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WARPED PRODUCT SUBMANIFOLDS OF LOCALLY PRODUCT RIEMANNIAN MANIFOLDS WITH SLANT FACTOR

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ABSTRACT. In this paper, we delve into almost product Riemannian manifolds featuring warped product submanifolds. We analyze the integrability of slant and invariant distributions, examining the geometric structure of their leaves. Moreover, we determine the conditions for a semi-slant submanifold within an almost product Riemannian manifold to become a semi-slant warped product submanifold, characterized by the canonical tensors ξ and η.

1. INTRODUCTION

R.L. Bishop and B. O'Neill [5] pioneered the concept of warped product manifolds while exploring manifolds with negative sectional curvature, analyzing their intrinsic geometric characteristics. Subsequently, it was recognized that these manifolds offer an ideal framework for modeling space-time in regions near black holes or objects with intense gravitational fields. This insight spurred investigations into warped product manifolds from an extrinsic geometric perspective. B.Y. Chen [10] was among the first to adopt this approach, focusing on CR-submanifolds of Kaehler manifolds as warped product structures. He examined their geometric properties and derived a characterization, based on the shape operator of the immersion, that determines when a CR-submanifold becomes a CR-warped product submanifold. Comparable studies were conducted by I. Hasegawa and I. Mihai [13] for contact CR-submanifolds within Sasakian manifolds. In the context of CR-submanifolds of Kähler manifolds, Chen [8] highlighted the critical role of the canonical structures ξ and η in elucidating various geometric properties of these submanifolds. Specifically, he established that a CR-submanifold of a Kaehler manifold is a CR-product—a special case of a CR-warped product, essentially a trivial warped product—if and only if ξ is parallel. However, this result does not extend to semi-slant submanifolds of locally product Riemannian manifolds.

The exploration of invariant and anti-invariant submanifolds in locally product Riemannian manifolds was pioneered by Adati [1]. Building on this, Bejancu [4] and Pitis [24] introduced a novel class of submanifolds, termed semi-invariant submanifolds, by blending properties of invariant and anti-invariant submanifolds. The study of slant and semi-slant submanifolds in locally product Riemannian manifolds was conducted

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by Şahin [25] and Li and Liu [19], who delved into their intrinsic geometric properties. This motivates us to study a more general problem of finding a characterization under which a semi-slant submanifold of locally product manifold reduces to a trivial product submanifold.
As B. Y. Chen demonstrated for warped product submanifolds of Kaehler manifolds, if the canonical form ξ is parallel, then the CR-warped product reduces to a trivial product. This naturally raises the question: in the context of semi-slant warped products within locally product manifolds, what occurs if ξ is parallel. Will such a warped product persist in its form, or will it degenerate into a trivial product? Motivated by this inquiry, the present paper investigates semi-slant warped product submanifolds of locally product Riemannian manifolds.
The structure of this paper is organized as follows. Section 2 presents the essential definitions and properties of locally product Riemannian manifolds relevant to this study. In Section 3, we review several significant results that facilitate the proofs of our main theorems, while also exploring the geometry of the leaves of the distributions and their totally geodesic properties. In Section 4, we establish our primary findings, focusing on the canonical structures ξ and η, which enable a semi-slant submanifold to be characterized as a semi-slant warped product submanifold. Furthermore, we investigate the conditions on the shape operator for semi-slant warped product submanifolds within locally product Riemannian manifolds.

2. Preliminaries

Let $\bar{M}$ be an m-dimensional manifold with tensor of type $(1,1)$ such that

$$\Psi^2 = I \ \ (\Psi \neq 0). \tag{2.1}$$

If we put $P = \frac{1}{2}(I + \Psi),\ \ Q = \frac{1}{2}(I - \Psi),\ \ P^2 = P,\ \ Q^2 = Q,\ \ PQ = QP = 0$, such that

$$g(\Psi\widehat{X}, \Psi\widehat{Y}) = g(\widehat{X}, \widehat{Y}). \tag{2.2}$$

A Riemannian manifold $\bar{M}$ is called locally product Riemannian manifold if

$$(\nabla_{\widehat{U}}\Psi)\widehat{V} = 0, \tag{2.3}$$

for any $\widehat{U}, \widehat{V} \in T\bar{M}$ [31]. Let M be a submanifold of a locally product Riemannian manifold $\bar{M}$ with TM and $TM^{\perp}$ as the tangent and normal bundles on M respectively. If ∇ and $\nabla^{\perp}$ are the induced Riemannian connections on TM and $TM^{\perp}$ then Gauss and Weingarton formulae are

$$\bar{\nabla}_{\widehat{U}}\widehat{V} = \nabla_{\widehat{U}}\widehat{V} + h(\widehat{U}, \widehat{V}), \tag{2.4}$$

$$\bar{\nabla}_{\widehat{U}}N = -A_N\widehat{U} + \nabla^{\perp}_{\widehat{U}}N, \tag{2.5}$$

for any $\widehat{U}, \widehat{V} \in TM$ and $N \in T^{\perp}M$. A_N and h respectively denote the shape operator (corresponding to the normal vector field N) and the second fundamental form of the immersion of M into $\bar{M}$. The two are related as

$$g(A_N\widehat{U}, \widehat{V}) = g(h(\widehat{U}, \widehat{V})), N), \tag{2.6}$$

where g denotes the Riemannian metric on $\bar{M}$ as well as the induced metric on M.
For any $\widehat{U} \in TM$, we write

$$\xi\widehat{U} = tan(\Psi\widehat{U}) \ \text{ and } \ \eta\widehat{U} = nor(\Psi\widehat{U}). \tag{2.7}$$

Similarly, for $N \in TM^{\perp}$, we write

$$tN = tan(\Psi N) \text{ and } fN = nor(\Psi N) \tag{2.8}$$

where $'tan'$ and $'nor'$ are the natural projections associated with the direct decomposition:

$$T_x\bar{M} = T_xM \oplus T_x^{\perp}M, \;\; x \in M.$$

The tensor fields on the manifold M, defined by the endomorphism ξ and the normal-valued 1-form η, are denoted by the same symbols, ξ and η, respectively. Likewise, the tensor fields t and f represent tangential and normal-valued $(1,1)$-tensor fields on the normal bundle of M. The covariant derivatives of the tensor fields ξ, η, t, and f are defined as follows:

$$(\bar{\nabla}_{\widehat{U}}\xi)\widehat{V} = \nabla_{\widehat{U}}\xi\widehat{V} - \xi\nabla_{\widehat{U}}\widehat{V}, \tag{2.9}$$

$$(\bar{\nabla}_{\widehat{U}}\eta)\widehat{V} = \nabla^{\perp}_{\widehat{U}}\eta\widehat{V} - \eta\nabla_{\widehat{U}}\widehat{V}. \tag{2.10}$$

$$(\bar{\nabla}_{\widehat{U}}t)N = \nabla_{\widehat{U}}tN - t\nabla^{\perp}_{\widehat{U}}N, \tag{2.11}$$

$$(\bar{\nabla}_{\widehat{U}}f)N = \nabla^{\perp}_{\widehat{U}}fN - f\nabla^{\perp}_{\widehat{U}}N. \tag{2.12}$$

For any $\widehat{U}, \widehat{V} \in \Gamma(TM)$ and on using covariant derivative formula of ξ and η, we obtained

$$(\nabla_{\widehat{U}}\xi)\widehat{V} = th(\widehat{U}, \widehat{V}) + A_{\eta\widehat{V}}\widehat{U} \tag{2.13}$$

$$(\nabla_{\widehat{U}}\eta)\widehat{V} = fh(\widehat{U}, \widehat{V}) - h(\widehat{U}, \xi\widehat{V}). \tag{2.14}$$

For any $x \in M$ and $\widehat{X} \in T_xM$, angel $\theta(\widehat{X}) \in [0, \frac{\pi}{2}]$ between $\Psi\widehat{X}$ and T_xM is well defined. If $\theta(\widehat{X})$ doses not defined on the choice of $x \in M$ and $\widehat{X} \in T_xM$, we say that M is slant in $\bar{M}$. The contact angel θ is called the slant angel of M in $\bar{M}$. The anti-invariant submanifold of locally product Riemannian manifold are slant submanifolds with slant angel $\frac{\pi}{2}$ and invariant submanifolds are slant submanifolds with slant angel 0. If slant angel $\theta \neq 0, \frac{\pi}{2}$, the slant submanifold is called a proper slant submanifold of locally product Riemannian manifold [19, 25]. Based on B. Şahin's findings [25], it can be stated that;

Theorem 2.1. *Let M be a submanifold of locally product Riemannian manifold $\bar{M}$. Then M is slant if and only if there exists a constant $\lambda \in [0,1]$ such that*

$$\xi^2\widehat{U} = \lambda\widehat{U},$$

for any $\widehat{U} \in \Gamma(\mathcal{H}^{\theta})$.

Furthermore, in such a case, if θ is the slant angel of M, then it verifies that $\lambda = cos^2\theta$. Thus one has the following consequence of the above formulae:

$$g(\xi\widehat{X}, \xi\widehat{Y}) = cos^2\theta g(\widehat{X}, \widehat{Y}) \tag{2.15}$$

$$g(\eta\widehat{X}, \eta\widehat{Y}) = sin^2\theta g(\widehat{X}, \widehat{Y}). \tag{2.16}$$

R. L. Bishop and B. O'Neill [5] introduced the concept of warped products (or, more generally, warped bundles) to construct a diverse class of manifolds with negative sectional curvature, generalizing the notion of the Riemannian product of two manifolds. In the subsequent sections, we will examine the concept of warped product manifolds and explore their intrinsic geometric properties.

Let (M_1, g_1) and (M_2, g_2) be two Reimannian manifolds with Reimannian metric g_1 and g_2 respectively and f be a positive differentiable function on M_1. Then the warped product $M_1 \times_f M_2$ is the manifold $M_1 \times M_2$ endowed with the Reimannian metric g given by

$$g = \pi^*(g_1) + (fo\pi_1)^2\pi_2^*(g_2) \tag{2.17}$$

where $\pi = (1, 2)$ are the projection maps of M on to M_1 and M_2 respectively. Then the function f, in this case known as the warping function. If the warping function is just a constant, the warped product is simply a Reimannian product, known as trivial warped product.

A warped product submanifold is defined as a warped product manifold that is isometrically immersed in a Riemannian manifold. In their seminal work, R. L. Bishop and B. O'Neill [5] derive several significant observations and formulas that elucidate key geometric properties of warped product manifolds.

Theorem 2.2. [5] *Let $M = M_1 \times_f M_2$ be a warped product manifold. If $\widehat{U}_1, \widehat{V}_1 \in TM_1$ and $\widehat{U}_2, \widehat{V}_2 \in TM_2$ then*

(i) $\nabla_{\widehat{U}_1}\widehat{V}_1 \in TM_1$

(ii) $\nabla_{\widehat{U}_1}\widehat{U}_2 = \nabla_{\widehat{U}_2}\widehat{U}_1 = \widehat{U}_1(lnf)\widehat{U}_2$

(iii) $nor(\nabla_{\widehat{U}_2}\widehat{V}_2) = -g(\widehat{U}_2, \widehat{V}_2)\nabla lnf,$

where $nor(\nabla_{\widehat{U}_2}\widehat{V}_2)$ denote the component of $\nabla_{\widehat{U}_2}\widehat{V}_2$ in TM_1 and ∇f is the gradient of f defined as

$$g(\nabla f, \widehat{U}_1) = \widehat{U}_1 f, \tag{2.18}$$

for any $\widehat{U}_1 \in TM_1$. A couple of significant consequences of the aforementioned theorem can be articulated as follows::

Corollary 2.1. *On a warped product manifold $M = M_1 \times_f M_2$;*

(i) *M_1 is totally geodesic.*

(ii) *M_2 is totally umbilical.*

3. Semi-slant submanifold of almost product Riemannian manifold

In this section, we examine semi-slant submanifolds within the context of a locally product Riemannian manifold. Specifically, we analyze the integrability of the distributions and the totally geodesic nature of the leaves associated with the invariant and slant distributions defined in the context of semi-slant submanifolds. Furthermore, we explore the conditions under which a semi-slant submanifold of a locally product Riemannian manifold reduces to a warped product manifold. It is noted that slant and semi-slant submanifolds in locally product Riemannian manifolds have been investigated by [25, 19].

Definition 3.1. [25] *A semi-slant submanifold M of a locally product Riemannian manifold $\bar{M}$ is a submanifold which admits two orthogonal complementary distribution $\mathcal{H}$ and $\mathcal{H}^\theta$ such that*

(i) $TM = \mathcal{H} \oplus \mathcal{H}^\theta$

(ii) $\mathcal{H}$ is invariant. i.e., $\Psi\mathcal{H} = \mathcal{H}$

(iii) $\mathcal{H}^\theta$ is slant with slant angel $\theta \neq 0$.

The orthogonal complement of $\eta\mathcal{H}^\theta$ in the normal bundle $TM^\perp$, is an invariant subbundle of $TM^\perp$ and is denoted by μ. Thus we have

$$TM^\perp = \Psi\mathcal{H}^\theta \oplus \mu \tag{3.1}$$

It is also observe that for any $N \in TM^\perp$, $tN \in \mathcal{H}^\theta$. Consider the warped product submanifold of the form $M^\theta \times_f M^T$, then from the formula of warped product , we have

$$\nabla_{\widehat{U}_1}\widehat{U}_2 = \nabla_{\widehat{U}_2}\widehat{U}_1 = (\widehat{U}_1 \ln f)\widehat{U}_2 \tag{3.2}$$

for any $\widehat{U}_1 \in \Gamma(TM^\theta), \widehat{U}_2 \in \Gamma(TM^T)$. For any $\widehat{U} \in TM$, we have

$$\widehat{U} = \mathcal{B}\widehat{U} + \mathcal{C}\widehat{U} \tag{3.3}$$

where $\mathcal{B}\widehat{U} \in \mathcal{H}^\theta, \mathcal{C}\widehat{U} \in \mathcal{H}$.

Lemma 3.1. *Let M be a semi-slant submanifold of a locally product Riemannian manifold $\bar{M}$. Then*

$$A_{\eta\widehat{U}_2}\widehat{V}_2 = A_{\eta\widehat{V}_2}\widehat{U}_2$$

for $\widehat{U}_2, \widehat{V}_2 \in \Gamma(\mathcal{H}^\theta)$.

Proof. For any $\widehat{U}_2, \widehat{V}_2 \in \Gamma(\mathcal{H}^\theta)$ and $\widehat{U}_1 \in \Gamma(\mathcal{H})$, on using 2.6, we have

$$g(A_{\eta\widehat{U}_2}\widehat{V}_2, \widehat{U}_1) = g(h(\widehat{U}_1, \widehat{V}_2), \eta\widehat{U}_2)$$

Again using equations 2.2, 2.5 and 2.6, we have the desired result.

$$A_{\eta\widehat{U}_2}\widehat{V}_2 = A_{\eta\widehat{V}_2}\widehat{U}_2.$$

□

Now we discussed integrability of the invariant distribution $\mathcal{H}$ and slant distribution $\mathcal{H}^\theta$. Firstly the integrability of invariant distribution $\mathcal{H}$.

Theorem 3.1. *Let M be a semi-slant submanifold of a locally product Riemannian manifold $\bar{M}$. Then the invariant distribution $\mathcal{H}$ is integrable if and only if*

$$h(\widehat{U}_1, \xi\widehat{V}_1) = h(\widehat{V}_1, \xi\widehat{U}_1)$$

for any $\widehat{U}_1, \widehat{V}_1 \in \Gamma(\mathcal{H})$.

Since M is a semi-slant submanifold of the almost product Riemannian manifold $\bar{M}$, it possesses two orthogonal complementary distributions invariant $\mathcal{H}$ and slant distribution $\mathcal{H}^\theta$. Now, we will explore the necessary and sufficient conditions for the integrability of the slant distribution $\mathcal{H}^\theta$.

Theorem 3.2. *Let M be a semi-slant submanifold of a locally product Riemannian manifold $\bar{M}$. Then the slant distribution $\mathcal{H}^\theta$ is integrable if and only if*

$$\nabla_{\widehat{U}_2}\xi\widehat{V}_2 - \nabla_{\widehat{V}_2}\xi\widehat{U}_2 + A_{\eta\widehat{V}_2}\widehat{U}_2 - A_{\eta\widehat{U}_2}\widehat{V}_2 \in \Gamma(D_\theta)$$

for any $\widehat{U}_2, \widehat{V}_2 \in \Gamma(\mathcal{H}^\theta)$ and $\widehat{U}_1 \in \Gamma(\mathcal{H})$.

Proof. for any $\widehat{U}_2, \widehat{V}_2 \in \Gamma(\mathcal{H}^\theta)$ and $\widehat{U}_1 \in \Gamma(\mathcal{H})$ and by using equation 2.7, 2.4 and 2.5, we reached

$$g([\widehat{U}_2, \widehat{V}_2], \widehat{U}_1) = g(\nabla_{\widehat{U}_2} \xi \widehat{V}_2 - \nabla_{\widehat{V}_2} \xi \widehat{U}_2 + A_{\eta \widehat{V}_2} \widehat{U}_2 - A_{\eta \widehat{U}_2} \widehat{V}_2, \xi \widehat{U}_1),$$

from which we can easily deduce that the slant distribution D_θ is integrable if and only of

$$\nabla_{\widehat{U}_2} \xi \widehat{V}_2 - \nabla_{\widehat{V}_2} \xi \widehat{U}_2 + A_{\eta \widehat{V}_2} \widehat{U}_2 - A_{\eta \widehat{U}_2} \widehat{V}_2 \in \Gamma(\mathcal{H}^\theta).$$

□

This completes the proof of theorem.

Theorem 3.3. *Let M be a proper semi-slant submanifold of a locally product Riemannian manifold $\bar{M}$ then the distribution $\mathcal{H}^\theta$ is parallel if and only if*

$$(\nabla_{\widehat{U}_1} \xi) \widehat{V}_1 \in \Gamma(\mathcal{H}^\theta)$$

for any $\widehat{U}_1, \widehat{V}_1 \in \Gamma(\mathcal{H}^\theta)$.

Proof. Suppose that $\mathcal{H}^\theta$ is parallel then by virtue of covariant derivative of ξ, i.e., from equation 2.9, $(\nabla_{\widehat{U}_1} \xi) \widehat{V}_1 \in \Gamma(\mathcal{H}^\theta)$ for all $\widehat{U}_1, \widehat{V}_1 \in \Gamma(\mathcal{H}^\theta)$. Conversely suppose that $(\nabla_{\widehat{U}_1} \xi) \widehat{V}_1 \in \Gamma(\mathcal{H}^\theta)$ for all $\widehat{U}_1, \widehat{V}_1 \in \Gamma(\mathcal{H}^\theta)$, then from equation 2.9, we have

$$g(\nabla_{\widehat{U}_1} \xi \widehat{V}_1 - \xi \nabla_{\widehat{U}_1} \widehat{V}_1, \widehat{U}_2) = 0,$$

for any $\widehat{U_2} \in \Gamma(\mathcal{H})$. By interchanging $\widehat{V}_1$ by $\xi \widehat{V}_1$ and $\widehat{U}_2$ by $\xi \widehat{U}_2$ with equation 2.15, we have

$$-\cos^2 \theta g(\nabla_{\widehat{U}_1} \widehat{V}_1, \xi \widehat{U}_2) - \cos^2 \theta g(\nabla_{\widehat{U}_1} \xi \widehat{V}_1, \widehat{U}_2) = 0.$$

Since $\widehat{V}_1 \in \Gamma(\mathcal{H}^\theta)$ and again replace $\widehat{V}_1$ by $\xi \widehat{V}_1$, we yields that

$$g(\nabla_{\widehat{U}_1} \widehat{V}_1, \widehat{U}_2) = 0.$$

From this, we deduce that $\mathcal{H}^\theta$ is parallel, leading to our final result. □

We now investigate the geometry of the leaves of the slant distribution $\mathcal{H}^\theta$ and identify the necessary and sufficient conditions for the slant distribution to be totally geodesic.

Theorem 3.4. *Let M be a proper semi-slant submanifold of a locally product Riemannian manifold $\bar{M}$, then the distribution $\mathcal{H}^\theta$ is defines totally geodesic foliation on M if and only if*

$$(\nabla_{\widehat{Z}} \eta \xi \widehat{W} - \nabla_{\widehat{Z}} t \eta \widehat{W} + \nabla_{\widehat{Z}} f \eta \widehat{W}) \in \Gamma(\mathcal{H}^\theta) \tag{3.4}$$

for any $\widehat{U} \in \Gamma(\mathcal{H})$ and $\widehat{Z}, \widehat{W} \in \Gamma(\mathcal{H}^\theta)$.

Proof. In the light of equations (2.2) and (2.7), we have

$$g(\nabla_{\widehat{Z}} \widehat{W}, \widehat{U}) = g(\nabla_{\widehat{Z}} \xi \widehat{W}, \eta \widehat{U}) + g(\nabla_{\widehat{Z}} \eta \widehat{W}, \eta \widehat{U}),$$

for any $\widehat{U} \in \Gamma(\mathcal{H})$ and $\widehat{Z}, \widehat{W} \in \Gamma(\mathcal{H}^\theta)$. By using equations (2.2), (2.7), (2.8) with (i) part of theorem 2.1, the above equation takes the form

$$sin^2\theta \, g(\nabla_{\widehat{Z}} \widehat{W}, \widehat{U}) = g(\nabla_{\widehat{Z}} \eta \xi \widehat{W}, \widehat{U}) - g(\nabla_{\widehat{Z}} t \eta \widehat{W}, \widehat{U}) + g(\nabla_{\widehat{Z}} f \eta \widehat{W}, \widehat{U}).$$

This completes the proof of theorem. □

4. Semi-slant warped product submanifold

Let M be a submanifold of a locally product Riemannian manifold $\bar{M}$. In this section, we analyze warped product semi-slant submanifolds of the form $M^\theta \times_f M^T$, where M^θ is a proper slant submanifold and M^T is an invariant submanifold of $\bar{M}$.
From theorem 2.2, we have

$$\nabla_{\widehat{U}_1}\widehat{U}_2 = \nabla_{\widehat{U}_2}\widehat{U}_1 = \widehat{U}_1(\ln f)\widehat{U}_2 \tag{4.1}$$

for any $\widehat{U}_1 \in \Gamma(\mathcal{H}^\theta), \widehat{U}_2 \in \Gamma(\mathcal{H})$. We have now presented several key results that facilitate the derivation of our main result as follows:

Theorem 4.1. *Let $M = M^\theta \times_f M^T$ be a semi-slant warped submanifold of a locally product Riemannian manifold where M^T is invariant and M^θ is slant submanifold. Then for any $\widehat{U}_1, \widehat{V}_1 \in \Gamma(TM^\theta)$ and $\widehat{U}_2, \widehat{V}_2 \in \Gamma(TM^T)$, we have the following*

(i) $(\nabla_{\widehat{U}_1}\xi)\widehat{U}_2 = 0$

(ii) $(\nabla_{\widehat{U}_2}\xi)\widehat{U}_1 = \xi\widehat{U}_1(\ln f)\widehat{U}_2 - \widehat{U}_1(\ln f)\xi\widehat{U}_2$

(iii) $(\nabla_{\widehat{U}}\xi)\widehat{U}_1 = A_{\eta\widehat{U}_1}B\widehat{U} + th(\mathcal{B}\widehat{U}, \widehat{U}_1) + \xi\widehat{U}_1(\ln f)\mathcal{C}\widehat{U} - \widehat{U}_1(\ln f)\xi\mathcal{C}\widehat{U}.$

Proof. By using equations (2.9), (4.1), we directly have (i). For part (ii), we apply the same equation as in part (i), yielding the result for part (ii). For part (iii), by utilizing equation 2.9, we obtain

$$(\nabla_{\widehat{U}}\xi)\widehat{U}_1 = (\nabla_{\mathcal{B}\widehat{U}}\xi)\widehat{U}_1 + (\nabla_{\mathcal{C}\widehat{U}}\xi)\widehat{U}_1.$$

In the light of equation 2.13, we yields

$$(\nabla_{\widehat{U}}\xi)\widehat{U}_1 = th(\mathcal{B}\widehat{U}, \widehat{U}_1) + \xi\widehat{U}_1(\ln f)\mathcal{C}\widehat{U} - \widehat{U}_1(\ln f)\xi\mathcal{C}\widehat{U} + A_{\eta\widehat{U}_1}B\widehat{U}.$$

This proves the theorem completely. □

This section aims to characterize a semi-slant submanifold in a locally product Riemannian manifold that reduces to a warped product submanifold. There has been a longstanding interest in finding an analogue of the classical De Rham theorem for warped products. Hiepko [14] proved the following theorem, which we will employ to characterize warped product submanifolds.

Theorem 4.2. [14] *Let F be a vector sub bundle in the tangent bundle of a Riemannian manifold M and let $F^\perp$ be its normal bundle. Assume that the two distributions are both involutive and the integral manifold of F (resp.$F^\perp$) are extrinsic spheres (resp. totally geodesic). Then M is locally isometric to a warped product $M_1 \times_f M^\theta$. Moreover, if M is simply connected and complete there exists a global isometry of M with a warped product.*

We are now positioned to derive our main result in terms of the covariant tensor ξ as follows:

Theorem 4.3. *A semi-slant submanifold of a locally product Riemannian manifold $\bar{M}$ is locally semi-slant warped product submanifold if and only if*

$$\begin{aligned}(\nabla_{\widehat{U}}\xi)\widehat{V} = \xi\mathcal{B}\widehat{V}(\ln f)\mathcal{C}\widehat{U} - \mathcal{B}\widehat{V}(\ln f)\xi\mathcal{C}\widehat{U} + A_{\eta\mathcal{C}\widehat{V}}\mathcal{C}\widehat{U} \\ + th(\mathcal{B}\widehat{U}, \mathcal{B}\widehat{V}) + th(\mathcal{C}\widehat{U}, \mathcal{C}\widehat{V})\end{aligned} \tag{4.2}$$

for any $\widehat{U}, \widehat{V} \in \Gamma(TM)$.

Proof. Let $M = M^\theta \times_f M^T$ be a semi-slant warped product submanifold of a locally product Riemannian manifold $\bar{M}$. For any $\widehat{U}, \widehat{V} \in \Gamma(TM)$, utilizing the decomposition (3.3), we obtain that

$$(\nabla_{\widehat{U}}\xi)\widehat{V} = (\nabla_{\mathcal{B}\widehat{U}}\xi)\mathcal{B}\widehat{V} + (\nabla_{\mathcal{B}\widehat{U}}\xi)\mathcal{C}\widehat{V} + (\nabla_{\mathcal{C}\widehat{U}}\xi)\mathcal{B}\widehat{V} + (\nabla_{\mathcal{C}\widehat{U}}\xi)\mathcal{C}\widehat{V}. \tag{4.3}$$

By applying equation (2.13) to the first and last terms on the right-hand side of equation (4.3), we obtain that

$$(\nabla_{\mathcal{B}\widehat{U}}\xi)\mathcal{B}\widehat{V} = th(\mathcal{B}\widehat{U}, \mathcal{B}\widehat{V}) + A_{\eta\mathcal{B}\widehat{V}}\mathcal{B}\widehat{U} \quad \text{and}$$

$$(\nabla_{\mathcal{C}\widehat{U}}\xi)\mathcal{B}\widehat{V} = \xi\mathcal{B}\widehat{V}(\ln f)\mathcal{C} - \mathcal{B}\widehat{V}(\ln f)\xi\mathcal{C}\widehat{U}.$$

The second term of equation (4.3) becomes zero upon application of equation (2.9) and Theorem 2.2. Furthermore, for the fourth term, by employing equation (2.9) and part (i) of Theorem 2.2, we derive that

$$(\nabla_{\mathcal{C}\widehat{U}}\xi)\mathcal{C}\widehat{V} = th(\mathcal{C}\widehat{U}, \mathcal{C}\widehat{V}).$$

Building upon the aforementioned results, equation (4.3) can be formulated as follows:

$$(\nabla_{\widehat{U}}\xi)\widehat{V} = \xi\mathcal{B}\widehat{V}(\ln f)\mathcal{C}\widehat{U} - \mathcal{B}\widehat{V}(\ln f)\xi\mathcal{C}\widehat{U} + A_{\eta\mathcal{B}\widehat{V}}\mathcal{B}\widehat{U}$$
$$+ th(\mathcal{B}\widehat{U}, \mathcal{B}\widehat{V}) + th(\mathcal{C}\widehat{U}, \mathcal{C}\widehat{V}).$$

Conversely, assume that equation (4.3) holds. By considering $\widehat{U}_1, \widehat{V}_1 \in \Gamma(\mathcal{H}^\theta)$ in equation (4.3), we obtain $(\nabla_{\widehat{U}_1}\xi)\widehat{V}_1 = th(\mathcal{B}\widehat{U}_1, \mathcal{B}\widehat{V}_1) + A_{\eta\mathcal{B}\widehat{V}_1}\mathcal{B}\widehat{U}_1$. This implies that

$$g((\nabla_{\widehat{U}_1}\xi)\widehat{V}_1, \widehat{U}_2) = g(h(\mathcal{B}\widehat{U}_1, \mathcal{B}\widehat{V}_1), \eta\widehat{U}_2) + g(h(\mathcal{B}\widehat{U}_1, \widehat{U}_1), \eta\mathcal{B}\widehat{V}_1)$$

for all $\widehat{U}_2 \in \Gamma(\mathcal{H})$. Considering Theorem 3.3, we deduce that $\mathcal{H}^\theta$ is parallel. Consequently, $\mathcal{H}^\theta$ is integrable, and its leaves are totally geodesic in M. For $\widehat{U}_1 \in \Gamma(TM^\theta)$ and $\widehat{U}_2 \in \Gamma(TM^T)$, it follows directly from equation 4.3 and part (ii) of Theorem 2.2 that

$$(\nabla_{\widehat{U}_1}\xi)\widehat{U}_2 + (\nabla_{\widehat{U}_2}\xi)\widehat{U}_1 = \xi\mathcal{B}\widehat{U}_1(\ln f)\mathcal{C}\widehat{U}_2 - \mathcal{B}\widehat{U}_1(\ln f)\xi\mathcal{C}\widehat{U}_2. \tag{4.4}$$

On the other hand, in the light of equation (2.13), we yields that

$$(\nabla_{\widehat{U}_1}\xi)\widehat{U}_2 + (\nabla_{\widehat{U}_2}\xi)\widehat{U}_1 = 2th(\widehat{U}_1, \widehat{U}_2) + A_{\eta\widehat{U}_1}\widehat{U}_2. \tag{4.5}$$

From equations 4.4 and 4.5, by taking the inner product with $\widehat{V}_2 \in \Gamma(\mathcal{H})$, we obtain that

$$\xi\mathcal{B}\widehat{U}_1(\ln f)g(\mathcal{C}\widehat{U}_2, \widehat{V}_2) - \mathcal{B}\widehat{U}_1(\ln f)g(\xi\mathcal{C}\widehat{U}_2, \widehat{V}_2)$$
$$= 2g(th(\widehat{U}_1, \widehat{U}_2), \widehat{V}_2) + g(h(\widehat{U}_1, \widehat{V}_2), \eta\widehat{U}_1).$$

Since $\widehat{U}_2 \in \Gamma(D_\theta)$ this implies that $\widehat{U}_2 = \mathcal{C}\widehat{U}_2$, we have

$$\xi\mathcal{B}\widehat{U}_1(\ln f)g(\widehat{U}_2, \widehat{V}_2) - \mathcal{B}\widehat{U}_1(\ln f)g(\xi\widehat{U}_2, \widehat{V}_2)$$
$$= -2g(h(\widehat{U}_1, \widehat{U}_2)\eta\widehat{V}_1), \widehat{V}_2 + g(h(\widehat{U}_1, \widehat{V}_2), \eta\widehat{U}_1).$$

By interchanging $\widehat{U}_2$ and $\widehat{V}_2$ and summing the resulting equations, we obtain that

$$2\xi\widehat{U}_1(\ln f)g(\widehat{U}_2, \widehat{V}_2) = -g(h(\widehat{U}_1, \widehat{U}_2), \eta\widehat{V}_2) - g(h(\widehat{U}_2, \widehat{V}_2), \eta\widehat{U}_1).$$

By using Lemma 3.1, we have

$$\xi\widehat{U}_1(\ln f)g(\widehat{U}_2, \widehat{V}_2) = -g(h(\widehat{U}_2, \widehat{V}_2), \eta\widehat{V}_2).$$

Let h' denote the second fundamental form of M^T in $\bar{M}$, and let ∇' be the Riemannian connection on M^T. By applying the Gauss formula and the covariant derivative of Ψ, we obtain that

$$h^{'}(\widehat{U}_2, \widehat{V}_2) = g(\widehat{U}_2, \widehat{V}_2)\nabla\mu.$$

This implies that M^T is totally umbilical in M, with $\nabla\mu$ representing the mean curvature vector with respect to the immersion of M^T in M. Furthermore, $\nabla\mu$ is parallel, as $\widehat{V}_2\mu = 0$ for all $\widehat{V}_2 \in \Gamma(\mathcal{H})$. Consequently, the leaves of $\mathcal{H}$ are extrinsic spheres in M. Therefore, by applying Theorem 4.2 of Hiepko, $M = M^\theta \times_f M^T$ is a warped product submanifold of a locally product Riemannian manifold $\bar{M}$. □

A characterization of a semi-slant warped product submanifold in terms of the canonical structure η is established in the following theorem.

Theorem 4.4. *A semi-slant submanifold M of a locally product Riemannian manifold $\bar{M}$ is locally isometric to semi-slant warped product if and only if*

$$(\bar{\nabla}_{\widehat{U}}\eta)\widehat{V} = fh(\mathcal{C}\widehat{U}, \widehat{V}) - h(\mathcal{C}\widehat{U}, \xi\widehat{V}) - (\mathcal{B}\widehat{U}\mu)\mathcal{C}\widehat{V}, \tag{4.6}$$

fora any $\widehat{U}, \widehat{V} \in TM$ where μ is a $\mathbb{C}^\infty$-function on M such that $\widehat{Z}\mu = 0$ for any $\widehat{Z} \in \mathcal{H}$.

Proof. Let $M = M^\theta \times_f M^T$ be a semi-slant warped product submanifold of $\bar{M}$. Then, by applying equation (2.14), we deduce that

$$(\nabla_{\widehat{U}}\eta)\mathcal{C}\widehat{V} = fh(\widehat{U}, \widehat{V}) - h(\widehat{U}, \xi\widehat{V}). \tag{4.7}$$

By employing equation 2.10 in conjunction with part (ii) of Theorem 2.2, we obtain that

$$(\nabla_{\mathcal{B}\widehat{U}}\eta)\mathcal{C}\widehat{V} = -\mathcal{B}\widehat{U}(lnf)\mathcal{C}\widehat{V}. \tag{4.8}$$

In the light of equation (3.3), $(\bar{\nabla}_{\widehat{U}}\eta)\widehat{V}$ can be expressed as

$$(\bar{\nabla}_{\widehat{U}}\eta)\widehat{V} = (\bar{\nabla}_{\mathcal{B}\widehat{U}}\eta)\mathcal{B}\widehat{V} + (\bar{\nabla}_{\mathcal{B}\widehat{U}}\eta)\mathcal{C}\widehat{V} + (\bar{\nabla}_{\mathcal{C}\widehat{U}}\eta)\widehat{V}. \tag{4.9}$$

The first term on the right-hand side of the aforementioned equation vanishes since M^T is totally geodesic in M. Furthermore, by applying equations (4.7), (4.8), Theorem 2.2, and equation (2.14) to equation (4.9), we obtain that

$$(\bar{\nabla}_{\widehat{U}}\eta)\widehat{V} = fh(\mathcal{C}\widehat{U}, \widehat{V}) - h(\mathcal{C}\widehat{U}, \xi\widehat{V}) - (\mathcal{B}\widehat{U}\ln f)\mathcal{C}\widehat{V}. \tag{4.10}$$

Conversely, assume that equation (4.6) holds on a semi-slant submanifold of a locally product Riemannian manifold $\bar{M}$. Then, for any $\widehat{U}_1, \widehat{V}_1 \in \Gamma(\mathcal{H}^\theta)$, equation (2.10) yields $(\bar{\nabla}_{\widehat{U}_1}\eta)\widehat{V}_1 = 0$, which, upon applying equation (2.10), implies that $\nabla_{\widehat{U}_1}\widehat{V}_1 \in \Gamma(\mathcal{H}^\theta)$. This demonstrates that $\mathcal{H}^\theta$ is involutive on M, and its leaves are totally geodesic in M. Furthermore, for $\widehat{U}_2 \in \Gamma(\mathcal{H}^\theta)$ and $\widehat{U}_1 \in \Gamma(\mathcal{H})$, by equation (4.6), we obtain that

$$(\bar{\nabla}_{\widehat{U}_1}\eta)\widehat{U}_2 = -(\mathcal{B}\widehat{U}_1\mu)\mathcal{C}\widehat{U}_2.$$

By taking the inner product with $\widehat{V}_2 \in \Gamma(\mathcal{H})$ on both sides of the aforementioned equation, we obtain that

$$g(\nabla_{\widehat{U}_2}\widehat{U}_1, \widehat{V}_2) = \widehat{U}_1(\mu)g(\mathcal{C}\widehat{U}_2, \widehat{V}_2).$$

If h' denotes the second fundamental form of M^T in M, then it follows from the preceding result that

$$h^{'}(\widehat{U}_2, \widehat{V}_2) = g(\widehat{U}_2, \widehat{V}_2)\nabla\mu.$$

It follows that M^T is a totally umbilical submanifold of M with mean curvature vector $\nabla\mu$. Moreover, since $\nabla\mu$ is parallel (as $\widehat{U}_2\mu = 0$ for all $\widehat{U}_2 \in \mathcal{H}$), the leaves of $\mathcal{H}$ are extrinsic spheres in M. Consequently, by Theorem [4.2], M is locally isometric to a semi-slant warped product submanifold of the form $M = M^\theta \times_f M^T$. $\square$

Building on the relationship between the shape operator A and the second fundamental form h, together with our earlier analysis of the necessary and sufficient conditions involving the tensors ξ and η, we observe that a semi-slant submanifold of a locally product Riemannian manifold may reduce to a semi-slant warped product. It is therefore natural to investigate the condition, expressed in terms of the shape operator A, under which a semi-slant submanifold of a locally product Riemannian manifold becomes a semi-slant warped product. The following result establishes that this condition is both necessary and sufficient.

Theorem 4.5. *A semi-slant submanifold M of a locally product Riemannian manifold $\bar{M}$ is locally isometric to a semi-slant warped product $M = M^T \times_f M^\theta$ if and only if*

$$A_{\eta\widehat{Z}}\widehat{X} = (\xi Z \ln f)X, \tag{4.11}$$

and $((\bar{\nabla}_{\widehat{Z}}\xi)\widehat{W}, \widehat{X}) = 0$ where α is a C^∞- function on M such that $\widehat{Z}\alpha = 0$ for any $\widehat{X} \in \Gamma(\mathcal{H})$ and $\widehat{Z}, \widehat{W} \in \Gamma(\mathcal{H}^\theta)$.

Let M be a semi-slant warped product submanifold of a locally product Riemannian manifold $\bar{M}$. Then M^θ is totally geodesic in M, $(\nabla_{\widehat{Z}}\xi)\widehat{W} \in \Gamma(\mathcal{H}^\theta)$, for $\widehat{Z}, \widehat{W} \in \Gamma(\mathcal{H}^\theta)$, then we have

$$g((\nabla_{\widehat{Z}}\xi)\widehat{W}, \widehat{X}) = -g(h(\widehat{Z}, \widehat{W}), \eta\widehat{X}) = 0.$$

Now, $(\nabla_{\widehat{Z}}\xi)\widehat{W} \in \Gamma(\mathcal{H}^\theta)$, we have $g(h(\widehat{Z}, \widehat{W}), \eta\widehat{X}) = 0$ for $\widehat{X}, \widehat{Y} \in \Gamma(\mathcal{H})$ and $\widehat{Z} \in \Gamma(\mathcal{H}^\theta)$.

Proof. **of Theorem 4.5.** Let $M = M^\theta \times_f M^T$ be a semi-slant warped product submanifold of a locally product Riemannian manifold $\bar{M}$. According to part (i) of Corollary 2.1, M^θ is totally geodesic. Furthermore, from formula (2.13), we obtain

$$(\nabla_{\widehat{X}}\xi)\widehat{Z} + (\nabla_{\widehat{Z}}\xi)\widehat{X} = 2th(\widehat{X}, \widehat{Z}) + A_{\eta\widehat{Z}}\widehat{X}. \tag{4.12}$$

Based on part (ii) of Theorem 2.2 in conjunction with equation (2.9), we have

$$(\nabla_{\widehat{X}}\xi)\widehat{Z} = (\xi\mathcal{B}\widehat{Z}\ln f)\mathcal{C}\widehat{X} - (\mathcal{B}\widehat{Z}\ln f)\xi\mathcal{C}\widehat{X}, \text{ and } (\nabla_{\widehat{Z}}\xi)\widehat{X} = 0. \tag{4.13}$$

In the light of equations (4.12) and (4.13), we get

$$(\xi\mathcal{B}\widehat{Z}\ln f)\mathcal{C}\widehat{X} - (\mathcal{B}\widehat{Z}\ln f)\xi\mathcal{C}\widehat{X} = 2th(\widehat{X}, \widehat{Z}) + A_{\eta\widehat{Z}}\widehat{X}. \tag{4.14}$$

Taking the inner product in equation (4.14) with $\widehat{Y} \in \mathcal{H}$, we obtain

$$g(h(\widehat{X}, \widehat{Y}), \eta\widehat{Z})) = (\xi\mathcal{B}\widehat{Z}\ln f)g(\mathcal{C}\widehat{X}, \widehat{Y}) - (\mathcal{B}\widehat{Z}\ln f)g(\xi\mathcal{C}\widehat{X}, \widehat{Y}),$$

By using the fact $\mathcal{C}\widehat{X} = \widehat{X}, \mathcal{B}\widehat{Z} = \widehat{Z}$ and interchanging the roles of $\widehat{X}$ and $\widehat{Y}$, we can write

$$g(h(\widehat{X}, \widehat{Y}), \eta\widehat{Z})) = (\xi\widehat{Z}\ln f)g(\mathcal{C}\widehat{X}, \widehat{Y}) - (\mathcal{B}\widehat{Z}\ln f)g(\xi\widehat{Y}, \widehat{X}),$$

The aforementioned equation can also be expressed as:

$$A_{\eta\widehat{Z}}\widehat{X} = (\xi\widehat{Z}\ln f)\widehat{X},$$

$\square$

from which we derive equation (4.11).
Conversely, suppose that M is a semi-slant submanifold of a locally product Riemannian manifold such that $(\nabla_{\widehat{Z}}\xi)\widehat{W} \in \Gamma(\mathcal{H}^\theta)$ for all $\widehat{Z}, \widehat{W} \in \Gamma(\mathcal{H}^\theta)$, equipped with a C^∞-function α on M satisfying $\widehat{X}\alpha = 0$ for every $\widehat{X} \in \mathcal{H}$, and that equations (4.11) hold.
Since $(\nabla_{\widehat{Z}}\xi)\widehat{W} \in \Gamma(\mathcal{H}^\theta)$ for all $\widehat{Z}, \widehat{W} \in \Gamma(\mathcal{H}^\theta)$, it follows from Theorem 3.3 that $\mathcal{H}^\theta$ is parallel. Consequently, $\mathcal{H}^\theta$ is involutive and defines a totally geodesic foliation on M. Moreover, by assumption, the slant distribution $\mathcal{H}^\theta$ is involutive, and its leaves are totally geodesic on M.
Now, Let M^T be a leaf of $\mathcal{H}$ and h^0 be the second fundamental form of M^T into M. Then for any $\widehat{X}, \widehat{Y} \in \Gamma(\mathcal{H})$ and $\widehat{Z} \in \mathcal{H}^\theta$ with using equations (2.4), (2.5) and (2.7), we have

$$g(h^0(\widehat{X}, \widehat{Y}), J\widehat{Z}) = g(\nabla_{\widehat{X}}\xi\widehat{Y}, \widehat{Z}) + g(A_{\eta\widehat{Z}}\widehat{X}, \widehat{Y}).$$

By interchanging the roles of $\widehat{X}$ and $\widehat{Y}$, the aforementioned equation can be expressed as:

$$2g(h^0(\widehat{X}, \widehat{Y}), J\widehat{Z}) = g(\nabla_{\widehat{X}}\xi\widehat{Y} + \nabla_{\widehat{Y}}\xi\widehat{X}, \widehat{Z}) + g(A_{\eta\widehat{Z}}\widehat{X}, \widehat{Y}) + g(A_{\eta\widehat{Z}}\widehat{Y}, \widehat{X}).$$

Upon simplification using equations (4.11) and assumption of theorem, we deduce that

$$g(h^0(\widehat{X}, \widehat{Y})) = g(\widehat{X}, \widehat{Y})g(\nabla\mu, J\widehat{Z}),$$

from which we finally obtain

$$h^0(\widehat{X}, \widehat{Y}) = g(\widehat{X}, \widehat{Y})\nabla\mu$$

for all $\widehat{X}, \widehat{Y} \in \Gamma(\mathcal{H})$ and $\widehat{Z} \in \mathcal{H}^\theta$. This implies that M^T is totally umbilical in M with mean curvature vector $\nabla\mu$. Furthermore, $\nabla\mu$ is parallel since $\widehat{X}\mu = 0$ for all $\widehat{X} \in \mathcal{H}$. Consequently, the leaves of $\mathcal{H}$ are extrinsic spheres in M. Therefore, by virtue of Theorem 4.2, M is locally isometric to a semi-slant warped product submanifold $M = M^\theta \times_f M^T$. This completes the proof of the theorem.

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