



ON THE HARMONIC QUADRILATERAL

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ABSTRACT. In this article we show that a harmonic quadrilateral appears always as a member of a triplet comprising two more harmonic quadrilaterals and creating a structure based on an acute triangle and a related to it system of three mutually orthogonal circles.

1. DEFINITION, FIRST PROPERTIES

“*Harmonic*” is said a *cyclic* quadrilateral $q = ABCD$, inscribed in a circle $\kappa(O)$, whose pairs of opposite sides (AB, CD) and (BC, AD) have equal products of lengths:

$$|AB| \cdot |CD| = |BC| \cdot |DA|. \quad (1.1)$$

Writing this in the form $|BA|/|BC| = |DA|/|DC| = k_1$, we see that the opposite vertices $\{B, D\}$ are on the Apollonian circle $\kappa_1(O_1)$ of the diagonal segment AC , whose points X are characterized by the equality of the ratios $|XA|/|XC| = k_1$ (see Figure 1). Analogously writing equation (1.1) in the form $|AB|/|AD| = |CB|/|CD| = k_2$ we see that the opposite vertices $\{A, C\}$ are on the Apollonian circle $\kappa_2(O_2)$ of the diagonal segment BD , whose points X are characterized by the equality $|XB|/|XD| = k_2$.

By the properties of Apollonian circles, it follows that $\{A, C\}$ are inverse w.r.t. $\kappa_1(O_1)$ and $\{\kappa, \kappa_2\}$ passing through these two inverse points are orthogonal to κ_1 . Analogously $\{\kappa, \kappa_1\}$ are orthogonal to κ_2 . In total we have three circles, which are mutually orthogonal to each other. This implies that the *radical center* E of the three circles, coinciding with the intersection point of the diagonals, it is also the orthocenter of the triangle $o = OO_1O_2$ of the centers of the three circles, the triangle o is acute, the three intersection points $\{B, C, F\}$ of the circles are inside the triangle and the three second intersections $\{A, D, G\}$ are outside the triangle.

We notice also that the quadrilateral is *convex*, since O_1 is outside the diagonal segment AC and consequently its harmonic conjugate $E = O_1(AC)$, coinciding with the intersection point of the diagonals, is inside AC , and analogously it is also an inner point of the diagonal BD .

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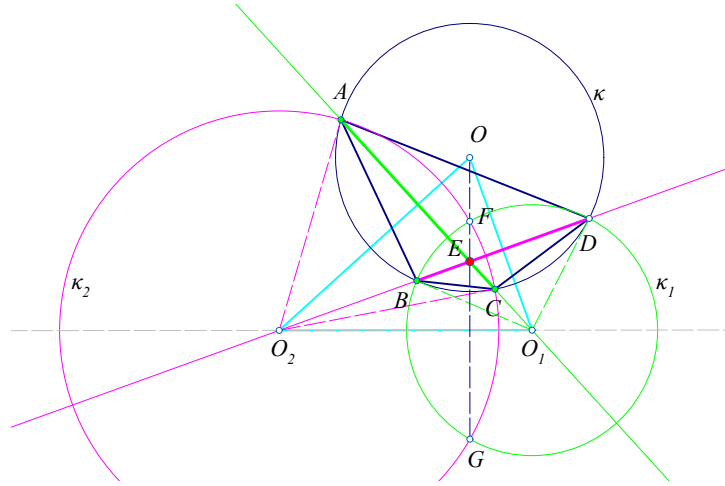


Figure 1. Vertices of the harmonic quadrilateral on two Apollonian circles

2. THEY COME IN TRIPLETS

The preceding discussion shows that a harmonic quadrilateral $q = ABCD$ determines a unique triple of mutually orthogonal circles $\{\kappa, \kappa_1, \kappa_2\}$ and an acute triangle $o = OO_1O_2$ of their centers. In a sense of *converse*, figure 2-(I) shows, how we can start from an acute

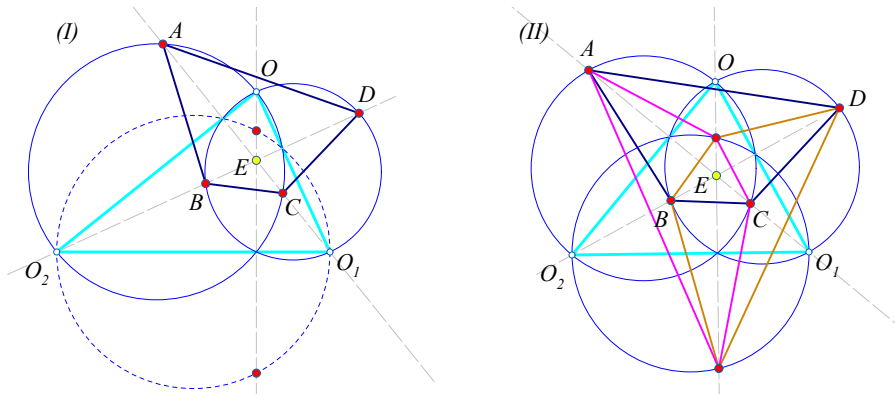


Figure 2. Determining a harmonic quadrilateral from a triangle

triangle $o = OO_1O_2$ and create a harmonic quadrilateral: (i) select two of its sides (in the figure $\{OO_1, OO_2\}$), and (ii) draw circles with these sides as diameters. The two circles intersect each of the corresponding to these sides altitudes in two points (B, D) and (A, C) , the four points, seen easily, to form a harmonic quadrilateral.

We can however select two of the sides of the triangle in three different ways. Figure 2-(II) shows the result. There are three harmonic quadrilaterals intimately related, associated to three mutually orthogonal circles (not drawn in the figure) and to the acute triangle of the centers of these circles. They result in the way of the construction of figure 2-(I), by permuting the roles of $\{O, O_1, O_2\}$.

This relation of the harmonic quadrilateral to a configuration based on an acute triangle, can be considered as an aspect of the subject from the viewpoint of triangle geometry. We'll supply some quantitative relations in this context in the next section.

We call "*harmonic triplet*" (in short "*h-triplet*") the configuration of *three harmonic quadrilaterals* uniquely determined by three mutually orthogonal circles, or/and equivalently, by an acute triangle, through its orthocenter and the three circles with diameters the sides of the triangle, as suggested by figure 2-(II). We call the vertices of the quadrilaterals of a triplet, the "*vertices of the triplet*" and the circumcircles and their centers the "*circles of the triplet*" resp. "*centers of the triplet*".

3. SOME FORMULAS IN BARYCENTRICS

We'll be using barycentric coordinates ([3]) to express quantitative relations between points, lines and circles appearing in the configuration of the preceding section. In order to be able to conveniently apply well known formulas we change to the standard notation and labeling of figures used in barycentric coordinates and suggested by figure 3. In this, the triangle of reference ABC has vertices the centers of the h-triplet, the corresponding circles being labeled with $\{\kappa_A(r_A), \kappa_B(r_B), \kappa_C(r_C)\}$. We start with the following

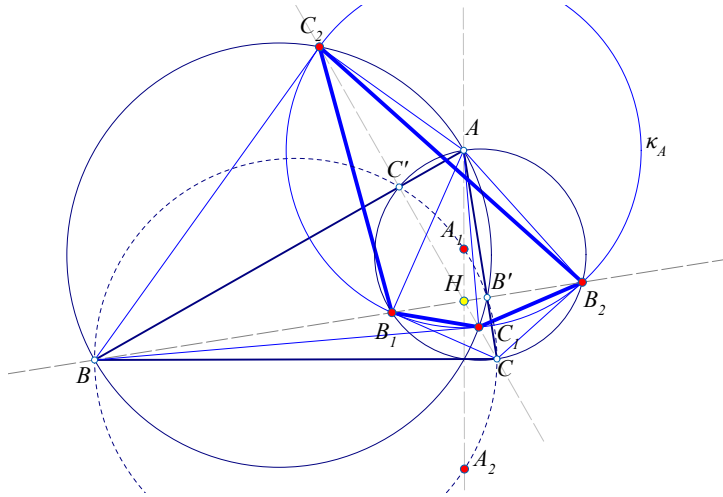


Figure 3. Construction of the harmonic $B_1C_1B_2C_2$ based on the triangle ABC

well known theorem ([3]), in which appear the *Conway triangle symbols*[4]:

$$S_A = \frac{1}{2}(b^2 + c^2 - a^2), \quad S_B = \frac{1}{2}(c^2 + a^2 - b^2), \quad S_C = \frac{1}{2}(a^2 + b^2 - c^2),$$

and the double S of the area of the triangle of reference ABC .

Theorem 3.1. *The points $\{U(u, v, w)\}$ of the circle on diameter U_1U_2 satisfy the equation in absolute barycentrics:*

$$S_A(u - u_1)(u - u_2) + S_B(v - v_1)(v - v_2) + S_C(w - w_1)(w - w_2) = 0. \quad (3.1)$$

Remark 3.1. *By the way, referring to figure 3, we notice that the radius r_A of the circum-circle κ_A of the harmonic quadrilateral $B_1C_1B_2C_2$ with center at A satisfies $r_A^2 = S_A$, and analogously $\{r_B^2 = S_B, r_C^2 = S_C\}$. This follows from the orthogonality of κ_A to the circle with*

diameter BC , later passing through the couple of points $\{(A_1, A_2), (B', C)\}$ which are inverse w.r.t. κ_A , point B' being the foot of the altitude BH on AC . This implies the equality

$$r_A^2 = AC \cdot AB' = AC \cdot AB \cdot \cos(\hat{A}) = S_A.$$

In order to find the barycentric coordinates $U(u, v, w)$ of the vertices of the harmonic quadrilaterals we apply formula (3.1) and compute the expressions of the circles with diameters the sides of the triangle ABC , the coordinates of the vertices of the triangle of reference ABC being $\{A(1, 0, 0), B(0, 1, 0), C(0, 0, 1)\}$:

$$\text{circle on diameter } CA : f_{CA}(U) = S_A u^2 + S_B v^2 + S_C w^2 - S_A u - S_C w = 0,$$

$$\text{circle on diameter } AB : f_{AB}(U) = S_A u^2 + S_B v^2 + S_C w^2 - S_B v - S_A u = 0,$$

$$\text{circle on diameter } BC : f_{BC}(U) = S_A u^2 + S_B v^2 + S_C w^2 - S_C w - S_B v = 0.$$

The vertices of the three harmonic quadrilaterals, associated to the triangle of reference ABC , are intersections of these circles with corresponding altitudes of the triangle. Knowing the absolute barycentrics $H = \frac{1}{S^2}(S_B S_C, S_C S_A, S_A S_B)$ of the orthocenter, we start with points $\{B_1, B_2\}$ as intersections of the circle on diameter CA with line $BH : (1-t)B + tH$. Points $\{B_1, B_2\}$ correspond to the roots of the quadratic equation:

$$\begin{aligned} f_{CA}((1-t)B + tH) &= 0 \quad \Leftrightarrow \quad S_B(S_A + S_C)t^2 - 2S^2t + S^2 = 0 \quad \Leftrightarrow \\ t_{1,2} &= S \frac{S \mp \sqrt{S_C S_A}}{S_B(S_C + S_A)} \Rightarrow \\ B_1 &= \begin{pmatrix} S_C(\sqrt{S_A S_C} - S) \\ -(S_C + S_A)\sqrt{S_A S_C} \\ S_A(\sqrt{S_A S_C} - S) \end{pmatrix}, \quad B_2 = \begin{pmatrix} S_C(\sqrt{S_A S_C} + S) \\ -(S_C + S_A)\sqrt{S_A S_C} \\ S_A(\sqrt{S_A S_C} + S) \end{pmatrix}. \end{aligned} \quad (3.2)$$

Working analogously, we obtain the coordinates of points $\{C_1, C_2\}$ as intersections of the circle on diameter AB with line CH :

$$C_1 = \begin{pmatrix} S_B(\sqrt{S_A S_B} - S) \\ S_A(\sqrt{S_A S_B} - S) \\ -(S_B + S_A)\sqrt{S_A S_B} \end{pmatrix}, \quad C_2 = \begin{pmatrix} S_B(\sqrt{S_A S_B} + S) \\ S_A(\sqrt{S_A S_B} + S) \\ -(S_B + S_A)\sqrt{S_A S_B} \end{pmatrix}, \quad (3.3)$$

and points $\{A_1, A_2\}$ as intersections of the circle on diameter BC with line AH :

$$A_1 = \begin{pmatrix} -(S_C + S_B)\sqrt{S_C S_B} \\ S_C(\sqrt{S_C S_B} - S) \\ S_B(\sqrt{S_C S_B} - S) \end{pmatrix}, \quad A_2 = \begin{pmatrix} -(S_C + S_B)\sqrt{S_C S_B} \\ S_C(\sqrt{S_C S_B} + S) \\ S_B(\sqrt{S_C S_B} + S) \end{pmatrix}. \quad (3.4)$$

The identity $S^2 = S_A S_B + S_B S_C + S_C S_A$ was repeatedly used in doing these calculations, and according to remark 3.1 this translates to the identity

$$r_A^2 r_B^2 + r_B^2 r_C^2 + r_C^2 r_A^2 = S^2.$$

Next theorem summarizes our discussion we have done so far (see Figure 3).

Theorem 3.2. *A harmonic quadrilateral defines in a unique way two other harmonic quadrilaterals, all three inscribed in corresponding mutually orthogonal circles. Thus, harmonic quadrilaterals appear in triplets, related to an acute triangle ABC , the vertices of this triangle being the centers of their, mutually orthogonal circumcircles. The pairs of opposite vertices of each quadrilateral of the triplet lie on the two altitudes of ABC not passing through the vertex-center*

of its circumcircle and are intersections of these altitudes with the circles on diameters the sides orthogonal to them. The pairs of vertices (A_1, A_2) , (B_1, B_2) and (C_1, C_2) , lying respectively on the altitudes $\{AH, BH, CH\}$, H being the orthocenter, are expressed in barycentrics w.r.t. the triangle ABC respectively through formulas (3.4), (3.2), (3.3).

Remark 3.2. Using remark 3.1 we can express formulas (3.4), (3.2), (3.3) in terms of the radii $\{r_A, r_B, r_C\}$ of the circumcircles of the triplet. Applying this remark to the identities involving the Conway triangle symbols ([4]), we also obtain a wealth of identities for these radii, as for example the one involving the “Brocard angle” ω ([1, p.274], [6, p.84]) of the triangle of reference ABC :

$$r_A^2 + r_B^2 + r_C^2 = \frac{1}{2}(a^2 + b^2 + c^2) = S \cot(\omega),$$

with $\{a = |BC|, b = |CA|, c = |AB|; a^2 = r_B^2 + r_C^2, b^2 = r_C^2 + r_A^2, c^2 = r_A^2 + r_B^2\}$.

The preceding formulas, combined with the remarks, enable the expression of the various elements of the h-triplet in terms of the triangle of reference ABC and the radii of the circumcircles $\{r_A, r_B, r_C\}$. In fact, setting

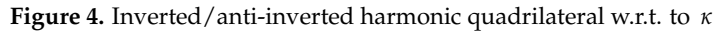
$$\begin{aligned} S_{AB}^+ &= \frac{S + r_A r_B}{c^2}, \quad S_{AB}^- = \frac{S - r_A r_B}{c^2}, \quad S_{BC}^+ = \frac{S + r_B r_C}{a^2}, \\ S_{BC}^- &= \frac{S - r_B r_C}{a^2}, \quad S_{CA}^+ = \frac{S + r_C r_A}{b^2}, \quad S_{CA}^- = \frac{S - r_C r_A}{b^2}, \end{aligned}$$

we have

$$\begin{aligned} |B_1 C_1|^2 &= 2r_A^2 S_{CA}^- S_{AB}^-, \quad |C_1 A_1|^2 = 2r_B^2 S_{AB}^- S_{BC}^-, \quad |A_1 B_1|^2 = 2r_C^2 S_{BC}^- S_{CA}^-, \\ |A_2 B_2|^2 &= 2r_C^2 S_{BC}^+ S_{CA}^+, \quad |B_2 C_2|^2 = 2r_A^2 S_{CA}^+ S_{AB}^+, \quad |C_2 A_2|^2 = 2r_B^2 S_{AB}^+ S_{BC}^+, \\ |C_1 B_2|^2 &= 2r_A^2 S_{AB}^- S_{CA}^+, \quad |B_1 C_2|^2 = 2r_A^2 S_{CA}^- S_{AB}^+, \quad |A_1 C_2|^2 = 2r_B^2 S_{BC}^- S_{AB}^+, \\ |C_1 A_2|^2 &= 2r_B^2 S_{AB}^- S_{BC}^+, \quad |A_1 B_2|^2 = 2r_C^2 S_{BC}^- S_{CA}^+, \quad |B_1 A_2|^2 = 2r_C^2 S_{CA}^- S_{BC}^+, \\ |A_1 A_2| &= \frac{2r_B r_C}{a}, \quad |B_1 B_2| = \frac{2r_C r_A}{b}, \quad |C_1 C_2| = \frac{2r_A r_B}{c}, \\ \text{area}(B_1 C_1 B_2 C_2) &= \frac{2S r_A^2 r_B r_C}{b^2 c^2}, \quad \sin(\widehat{C_1}) = \frac{r_C}{b}, \quad \sin(\widehat{B_1}) = \frac{r_B}{c}. \end{aligned}$$

4. SIX SQUARES

A characteristic of harmonic quadrilaterals is that “their vertices form a harmonic quadruple” on their circumcircle ([2, p.41]). Since inversions and anti-inversions ([5, I, p.391]) preserve the cross ratio, it follows that they also preserve the harmonic properties of quadrilaterals, as long as the inversion is “generic”, i.e. the center of inversion is not contained in the circumscribing circle of the quadrilateral. A case in which this circumcircle inverts in a line and the quadrilateral’s vertices map to a harmonic quadruple on this line. Figure 4 shows the inverted images $\{A'B'C'D', A''B''C''D''\}$ of a harmonic quadrilateral w.r.t. to an inversion resp. anti-inversion w.r.t the same circle κ . We remind that the anti-inversion is a composition of an ordinary inversion and the symmetry in the center of the inverting circle. Inversions leave the points of their inversion circle fixed, and preserve any other circle which is orthogonal to the circle of inversion. Anti-inversions map each point of their circle of inversion to its diametral and preserve any circle which passes through two diametral points of the circle of inversion. It is also known that any



In fact, to invert the quadrilateral $q = B_1C_1B_2C_2$ to a square, consider the circle $\lambda(A_1)$ with radius $r_\lambda = \sqrt{|A_1B_1||A_1B'_1|}$, the product being the power $p_A = r_A^2 - |AA_1|^2$ of A_1 w.r.t. κ_A . Let f_λ be also the anti-inversion w.r.t. to λ . Obviously for a point $X \in \lambda$ the anti-inverse $X' = f_\lambda(X)$ is the second intersection point of the line XA_1 with κ_A (the directed segments must satisfy $A_1X \cdot A_1X' = -p_A$). Applying this to the vertices of q , we have to show that the points $\{B'_1, C'_1, B'_2, C'_2\}$ are on two orthogonal diameters of κ_A , as seen in figure 5. The orthogonality of these two lines $\{f_\lambda(\kappa_B), f_\lambda(\kappa_C)\}$ results from that of the circles $\{\kappa_B, \kappa_C\}$ and the fact that f_λ is *conformal*, i.e. it preserves angle measures. To show that they intersect precisely at the center A of κ_A it suffices to prove that $f_\lambda(A_2) = A$. This follows from a short calculation:

Considering the inversion $X'' = f_\mu(X)$ w.r.t. to the circle μ orthogonal to κ_A and with center A_2 , we show analogously that the inverses $\{B_1'', C_1'', B_2'', C_2''\}$ are vertices of

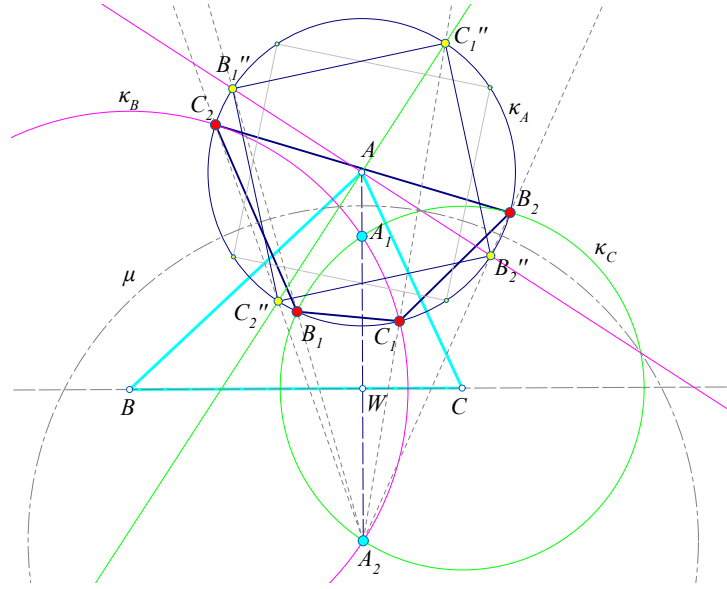


Figure 6. A second inversion of $B_1C_1B_2C_2$ to a square

a square inscribed in the circle κ_A (see Figure 6). Doing the analogous constructions for all three members of the harmonic triplet we deduce the following theorem.

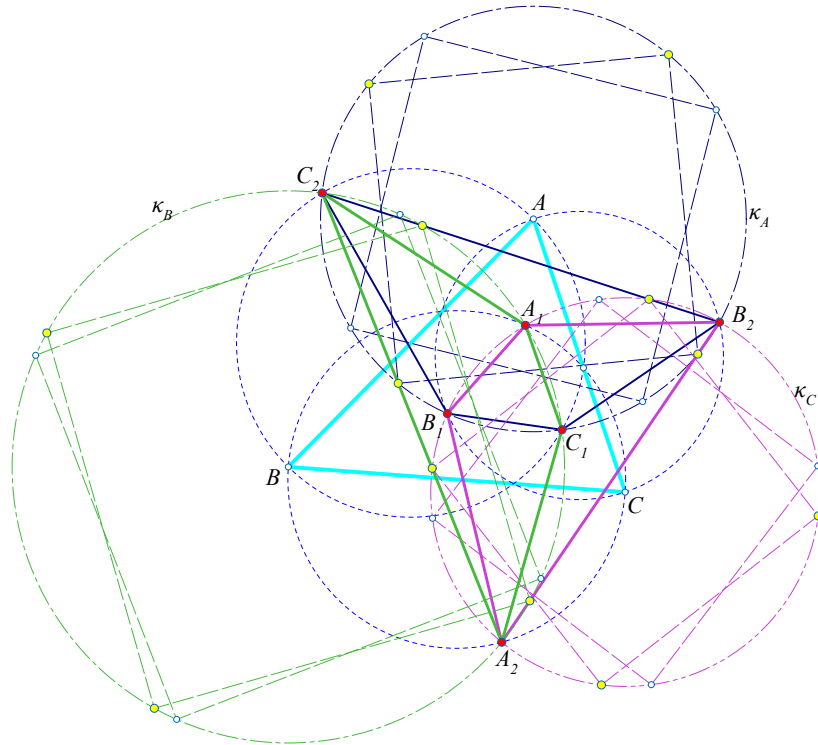


Figure 7. The six squares resulting from the inversions of theorem 4.1

Theorem 4.1. *To the quadrilateral $q = B_1C_1B_2C_2$ of the h -triplet, inscribed in the circle κ_A corresponds an anti-inversion with center A_1 and an inversion with center A_2 which map respectively the vertices of q to the vertices of two squares $\{q' = A'_1B'_1A'_2B'_2, q'' = A''_1B''_1A''_2B''_2\}$ inscribed in the circumcircle κ_A of q . Analogous properties hold also for the other two quadrilaterals of the h -triplet.*

Figure 7 shows the six squares, resulting from the six inversions referred to in the theorem, applied to the three quadrilaterals of a h -triplet.

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