \lambda t_a + \frac{t_b + t_c}{\lambda}.$$

The right hand side is minimized at $\lambda = \sqrt{(t_b + t_c)/t_a}$, which belongs to $(0, 1]$ exactly under the assumption $t_a \geq t_b + t_c$. The minimum is $2\sqrt{t_a(t_b + t_c)}$. Thus the original unweighted choice $\lambda = 1$ can be improved precisely when one tangent distance is greater

than the sum of the other two. Formula (1.3) collects the three possible dominant cases into one symmetric expression.

References

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