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# CONSTANT SECTIONAL CURVATURE SURFACES WITH A SEMI-SYMMETRIC NON-METRIC CONNECTION ON A WALKER 3-MANIFOLD

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ABSTRACT. This paper investigates the geometry of surfaces in a three-dimensional Walker manifold endowed with a canonical semi-symmetric non-metric connection. We establish fundamental properties of this connection and derive explicit formulas for the corresponding sectional and scalar curvatures. Special attention is given to cylindrical surfaces, for which we provide complete classifications when the sectional curvature is constant. The main results include detailed parametrizations of generating curves for non-planar cylindrical surfaces with constant sectional curvature, with distinct cases for zero, positive, and negative curvature values. These findings extend our understanding of geometric structures under non-metric connections and reveal new insights into the interplay between curvature properties and connection characteristics in pseudo-Riemannian geometry.

## 1. INTRODUCTION AND MOTIVATIONS

In 1924, Friedmann and Schouten introduced semi-symmetric connections within the framework of Riemannian geometry, thereby proposing a significant generalization of the Levi-Civita connection. An affine connection $\tilde{\nabla}$ on a Riemannian manifold $(\tilde{M}, \tilde{g})$ is characterized as semi-symmetric when its torsion tensor $\tilde{T}$ satisfies the following condition:

$$\tilde{T}(X,Y) = \tilde{g}(U,Y)X - \tilde{g}(U,X)Y = u(Y)X - u(X)Y, \quad X,Y \in \mathfrak{X}(\tilde{M}),$$

where the 1-form $u$ is associated with the vector field $U$ on $M$ through the relation:

$$u(X) = g(X,U).$$

In 1932, Hayden developed the notion of semi-symmetric metric connection. A semi-symmetric connection $\widetilde{\nabla}$ is called metric if it preserves the metric, that is:

$$\widetilde{\nabla} g = 0.$$

A metric connection with vanishing torsion corresponds to the Levi-Civita connection; otherwise, it is referred to as a Hayden connection. Consequently, metric connections

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encompass both of these categories. In 1970, Yano investigated the properties of semi-symmetric metric connections, notably demonstrating that on a conformally flat Riemannian manifold admitting such a connection, the curvature tensor vanishes. Subsequent research by Imai [8], Mishra and Pandey [12], De and De [5], along with other investigators [10, 19], has further advanced the analysis of these geometric structures.
The concept of semi-symmetric non-metric connection was formulated by Agashe and Chafle [1] in 1992 for Riemannian manifolds $(M, g)$. A semi-symmetric connection $\widetilde{\nabla}$ is designated as non-metric when:

$$\widetilde{\nabla} g \neq 0. \tag{1.1}$$

In this framework, there exists a relationship between $\tilde{\nabla}$ and the Levi-Civita connection $\tilde{\nabla}^0$ of $(\tilde{M}, \tilde{g})$, expressed as:

$$\tilde{\nabla}_X Y = \tilde{\nabla}^0_X Y + \tilde{g}(U, Y)X, \quad X, Y \in \mathfrak{X}(\tilde{M}). \tag{1.2}$$

In 1994, Agashe and Chafle [2] examined the characteristics of submanifolds within a Riemannian manifold equipped with semi-symmetric non-metric connections. Building upon this foundation, Sengupta, De, and Binh [15] generalized in 2000 the framework introduced by Agashe and Chafle. They established the relationships connecting the curvature tensor $\widetilde{R}$ of $M$ with respect to the semi-symmetric non-metric connection $\widetilde{\nabla}$ and the curvature tensor field $R$ associated with the Riemannian connection. Additionally, they derived the first and second Bianchi identities for the semi-symmetric non-metric connection and investigated several properties of the Weyl projective curvature tensor under this connection.
In a recent investigation [3], the authors considered Euclidean space $\mathbb{R}^3$ endowed with a canonical semi-symmetric non-metric connection determined by a vector field $U \in \mathfrak{X}(\mathbb{R}^3)$. They analyzed surfaces for which the sectional curvature with respect to this connection remains constant. For cylindrical surfaces, they obtained a complete classification when the rulings are either orthogonal or parallel to $U$. Regarding rotational surfaces, they established that the rotation axis must be parallel to $U$ and classified all conical rotational surfaces possessing constant sectional curvature. Furthermore, for the particular case of sectional curvature equal to $\frac{1}{2}$, they demonstrated the existence of rotational surfaces that orthogonally intersect the rotation axis.
Drawing inspiration from these developments, we direct our attention to the study of Walker manifolds. These manifolds, distinguished by the presence of a parallel null distribution, provide an exceptionally rich setting for pseudo-Riemannian geometry. In three dimensions, these manifolds admit a canonical form whose geometric properties are encoded within a single function. The present work focuses on three-dimensional strict Walker manifolds and investigates the geometry of their surfaces under a canonical snm-connection determined by a distinguished vector field.
The objectives of this article are threefold:

(1) To establish the fundamental properties of the canonical snm-connection on Walker manifolds and to derive explicit formulas for the corresponding sectional and scalar curvatures.

(2) To analyze the geometry of cylindrical surfaces within these manifolds, with particular emphasis on those whose rulings are orthogonal or parallel to the vector field defining the connection.
(3) To classify cylindrical surfaces possessing constant sectional curvature by providing explicit parametrizations of their generating curves.

These results enhance our understanding of geometric structures under non-metric connections and reveal novel insights into the interplay between curvature properties and connection characteristics in pseudo-Riemannian geometry.

## 2. Preliminaries

2.1. **Three Dimensional Walker Spaces.** Walker manifolds in general are pseudo-Riemannian structures $(M, g, D)$ featuring a distribution $D$ on which the metric $g$ vanishes (forming a lightlike distribution) and which remains parallel under the Levi-Civita connection of $g$. A distinctive characteristic of 3-dimensional Walker manifolds is that all their geometric information can be captured by a single function. More generally, for a $(2r+1)$-dimensional pseudo-Riemannian manifold $M$ possessing a parallel field of null $r$-dimensional planes $D$, a canonical representation of the metric tensor in matrix notation is:

$$(g_{ij}) = \begin{pmatrix} 0 & 0 & I_r \\ 0 & \epsilon & 0 \\ I_r & 0 & B \end{pmatrix}$$

where $I_r$ denotes the $r \times r$ identity matrix, $B$ represents a symmetric $r \times r$ matrix with entries depending on the coordinates $x_1, \ldots, x_{2r+1}$, and $\epsilon = \pm 1$ [16]. Consequently, any Walker manifold in three dimensions is locally isometric to the manifold $M_f^3$ whose metric possesses the matrix representation:

$$g_f^\epsilon = (g_{ij}) = \begin{pmatrix} 0 & 0 & 1 \\ 0 & \epsilon & 0 \\ 1 & 0 & f \end{pmatrix}$$

with respect to the natural coordinate frame $\{\partial_t, \partial_x, \partial_y\}$, where $f = f(t, x, y)$ denotes a smooth function defined on an open subset $O \subseteq \mathbb{R}^3$. Thus, we have:

$$g_f^\epsilon = 2\, dt\, dy + \epsilon\, dx^2 + f(t, x, y)\, dy^2. \tag{2.1}$$

The manifold $M_f^3$ carries signature $(1, 2)$ when $\epsilon = 1$ and signature $(2, 1)$ when $\epsilon = -1$. When $f(t, x, y) = f(x, y)$, we refer to $M_f^3$ as a strict Walker manifold. The Levi-Civita connection $\nabla$ is well-established, with the non-vanishing Christoffel symbols given by:

$$\nabla_{\partial_t}\partial_y = \frac{1}{2} f_t\, \partial_t, \quad \nabla_{\partial_x}\partial_y = \frac{1}{2} f_x\, \partial_t, \quad \nabla_{\partial_y}\partial_y = \frac{1}{2}(f f_t + f_y)\, \partial_t - \frac{\epsilon}{2} f_x\, \partial_x - \frac{1}{2} f_t\, \partial_y. \tag{2.2}$$

The cross product operation $\times$, satisfying the property $g_f^\epsilon(U \times V, W) = \det(U, V, W)$ for vectors $U, V, W \in \mathbb{R}^3$, takes the form:

$$U \times V = (u_1v_2 - u_2v_1 - (u_2v_3 - u_3v_2)f, -\epsilon(u_1v_3 - u_3v_1), u_2v_3 - u_3v_2). \tag{2.3}$$

2.2. **Geometry of Surfaces in a Walker 3-Manifold.** In this subsection, we examine the differential geometry of surfaces embedded in a Walker manifold. Consider an open subset $U$ of the plane $\mathbb{R}^2$ such that any horizontal or vertical line intersects $U$ in intervals (when such intersection occurs). A two-parameter mapping is defined as a smooth map $\varphi : U \to M$. Consequently, $\varphi$ comprises two interrelated families of parameter curves:

(1) The $u$-parameter curves at constant $v = v_0$ are given by the mapping $u \mapsto \varphi(u, v_0)$.
(2) The $v$-parameter curves at constant $u = u_0$ are given by the mapping $v \mapsto \varphi(u_0, v)$.

Assuming that $\varphi$ represents an isometric immersion, the first fundamental form of this immersion is characterized by:

$$E = g_f^\epsilon(\varphi_u, \varphi_u), \quad F = g_f^\epsilon(\varphi_u, \varphi_v), \quad G = g_f^\epsilon(\varphi_v, \varphi_v). \tag{2.4}$$

The second fundamental form of $\varphi$ has coefficients:

$$\begin{cases} L &= \epsilon_1 g_f^\epsilon(\varphi_{uu}, \eta), \\ M &= \epsilon_1 g_f^\epsilon(\varphi_{uv}, \eta), \\ N &= \epsilon_1 g_f^\epsilon(\varphi_{vv}, \eta) \end{cases} \tag{2.5}$$

where $\epsilon_1 = g_f^\epsilon(\eta, \eta)$ represents the sign of the unit normal vector $\eta$ along $\varphi$.
The mean curvature $H$ of the surface $\varphi$ is expressed as:

$$H = \frac{\epsilon_1}{2} \frac{LG - 2MF + NE}{EG - F^2}. \tag{2.6}$$

For a surface $S$ embedded in $(M, g_f^\epsilon)$, the Gauss equation establishes a relationship between the sectional curvature $K(\partial_{e_1}, \partial_{e_2}) = K$ of $S$ and the sectional curvature $K^M(\partial_{e_1}, \partial_{e_2}) = K^M$ of $(M, g_f^\epsilon)$ as follows:

$$K = K^M + \epsilon_1 \frac{LN - M^2}{EG - F^2}. \tag{2.7}$$

2.3. **A canonical semi-symmetric non-metric connection.** Consider a Riemannian manifold $(\tilde{M}, \tilde{g})$ of dimension $\geq 2$ and an affine connection $\tilde{\nabla}$ on $\tilde{M}$. The torsion and curvature tensors of $\tilde{\nabla}$ are respectively defined as a $(1,2)$-tensor field $\tilde{T}$ and a $(1,3)$-tensor field $\tilde{R}$ through:

$$\tilde{T}(X, Y) = \tilde{\nabla}_X Y - \tilde{\nabla}_Y X - [X, Y],$$

$$\tilde{R}(X, Y)Z = \tilde{\nabla}_X \tilde{\nabla}_Y Z - \tilde{\nabla}_Y \tilde{\nabla}_X Z - \tilde{\nabla}_{[X,Y]} Z, \tag{2.8}$$

for all vector fields $X, Y, Z \in \mathfrak{X}(\tilde{M})$. Employing (2.8), the non-vanishing components of the curvature tensor for $(M, g_f^\epsilon)$ are expressed as:

$$\begin{aligned} R(\partial_x,\partial_z)\partial_x &= \frac{1}{2}f_{xx}\partial_x,\ R(\partial_x,\partial_z)\partial_y = \frac{1}{2}f_{xy}\partial_x,\ R(\partial_y,\partial_z)\partial_y = -\frac{1}{2}f_{yy}\partial_x, \\ R(\partial_x,\partial_z)\partial_z &= \frac{1}{2}ff_{xx}\partial_x - \frac{\epsilon}{2}ff_{xy}\partial_y - \frac{1}{2}ff_{xx}\partial_z,\ R(\partial_y,\partial_z)\partial_x = \frac{1}{2}f_{xy}\partial_x, \\ R(\partial_y,\partial_z)\partial_z &= \frac{1}{2}ff_{xy}\partial_x - \frac{\epsilon}{2}f_{yy}\partial_y - \frac{1}{2}f_{xy}\partial_z. \end{aligned} \tag{2.9}$$

Let $\tilde{\nabla}$ denote a semi-symmetric non-metric connection on $(\tilde{M},\tilde{g})$ determined by a vector field $U \in \mathfrak{X}(\tilde{M})$. Through equation (2), a relationship exists between $\tilde{R}$ and the Riemann curvature tensor $\tilde{R}_0$ associated with the Levi-Civita connection $\tilde{\nabla}_0$. Specifically, for orthonormal vectors $e_1, e_2 \in T_p\tilde{M}$, we obtain:

$$\tilde{g}(\tilde{R}(e_1,e_2)e_2,e_1) = \tilde{g}(\tilde{R}_0(e_1,e_2)e_2,e_1) - e_2(\tilde{g}(C,e_2)) + \tilde{g}(C,\tilde{\nabla}_0 e_2 e_2) + \tilde{g}(C,e_2)^2. \tag{2.10}$$

While the initial term on the right-hand side represents the sectional curvature of the plane section $\pi = \text{span}\{e_1,e_2\}$, the left-hand side term depends on the selected basis for $\pi$. Therefore, the quantity $\tilde{g}(\tilde{R}(e_1,e_2)e_2,e_1)$ does not constitute a sectional curvature in the classical sense. For an arbitrary basis $\{e_1,e_2\}$ of $\pi$, it follows directly that:

$$\tilde{K}(\pi) = \frac{\tilde{g}(\tilde{R}(e_1,e_2)e_2,e_1) + \tilde{g}(\tilde{R}(e_2,e_1)e_1,e_2)}{2(\tilde{g}(e_1,e_1)\tilde{g}(e_2,e_2) - \tilde{g}(e_1,e_2)^2)}. \tag{2.11}$$

## 3. Basic results

Now suppose that $\tilde{M}$ is a Walker 3-space $(M, g_f^\varepsilon)$. We compute the sectional curvature of a plane in.

**Proposition 3.1.** *The 3-dimensional Walker manifold, endowed with the metric*

$$g_f^\epsilon = 2\,dt\,dy + \epsilon\,dx^2 + f(t,x,y)\,dy^2.,$$

*where $\varepsilon = \pm 1$ and $f$ is a smooth function, admits a semi-symmetric non-metric connection.*

*Proof.* Let $(M, g_f^\varepsilon)$ be a 3-Walker manifold with coordinates $(t,x,y)$, and the expression of the matrix of the metric is given by

$$g_f^\varepsilon = \begin{pmatrix} 0 & 0 & 1 \\ 0 & \varepsilon & 0 \\ 1 & 0 & f \end{pmatrix}.$$

The vector field $\partial_x$ is isotropic and parallel. Define the 1-form $\omega = dx$. The connection $\widetilde{\nabla}$ is defined by

$$\widetilde{\nabla}_X Y = \nabla_X Y + \omega(Y)X, \tag{3.1}$$

where $\nabla$ is the Levi-Civita connection of $g_f^\varepsilon$.

(1) **Semi-symmetry:**
The torsion of $\widetilde{\nabla}$ is

$$\widetilde{T}(X,Y) = \widetilde{\nabla}_X Y - \widetilde{\nabla}_Y X - [X,Y] = \omega(Y)X - \omega(X)Y,$$

since $\nabla$ is torsion-free. Thus, $\widetilde{\nabla}$ is semi-symmetric.

(2) **Non-metricity:**
We have

$$(\widetilde{\nabla}_X g_f^\varepsilon)(Y,Z) = X(g_f^\varepsilon(Y,Z)) - g_f^\varepsilon(\widetilde{\nabla}_X Y, Z) - g_f^\varepsilon(Y, \widetilde{\nabla}_X Z).$$

Substituting $\widetilde{\nabla}_X Y = \nabla_X Y + \omega(Y)X$, and using $(\nabla_X g_f^\varepsilon)(Y,Z) = 0$, we obtain

$$(\widetilde{\nabla}_X g_f^\varepsilon)(Y,Z) = -\omega(Y) g_f^\varepsilon(X,Z) - \omega(Z) g_f^\varepsilon(X,Y).$$

This term is nonzero in general. For example, with $X = \partial_t$, $Y = \partial_y$, $Z = \partial_x$, we have

$$(\widetilde{\nabla}_{\partial_t} g_f^\varepsilon)(\partial_y, \partial_x) = -\omega(\partial_y) g_f^\varepsilon(\partial_t, \partial_x) - \omega(\partial_x) g_f^\varepsilon(\partial_t, \partial_y) = -0 \cdot 1 - 1 \cdot 1 = -1 \neq 0.$$

One can see also, if we take $X = \partial_y$, $Y = \partial_y$, $Z = \partial_x$, we have

$$(\widetilde{\nabla}_{\partial_y} g_f^\varepsilon)(\partial_y, \partial_x) = -\omega(\partial_y) g_f^\varepsilon(\partial_y, \partial_x) - \omega(\partial_x) g_f^\varepsilon(\partial_y, \partial_y) = -0 \cdot 1 - 1 \cdot f = -f \neq 0.$$

Hence, $\widetilde{\nabla}\widetilde{g} \neq 0$, and $\widetilde{\nabla}$ is non-metric.

Therefore, the connection $\widetilde{\nabla}$ is semi-symmetric and non-metric.

□

Now suppose that $\tilde{M}$ is the Walker space $(M, g_f^\varepsilon)$. We compute the sectional curvature of a plane in $M$.

**Proposition 3.2.** *let $\tilde{\nabla}$ be a canonical semi-symmetric non metric on $M$. If $\pi$ is a plane on $M$, then its sectional curvature is :*

$$\tilde{K}(\pi) = \frac{\langle U, e_1\rangle^2 + \langle U, e_2\rangle^2 - e_1(\langle U, e_1\rangle) - e_2(\langle U, e_2\rangle) - \frac{1}{4}\langle U, e_1\rangle f_x + \frac{1}{2} f_{xx}}{2\varepsilon}. \quad (3.2)$$

*where $\{\mathbf{e_1}, \mathbf{e_2}\}$ is an orthormal basis of $\pi$.*

*Proof.* Using $e_1 = \partial_x$, $e_2 = \frac{2-f}{2\sqrt{2}}\partial_t + \frac{1}{\sqrt{2}}\partial_y$ and $e_3 = \frac{2+f}{2\sqrt{2}}\partial_t - \frac{1}{\sqrt{2}}\partial_y$, which is an orthonormal frame field in $TM$. We have $g_f^\varepsilon(e_1, e_1) = \varepsilon$, $g_f^\varepsilon(e_2, e_2) = 1$ and $g_f^\varepsilon(e_3, e_3) = -1$.
It follows from (2.2), (2.9), and (2.10), that with respect to this pseudo-orthonormal frame field, the only non-zero components of $\nabla$ and $R$ are, respectively:

$$\nabla_{e_2} e_1 = -\nabla_{e_3} e_1 = \frac{1}{4} f_x (e_2 + e_3),$$
$$\nabla_{e_2} e_2 = -\nabla_{e_3} e_2 = -\nabla_{e_2} e_3 = \nabla_{e_3} e_3 = -\frac{1}{4} f_x e_1$$

$$R(e_1, e_2)e_1 = -R(e_1, e_3)e_1 = -\frac{1}{4} f_{xx}(e_2 + e_3),$$
$$R(e_1, e_2)e_2 = -R(e_2, e_1)e_3 = -R(e_1, e_3)e_3 = R(e_1, e_3)e_3 = \frac{\varepsilon}{4} f_{xx} e_1.$$

Thus, we have the following results:

$$\tilde{g}(\tilde{R}(e_1,e_2)e_2,e_1) = \langle U,e_2\rangle^2 - \frac{1}{4}f_x\langle U,e_1\rangle - e_2(\langle U,e_2\rangle) + \frac{1}{4}f_{xx},$$
$$\tilde{g}(\tilde{R}(e_2,e_1)e_1,e_2) = \langle U,e_1\rangle^2 - e_1(\langle U,e_1\rangle) + \frac{1}{4}f_{xx}.$$

This yields the formula for $\tilde{K}(\pi)$. The last statement follows directly from this formula. □

**Consequence 3.1.** *As a consequence, $\tilde{K}(\pi)$ is non-constant. Furthermore, $\tilde{K}(\pi) = A$ if only if $f = 2\varepsilon Ax^2 + B(y) + D(y)$ (resp. $\tilde{K}(\pi) = A_0$) if only if $f = c_1 + +c_2e^{\frac{1}{2}x} + (4 - 8\varepsilon A_0)x)$ or if only if $f = (2\varepsilon A_0 - 1)x^2 + E(y)x + F(y)$ if only if $\pi$ is perpendicular to $U$ (resp. $\pi$ est parallel to $U$) where $B$ , $D$ $E$ and $F$ are functions .*

**Corollary 3.1.** *let $\tilde{\nabla}$ be a canonical semi-symmetric non metric on $M$. if $\pi$ is a plane on $M$, $f$ is constant and $\varepsilon = 1$ then its sectional curvature is*

$$\tilde{K}(\pi) = \frac{\langle U,e_1\rangle^2 + \langle U,e_2\rangle^2 - e_1\langle U,e_1\rangle - e_2\langle U,e_2\rangle}{2}. \tag{3.3}$$

*where $\{\mathbf{e_1},\mathbf{e_2}\}$ is an orthonormal basis of $\pi$. As a consequence, $\tilde{K}(\pi)$ is constant with $0 \leq \tilde{K}(\pi) \leq \frac{1}{2}$. Furthermore, $\tilde{K}(\pi) = 0$ (resp. $\tilde{K}(\pi) = \frac{1}{2}$) if only if $\pi$ is perpendiculary to $U$ (resp. $\pi$ is parallel to $U$).*

**Corollary 3.2.** *let $\tilde{\nabla}$ be a canonical semi-symmetric non metric on $M$. if $\pi$ is a plane on , $f$ is constant and $\varepsilon = -1$ then its sectional ccurvature is*

$$\tilde{K}(\pi) = -\frac{\langle U,e_1\rangle^2 + \langle U,e_2\rangle^2 - e_1\langle U,e_1\rangle - e_2\langle U,e_2\rangle}{2}, \tag{3.4}$$

*where $\{\mathbf{e_1},\mathbf{e_2}\}$ is an orthonormal basis of $\pi$. As consequence, $\tilde{K}(\pi)$ is constant with $-\frac{1}{2} \leq \tilde{K}(\pi) \leq 0$ . Furthermore, $\tilde{K}(\pi) = 0$ (resp. $\tilde{K}(\pi) = -\frac{1}{2}$) if only if $\pi$ is perpendicular to $U$ (resp. $\pi$ is parallel to $U$).*

**Remark 3.2.** *The notion of scalar curvature at a point $p \in M$ with respect to semi-symmetric non-metric $\tilde{\nabla}$ can be introduced similarly as for the Levi-Civita connection. Let $\{\mathbf{e_1},\mathbf{e_2},\mathbf{e_3}\}$ be an orthonormal basis of $T_pM$, $p \in M$. The scalar curvature $\rho$ with respect to $\tilde{\nabla}$ is defined by :*

$$\rho(p) = \tilde{K}(\mathbf{e_1},\mathbf{e_2}) + \tilde{K}(\mathbf{e_1},\mathbf{e_3}) + \tilde{K}(\mathbf{e_2},\mathbf{e_3}), \quad p \in M.$$

*Using the fact that*

$$\tilde{K}(\mathbf{e_1},\mathbf{e_2}) = \frac{\langle U,e_1\rangle^2 + \langle U,e_2\rangle^2 - e_1(\langle U,e_1\rangle) - e_2(\langle U,e_3\rangle) - \frac{1}{4}\langle U,e_1\rangle f_x + \frac{1}{2}f_{xx}}{2\varepsilon}$$
$$\tilde{K}(\mathbf{e_1},\mathbf{e_3}) = -\frac{\langle U,e_1\rangle^2 + \langle U,e_3\rangle^2 - e_1(\langle U,e_1\rangle) - e_3(\langle U,e_3\rangle) - \frac{1}{4}\langle U,e_1\rangle f_x + \frac{1}{2}f_{xx}}{2\varepsilon}$$
$$\tilde{K}(\mathbf{e_2},\mathbf{e_3}) = -\frac{\langle U,e_2\rangle^2 + \langle U,e_3\rangle^2 - e_2(\langle U,e_2\rangle) - e_3(\langle U,e_3\rangle) - \frac{1}{2}\langle U,e_1\rangle f_x}{2},$$

*we get*

$$\begin{aligned}\rho(p) &= \frac{\langle U,e_2\rangle^2-\langle U,e_3\rangle^2-e_2(\langle U,e_2\rangle)+e_3(\langle U,e_3\rangle)}{2\varepsilon}\\ &+ -\frac{\langle U,e_2\rangle^2+\langle U,e_3\rangle^2-e_2(\langle U,e_2\rangle)-e_3(\langle U,e_3\rangle)-\frac{1}{2}\langle U,e_1\rangle f_x}{2}.\end{aligned}$$

**Proposition 3.3.** *Let $\tilde{\nabla}$ be a canonical semi-symmetric non-metric on $M$. If $\pi$ is a plane of $M$, then its scalar curvature:*

$$\begin{aligned}\rho(p) &= \frac{\langle U,e_2\rangle^2-\langle U,e_3\rangle^2-e_2(\langle U,e_2\rangle)+e_3(\langle U,e_3\rangle)}{2\varepsilon}\\ &\quad -\frac{\langle U,e_2\rangle^2+\langle U,e_3\rangle^2-e_2(\langle U,e_2\rangle)-e_3(\langle U,e_3\rangle)-\frac{1}{2}\langle U,e_1\rangle f_x}{2},\end{aligned} \tag{3.5}$$

*where $\{\mathbf{e_1},\mathbf{e_2},\mathbf{e_3}\}$ is an orthonormal basis of $\pi$.*

**Corollary 3.3.** *Let $\tilde{\nabla}$ be a canonical semi-symmetric non-metric on $M$. If $\pi$ is a plane of $M$, then its scalar curvature $\rho(p)$is determined according to the following two cases:*

i) *$\rho(p)=0$ if only if $\pi$ is perpendiculary to $U$;*
ii) *$\rho(p)=0$ or $\rho(p)=-1$ or $\rho(p)=\frac{1}{4}f_{xx}$. In the latter case, $\rho(p)$ is constant if $f$ is an affine function.*

We conclude this section by establishing a relation between the sectional curvatures $K$ and $\tilde{K}$ of a surface in $M$ in terms of the Curvature of $M$ and the mean curvatures of the surface. Let $S$ be an oriented surface immersed in M and $N$ its unit normal vector field. Let also $\tilde{\nabla}$ be a canonical semi-symmetric non metric on $M$ determined by an arbitrary vector field $U\in\mathfrak{X}(M)$. We have the decomposition of $U$ in its tangential and normal components with respect to $S$ :

$$U=U^{\top}+\langle U,N\rangle N.$$

If $X,Y,Z,W\in X(M)$, then the Gauss equation with respect to $\tilde{\nabla}$ is (see [2]) :

$$\begin{aligned}\langle R(X,Y)Z,V\rangle &= \langle\tilde{R}(X,Y)Z,V\rangle-\langle h(X,Z),h(Y,V)\rangle+\langle h(X,V),h(Y,Z)\rangle\\ &+ \langle V,N\rangle\left(\langle h(X,Z),N\rangle\langle Y,V\rangle-\langle h(Y,Z),N\rangle\langle X,V\rangle\right).\end{aligned} \tag{3.6}$$

**Proposition 3.4.** *Let $S$ be an oriented surface in $M$ and denoted by $K^M$ and $H$ the sectional curvature and the mean curvature of $M$, respectively, with respect the Levi-Civita connection . Then :*

$$(1+\varepsilon_2\varepsilon_1)K=\tilde{K}+\varepsilon_2\varepsilon_1\left(K^M-\langle U,N\rangle H\right), \tag{3.7}$$

*where $\varepsilon_2=\langle N,N\rangle=\pm1$*
*Moreover, if $p\in S$, then there is an orthonormal basis $\{e_1,e_2\}$ of $T_pS$ such that :*

$$\begin{aligned}\langle R(e_1,e_2)e_2,e_1\rangle &= \langle\tilde{R}(e_1,e_2)e_2,e_1\rangle+\varepsilon_2h_{11}h_{22}\ -\varepsilon_2g_{11}h_{22}\langle U,N\rangle\\ \langle R(e_2,e_1)e_1,e_2\rangle &= \langle\tilde{R}(e_2,e_1)e_1,e_2\rangle+\varepsilon_2h_{11}h_{22}-\varepsilon_2g_{22}h_{11}\langle U,N\rangle\end{aligned} \tag{3.8}$$

*where $g_{ij}=\langle e_i,e_j\rangle$ and $h_{ij}$ are the coefficients of the second fundamental form $h$.*

*Proof.* Since the codimension of $S$ in $M$ is equal to 1, the normal bundle is trivially flat. Let $\{e_1, e_2\}$ be an orthonormal basis of $T_pS$ such that $g_{12} = 0$ and $h^0_{12} = 0$, where $h^0_{ij}$ are the coefficients of the second fundamental form of $S$ with respect to the Levi-Civita connection: see [4, Props. 3.1 and 3.2]. Consequently, we have $h_{12} = 0$ since $h = h^0$. Using the Gauss equation (15), we obtain (17). With respect to this basis, we have

$$K = K^M + \varepsilon_1 \frac{h_{11}h_{22}}{g_{11}g_{22}}, \quad H = \varepsilon_1 \frac{g_{22}h_{11} + g_{11}h_{22}}{2g_{11}g_{22}}.$$

Identity (16) is a consequence of (17) and the formulas (9) for $K$ and $\tilde{K}$. □ □

**Remark 3.3.** *Identity (16) is satisfied for any vector field $U \in \mathfrak{X}(M)$. Note also that $K$ is invariant under translation in $M$. Indeed, $K^M$ and $H$ do not change, just like $\tilde{K}$, because a plane $\pi$ is not affected by translations. However, rigid motions change the value of $\tilde{K}$ and, consequently, that of $K$. This is due to the presence of the vector field $U$ in (2) for computing successive covariant derivatives.*
*Simple consequences of the relation (16) appear in the following result.*

**Corollary 3.4.** *(1) For a plane, we have $(1 + \frac{\varepsilon_2}{\varepsilon_1})K = \tilde{K}$. In particular, $K \geq 0$ and equality holds if and only if the plane is orthogonal to the vector field $U$, $f$ is constant and $\varepsilon_1 = \varepsilon_2 = 1$.*
*(2) For a cylindrical surface whose rulings are parallel to $U$, we have $(1 + \frac{\varepsilon_2}{\varepsilon_1})K = \tilde{K}$.*

*Proof.* This is immediate since for a plane, we have $K^M = H = 0$, and for a cylindrical surface with rulings parallel to $U$, we have $K^M = 0$ and $\langle U, N\rangle = 0$. □

## 4. Cylindrical surfaces on a Walker 3-manifold

Let $S$ be a cylindrical surface in $M$ whose rulings are parallel to a unit vector $\mathbf{w}$ in $M$. If $\gamma = \gamma(u)$ is the generating curve of $S$ contained in a plane orthogonal to $\mathbf{w}$, then a parametrization of $S$ is given by:

$$\psi(u, v) = \gamma(u) + v\mathbf{w}, \quad u \in I \subset \mathbb{R}, v \in \mathbb{R}. \tag{4.1}$$

Without loss of generality, we assume that $\gamma$ is parametrized by arc length. Let $n$ be the unit normal vector to $\gamma$ and let $\kappa$ be the Frenet curvature of $\gamma$ with $\gamma'' = -\epsilon_2\kappa n$. Since $\gamma$ is contained in a plane orthogonal to $\mathbf{w}$, we consider the orientation of $\gamma$ such that $(\gamma', \mathbf{w}, n) = \varepsilon_2$, where $(\mathbf{a}, \mathbf{b}, \mathbf{c})$ denotes the determinant of the matrix formed by three vectors $\mathbf{a}, \mathbf{b}, \mathbf{c}$ of $M$.

**Theorem 4.1.** *Let $\tilde{\nabla}$ be a canonical snm-connection on $M$. If $M$ is a cylindrical surface parametrized by* (4.1)*, then its sectional curvature $K$ with respect to $\tilde{\nabla}$ is given by:*
$(1 + \frac{\varepsilon_2}{\varepsilon_1})K = \frac{1}{2\varepsilon}\left(\langle U, \gamma'\rangle^2 + \langle U, w\rangle^2 - \gamma'(\langle U, \gamma'\rangle) - w(\langle U, w\rangle) - \frac{1}{4}\langle U, e_1\rangle f_x + \frac{1}{2}f_{xx} + \frac{\kappa}{\varepsilon_1}\langle U, N\rangle\right).$

*Proof.* The tangent plane to $S$ is spanned by an orthonormal basis $\{e_1, e_2\}$, where $e_1 = \psi_s = \gamma'$ and $e_2 = \psi_t = \mathbf{w}$. We know that the sectional curvature is $K^M = 0$. The normal to $S$ and the mean curvature are given by:

$$N = \gamma' \times \mathbf{w}, \quad H = \frac{(\gamma', \mathbf{w}, \gamma'')}{2\varepsilon} = -\frac{\kappa\varepsilon_2}{2\varepsilon}.$$

We calculate the following covariant derivatives

$$\nabla_{\mathbf{w}}\gamma' = \frac{1}{4}f_x(\mathbf{w}+e_3), \quad \nabla_{\mathbf{w}}\mathbf{w} = -\frac{1}{4}f_x\gamma'$$

$$R(\gamma',\mathbf{w})\gamma' = -\frac{1}{4}f_{xx}(\mathbf{w}+e_3),$$
$$R(\gamma',\mathbf{w})\mathbf{w} = \frac{\varepsilon}{4}f_{xx}\gamma'.$$

We obtain:

$$\tilde{g}(\tilde{R}(\gamma',w)w,\gamma') = \langle U,w\rangle^2 - \frac{1}{4}f_x\langle U,e_1\rangle - w(\langle U,w\rangle) + \frac{1}{4}f_{xx}.$$
$$\tilde{g}(\tilde{R}(w,\gamma')\gamma',w) = \langle U,\gamma'\rangle^2 - \gamma'(\langle U,\gamma'\rangle) + \frac{1}{4}f_{xx}..$$

Thus, we obtain the sectional curvature $\tilde{K}$:

$$\tilde{K} = \frac{\langle U,\gamma'\rangle^2 + \langle U,w\rangle^2 - \gamma'(\langle U,\gamma'\rangle) - w(\langle U,w\rangle) - \frac{1}{4}\langle U,\gamma'\rangle f_x + \frac{1}{2}f_{xx}}{2\varepsilon}.$$

Using $(1+\frac{\varepsilon_2}{\varepsilon_1})K = \tilde{K} + \frac{\varepsilon_2}{\varepsilon_1}\left(K^M - \langle U,N\rangle H\right)$, we find:

$$(1+\frac{\varepsilon_2}{\varepsilon_1})K = \frac{1}{2\varepsilon}\Big(\langle U,\gamma'\rangle^2 + \langle U,w\rangle^2 - \gamma'(\langle U,\gamma'\rangle) - w(\langle U,w\rangle) - \frac{1}{4}\langle U,e_1\rangle f_x + \frac{1}{2}f_{xx} + \frac{\kappa}{\varepsilon_1}\langle U,N\rangle\Big).$$

The result follows because $\gamma'\times\mathbf{w} = n$ and $\gamma'' = -\kappa\varepsilon_2 n$. □

**Corollary 4.1.** *Let $\tilde{\nabla}$ be a canonical snm-connection on $M$. If $M$ is a cylindrical surface parametrized by (9) and $f$ is a constant with $\varepsilon = \varepsilon_1 = \varepsilon_2 = 1$, then its sectional curvature $K$ with respect to $\tilde{\nabla}$ is given by:*

$$K = \frac{1}{4}\left(\langle U,\gamma'\rangle^2 + \langle U,w\rangle^2 - \gamma'(\langle U,\gamma'\rangle) - w(\langle U,w\rangle) + \kappa\langle U,N\rangle\right). \tag{4.2}$$

We distinguish two particular cases, when the rulings are parallel or orthogonal to the constant vector $U$.

**Corollary 4.2.** *Every cylindrical surface whose rulings are parallel to $U$ has constant sectional curvature $K = A_0$ iff $f = c_1 + c_2e^{\frac{1}{2}x} + (4-8\varepsilon A_0)x$ or iff $f = (4A_0-1)x^2 + E(y)x + F(y)$ with respect to a canonical snm-connection determined by $U$ .*

**Corollary 4.3.** *Every cylindrical surface whose rulings are parallel to $U$ and $f$ is a constant, $\varepsilon = \varepsilon_1 = \varepsilon_2 = 1$ has constant sectional curvature $K = \frac{1}{4}$ with respect to a canonical snm-connection determined by $U$*

**Corollary 4.4.** *Every cylindrical surface whose rulings are parallel to $U$ and $f$ is a constant, $\varepsilon = \varepsilon_1 = \varepsilon_2 = -1$ has constant sectional curvature $K = -\frac{1}{4}$ with respect to a canonical snm-connection determined by $U$*

Suppose now that the rulings are orthogonal to $U$. In the following result, we obtain explicit parametrizations of cylindrical surfaces with constant sectional curvature. Without loss of generality, we assume that $U = \partial_y$, $f = 1$, $\varepsilon = \varepsilon_2 = 1$ and $\mathbf{w} = (0,1,0)$. Then $\gamma$ is contained in the $ty$-plane, i.e., $\gamma(u) = (t(u), 0, y(u))$, for smooth functions $t$ and $y : I \to \mathbb{R}$. The case where $S$ is a plane is special. Any plane in $M$ perpendicular to $\partial_y$ can be seen as a cylindrical surface whose rulings are orthogonal to $\partial_y$. By Corollary 3.3, we know that its curvature $K$ is constant with $0 \leq K \leq \frac{1}{4}$ for $\varepsilon = \varepsilon_1 = \varepsilon_2 = 1$. We exclude this case.

**Corollary 4.5.** *Let $\tilde{\nabla}$ be the canonical snm-connection determined by $\partial_z$, and let $M$ be a non-planar cylindrical surface whose generators are orthogonal to $\partial_y$. If the sectional curvature $K$ with respect to $\tilde{\nabla}$ is constant and $f = \varepsilon = \varepsilon_1 = \varepsilon_2 = 1$, then the parametrization of the generating curve $\gamma$ is:*

*(1) If $K \neq 0$, then*

$$\gamma(u) = \left( \int e^{2\int a(u)\,du} \left( -4K \int e^{-2\int a(u)\,du}\,du + C_1 \right) du + C_2 \pm \int \sqrt{1 - [t'(u)]^2}\,du + C_3 \right)$$

.

*(2) If $K = 0$, then $\gamma(u) = \left( C_1 \int e^{2\int a(u)\,du}\,du + C_2, \pm \int \sqrt{1 - [t'(u)]^2}\,du + C_3 \right)$.*

*Proof.* Since $\gamma$ is parametrized by arc length, we know that $t'^2 + y'^2 = 1$ and that $\gamma' = (t', 0, y')$. Using the orientation of $\gamma$ given in Theorem 4.1, the normal vector is $n = (t' + y', 0, -y')$. Identity (10) becomes:

$$4K = (t' + y')^2 - t''.$$

The given equation is:

$$t''(u) - 2a(u)t'(u) = -4K$$

with $a(u) = y'(u)$.
Let $z(u) = t'(u)$, then:

$$z'(u) - 2a(u)z(u) = -4K$$

Thus:

$$z(u) = e^{2\int a(u)\,du} \left( -4K \int e^{-2\int a(u)\,du}\,du + C_1 \right)$$

Since $z(u) = t'(u)$, we integrate again:

$$t(u) = \int e^{2\int a(u)\,du} \left( -4K \int e^{-2\int a(u)\,du}\,du + C_1 \right) du + C_2.$$

**Special case** $K = 0$:

$$t'(u) = C_1 e^{2\int a(u)\,du}, \quad t(u) = C_1 \int e^{2\int a(u)\,du}\,du + C_2.$$

From the condition $t'(u)^2 + y'(u)^2 = 1$, we have:

$$y'(u) = \pm\sqrt{1 - [t'(u)]^2}$$

Therefore:

$$y(u) = \pm \int \sqrt{1 - [t'(u)]^2}\,du + C_3$$

where $t'(u)$ is known from the solution above. □

**Corollary 4.6.** *Let $\tilde{\nabla}$ be the canonical snm-connection determined by $\partial_y$, $M$ a non-planar cylindrical surface whose generators are orthogonal to $\partial_z$,$f = \varepsilon = \varepsilon_1 = \varepsilon_2 = 1$ and $K$ the sectional curvature of $M$ with respect to $\tilde{\nabla}$.*

*(1) If $K = \frac{1}{2}$, then $\gamma(u) = (t(u), y(u)$ where*
$t(u) = (\int e^{2\int a(u)du}\left(-2\int e^{-2\int a(u)du}du + C_1\right)du + C_2$ *and*
$y(u) = \pm\int\sqrt{1 - \left[e^{2\int a(u)du}\left(-2\int e^{-2\int a(u)du}du + C_1\right)\right]^2}du + C_3.$

*(2) If $K = -\frac{1}{2}$, then $\gamma(u) = (t(u), y(u))$ where*
$t(u) = \int e^{2\int a(u)du}\left(2\int e^{-2\int a(u)du}du + C_1\right)du + C_2$ *and*
$y(u) = \pm\int\sqrt{1 - \left[e^{2\int a(u)du}\left(2\int e^{-2\int a(u)du}du + C_1\right)\right]^2}du + C_3.$

*Proof.* For $K = \frac{1}{2}$, the equation is:

$$t''(u) - 2a(u)t'(u) = -2$$

with $a(u) = y'(u)$. The solution for $t(u)$ is

$$t(u) = \int e^{2\int a(u)du}\left(-2\int e^{-2\int a(u)du}du + C_1\right)du + C_2$$

and the solution for $y(u)$:

$$y(u) = \pm\int\sqrt{1 - \left[e^{2\int a(u)du}\left(-2\int e^{-2\int a(u)du}du + C_1\right)\right]^2}du + C_3.$$

For $K = -\frac{1}{2}$. The equation is:

$$t''(u) - 2a(u)t'(u) = 2$$

with $a(u) = y'(u)$, the solution for $t(u)$ is

$$t(u) = \int e^{2\int a(u)du}\left(2\int e^{-2\int a(u)du}du + C_1\right)du + C_2$$

the solution for $y(u)$ is

$$y(u) = \pm\int\sqrt{1 - \left[e^{2\int a(u)du}\left(2\int e^{-2\int a(u)du}du + C_1\right)\right]^2}du + C_3.$$

□

## 5. Conclusion

This work has established a comprehensive framework for studying the geometry of surfaces in three-dimensional Walker manifolds endowed with a semi-symmetric non-metric connection. The key contributions include:

(1) The construction of a canonical snm-connection on Walker manifolds and the derivation of explicit formulas for its sectional and scalar curvatures.
(2) The establishment of relationships between the curvature properties with respect to the snm-connection and those with respect to the Levi-Civita connection.

(3) The complete classification of cylindrical surfaces with constant sectional curvature, providing explicit parametrizations of their generating curves in all cases.

These results extend our understanding of geometric structures under non-metric connections and demonstrate the rich interplay between connection properties and curvature characteristics. The explicit parametrizations obtained for constant curvature surfaces provide concrete examples that illustrate the theoretical developments. Future work could explore applications of these results to physical theories in which non-metric connections play a role, or extend the analysis to higher-dimensional Walker manifolds and to more general connection types.

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