



## TRIADIC MORLEY: A SIZE-SHAPE-PHASE DECOMPOSITION OF A TRIANGLE

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**ABSTRACT.** Morley's theorem states that the intersections of adjacent internal angle trisectors of any nondegenerate triangle form an equilateral triangle. This configuration is recast as a triadic decomposition into a scalar *size*  $Sz$ , a *shape*  $Sh$  (similarity class), and a finite *phase* label  $\Phi$  that records trisector adjacency and orientation. On the *Size-Shape* edge, Morley acts as a natural projection: with  $Sz = R$  (circumradius), the Morley side length is

$$\text{side}(M(T)) = 8R \sin \frac{A}{3} \sin \frac{B}{3} \sin \frac{C}{3},$$

i.e. a multiplicative size-shape coupling and a projection of the 2-dimensional shape to a 1-dimensional equilateral orbit.

Finite  $\Phi = (\varepsilon, \sigma, \text{Adj})$  is formalized and a *generic local inverse* ("Morley Lift") is proved: for an oriented equilateral carrier  $E$  and compatible  $\Phi$ , the preimage triangle  $T$  is locally unique up to orientation-preserving similarity outside a proper real-analytic exceptional set. A constructive Jacobian-based system and numerical certificate are given. Complementary Morley-type couplings on the other edges— $(Sh, \Phi) \Rightarrow Sz$  (a single closure) and  $(Sz, \Phi) \Rightarrow Sh$  (discrete, generically unique)—complete the triad. The same lift applies to the standard *external* Morley triangle.

### 1. INTRODUCTION AND MOTIVATIONS

Morley's theorem asserts that, for any nondegenerate triangle, the three points of intersection of adjacent internal angle trisectors form an equilateral triangle; see, e.g., [1, 2]. The present paper examines this classical construction from a triadic point of view, separating the information carried by a triangle into three complementary components: a scalar *size*, a *shape* (its similarity class), and a finite *phase* datum extracted from the trisector configuration.

The guiding question is: *how does Morley's construction reorganize the information in a triangle?*

From the similarity viewpoint, a triangle is encoded by its angle triple

$$Sh(T) = (A, B, C) \in \mathcal{S} := \{(A, B, C) \in (0, \pi)^3 : A + B + C = \pi\},$$

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together with a positive size parameter  $Sz(T) \in \mathbb{R}_{>0}$  (e.g. the circumradius  $R$ ). Augmenting this by the finite phase label  $\Phi(T)$  from §2.4 gives a triadic “phase space”

$$\mathcal{M} := \mathbb{R}_{>0} \times \mathcal{S} \times \Phi,$$

and Morley’s construction can be viewed as a real-analytic, similarity-equivariant map

$$M : \mathcal{M} \longrightarrow \mathcal{E},$$

where  $\mathcal{E}$  is the 1-dimensional orbit of equilateral triangles (the “equilateral axis” of Fig. 2). The present work describes the fibres of this map and shows how the finite phase factor restores bijectivity on a generic subset.

Interpreting Morley as a map

$$T \longmapsto M(T),$$

one observes that the image has equilateral shape independently of the preimage, while its scale depends on both size and shape of  $T$ . With an explicit choice of size functional (e.g., circumradius  $R$ ), the scale of the Morley triangle admits a simple closed form in the vertex angles; see Eq. (3.1) in Section 3, [7, Ch. 12]. This exhibits Morley as a natural *size–shape projection* onto an equilateral carrier, and it makes clear that information is lost in passing from  $T$  to  $M(T)$ .

To account for the lost degrees of freedom, a canonical discrete *phase* label is introduced: a combinatorial record of the cyclic order of the trisectors at each vertex together with the adjacency data specifying which trisector pairs meet at the Morley vertices. The main structural statement is that the pair consisting of the equilateral carrier and this phase label recovers the original triangle up to similarity:

$$T \longleftrightarrow (M(T), \Phi(T)).$$

In particular, Morley’s construction provides a natural coupling on the *Size–Shape* edge of the triad, while the phase label supplies complementary couplings on the other two edges, where any two among {size, shape, phase} determine the third under suitable compatibility conditions.

This viewpoint is consonant with standard treatments of similarity and shape geometry [3, 4]—here specialized to the elementary Euclidean setting of a single triangle and kept strictly geometric. The remainder of the paper is organized as follows. Section 2 fixes notation (size functionals, shape encodings, trisectors, and phase labels). Section 3 formalizes Morley as a size–shape projection and quantifies the associated information loss. Section 4 introduces the phase datum and proves a reconstruction (“Morley Lift”) from equilateral carrier plus phase. Section 5 develops the complementary size–phase and shape–phase couplings. Section 6 collects brief generalizations and remarks; appendices contain explicit formulas.

The emphasis here is geometric and elementary: all arguments use Euclidean similarity, angle trisectors, and basic trigonometry. The novelty is structural—viewing Morley as a size–shape projection with an explicit shape factor and furnishing a finite phase label that restores a locally unique preimage. This keeps the scope within classical triangle geometry while isolating a clean inverse problem and its regularity.

Related work and novelty. Classical accounts focus on proofs and properties of the Morley triangle and on analytic size formulas; see [1, 2, 7]. To the author’s knowledge there is no prior formal treatment of an *inverse* Morley problem (reconstruction of  $T$  from the

equilateral carrier plus a finite combinatorial label), nor of a triadic organization separating *Size*, *Shape*, and a discrete *Phase* label with complementary couplings. The new contributions are: (i) a precise Size–Shape projection viewpoint with an explicit shape factor and information-fiber picture; (ii) a generic local inverse (“Morley Lift”) with a constructive solver and a rigorous rank condition; and (iii) complementary Shape–Phase and Size–Phase couplings that close the triad in an elementary, Euclidean framework. From the analytic viewpoint, the Morley map and its inverse lift can be read as a structured functional equation on size–shape data with a discrete  $\mathbb{Z}_3$ -parameter, sitting within the classical Euclidean theory of triangle transformations and moduli.

## 2. PRELIMINARIES: TRIANGLES, SIZE, SHAPE, AND TRIADIC DATA

**2.1. Similarity classes and shape.** Let  $T \subset \mathbb{R}^2$  be a nondegenerate oriented triangle with ordered vertices  $A, B, C$ . Two triangles are (positively) similar if they are related by an orientation-preserving similarity of the plane. The *shape space*  $\mathcal{S}$  is the set of equivalence classes under this relation; a convenient coordinate on  $\mathcal{S}$  is the ordered triple of angles  $(A, B, C)$  with  $A + B + C = \pi$ . (Other encodings—e.g. normalized side triples—are equivalent for our purposes; cf. [3, 4].) Note that we work modulo orientation-preserving similarities; mirror images are distinct unless stated otherwise.

**Remark 2.1.** *Absolute orientation (clockwise vs. counterclockwise) is not part of Size or (similarity) Shape. It will be recorded within the phase/Hodophase datum introduced in §2.4. In particular, absolute orientation is recorded in  $\Phi$  (the factor  $\varepsilon$ ), not in Sz or Sh.*

**2.2. Choice of size functional.** Fix a canonical size functional  $Sz : \{\text{triangles}\} \rightarrow \mathbb{R}_{>0}$  (for examples, circumradius  $R$ , a side length  $a$ , semiperimeter  $s$ , or area  $K$ ). We write  $Sh(T) \in \mathcal{S}$  for the similarity class. Thus a triangle is determined, up to an orientation-preserving Euclidean isometry, by the pair

$$(Sh(T), Sz(T)).$$

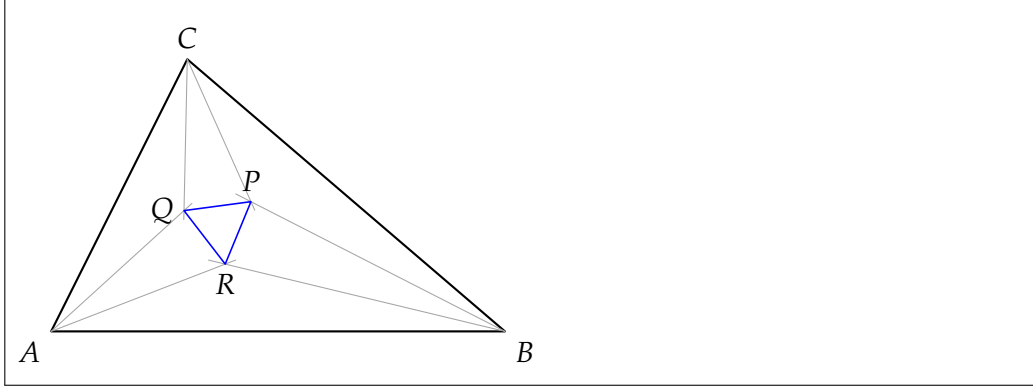
**2.3. Angle trisectors and the Morley configuration.** At each vertex, the internal angle is divided into three equal parts by two rays (the *internal trisectors*). For each side, consider the two trisectors adjacent to that side—one issued from each endpoint of the side. The intersections of these adjacent pairs define three interior points, which form the classical Morley triangle  $M(T)$ ; by Morley’s theorem,  $M(T)$  is equilateral (see Fig. 1 and [1, 2, 7]).

**2.4. Phase data (formal definition).** Fix an orientation and the vertex order  $A \rightarrow B \rightarrow C$ . At each vertex  $V \in \{A, B, C\}$  let  $(t_V^1, t_V^2)$  denote the ordered pair of *internal* trisector rays, where  $t_V^1$  is the one adjacent to the preceding side in the chosen cyclic order and  $t_V^2$  the one adjacent to the following side. Define the *phase label*

$$\Phi = (\varepsilon, \sigma, \text{Adj}),$$

with:

- $\varepsilon \in \{\pm 1\}$  (global chirality):  $+1$  for the chosen orientation,  $-1$  for its mirror;
- $\sigma \in \mathbb{Z}_3$  (rotation class): the cyclic shift aligning  $(P, Q, R)$  with the vertices of the equilateral carrier  $E$ ;
- $\text{Adj} = (\alpha_{BC}, \alpha_{CA}, \alpha_{AB})$  where, e.g.,  $\alpha_{BC} \in \{(t_B^i, t_C^j) : i, j \in \{1, 2\}\}$  specifies *which* adjacent pair meets on side  $BC$ , and similarly cyclically.



**Figure 1.** Classical Morley configuration and notation. Internal trisectors at  $A, B, C$ ; adjacent intersections form the equilateral  $M(T)$ .

Two phase labels are *compatible* with an oriented carrier  $E$  if the three prescribed adjacent pairs meet on the three sides of  $E$  in the cyclic order determined by  $\sigma$  and the sign  $\varepsilon$ .

**Example 2.1** (Standard adjacency). *With vertex order  $A \rightarrow B \rightarrow C$ , the standard choice is*

$$\text{Adj}_{\text{std}} = ((t_B^2, t_C^1), (t_C^2, t_A^2), (t_A^1, t_B^1)),$$

*corresponding to the usual adjacent pairs (near  $BC$ , near  $CA$ , near  $AB$ ).*

Thus  $\Phi \cong \{\pm 1\} \times \mathbb{Z}_3 \times (\text{local adjacency data})$ . This will be the sole discrete datum used throughout.

**Remark 2.2** (Internal vs. external trisectors). *Unless stated otherwise we use internal trisectors. The definition of  $\Phi$  and the lift mechanism extend verbatim to the standard external Morley configuration by changing the adjacency choices in  $\text{Adj}$  and the rotation class  $\sigma$ . Compatibility is always tested against the oriented carrier  $E$ .*

### 3. MORLEY AS A NATURAL SIZE–SHAPE PROJECTION

**3.1. Morley map.** Let  $M(T)$  denote the Morley triangle of  $T$ , with its induced orientation. By Morley’s theorem,  $M(T)$  is equilateral for every nondegenerate  $T$  (see, e.g., [1, 2]). We view  $M$  as a transformation of size–shape data, summarized schematically in Fig. 2.

#### 3.2. Naturality.

**Proposition 3.1** (Functoriality). *If  $T_1 \sim T_2$  are related by an orientation-preserving similarity with ratio  $\lambda > 0$ , then  $M(T_1) \sim M(T_2)$  with the same ratio  $\lambda$ .*

*Proof.* Similarities preserve angle measures and adjacency, sending internal trisectors to internal trisectors and adjacent intersections to adjacent intersections; lengths scale by  $\lambda$ .  $\square$

Hence  $M$  descends to a well-defined map on size–shape data:

$$(\text{Sh}(T), \text{Sz}(T)) \longmapsto (\text{Sh}(M(T)), \text{Sz}(M(T))),$$

with  $\text{Sh}(M(T))$  the equilateral shape.

**3.3. Explicit size formula.** Fix  $Sz$  to be the circumradius  $R$  of  $T$ . The side length of the Morley triangle satisfies

$$\text{side}(M(T)) = 8R \sin \frac{A}{3} \sin \frac{B}{3} \sin \frac{C}{3}, \quad (3.1)$$

see, for instance, [7, Ch. 12]. Thus the output scale factors into a scalar size  $R$  and a shape-dependent term in  $(A, B, C)$  with  $A + B + C = \pi$ .

Shape factor and fibers. Write

$$f(A, B, C) := \sin \frac{A}{3} \sin \frac{B}{3} \sin \frac{C}{3}, \quad A + B + C = \pi, \quad A, B, C \in (0, \pi). \quad (3.2)$$

Equation (3.1) is then  $\text{side}(M(T)) = 8R f(A, B, C)$ . On the normalized slice  $\{Sz(T) = R = 1\}$  the coordinate on the equilateral orbit (the “size axis” in Fig. 2) is

$$s_M(T) = 8f(A, B, C). \quad (3.3)$$

Thus *different* triangles land at *different* points of the axis precisely when their angle triple yields a different value of  $f$ . For a fixed output size  $s^*$ , the preimage is the level set

$$\{(A, B, C) : f(A, B, C) = s^*/8, \quad A + B + C = \pi\},$$

typically a one-dimensional curve in shape space (hence an infinite fiber), illustrated in Fig. 2. Moreover,

$$0 < f(A, B, C) \leq \sin^3\left(\frac{\pi}{9}\right), \quad (3.4)$$

with equality if and only if  $A = B = C = \pi/3$  (by symmetry/concavity of  $\log \sin$  on  $(0, \pi/3)$ ). Therefore  $0 < s_M \leq 8R \sin^3(\pi/9)$ , and  $s_M$  increases as the angles become more balanced. The phase components  $\varepsilon, \sigma \in \Phi$  rotate or mirror the equilateral carrier but do not affect  $s_M$ , so the axis encodes only size–shape data. Since  $A/3, B/3, C/3 \in (0, \pi/3) \subset (0, \pi/2)$ , the function  $\log \sin x$  is strictly concave on the relevant interval.

In particular, the Morley image forgets the two continuous shape parameters and retains only the scalar  $s_M = 8Sz f(A, B, C)$ ; level sets of  $f$  therefore describe equilateral fibers in shape space.

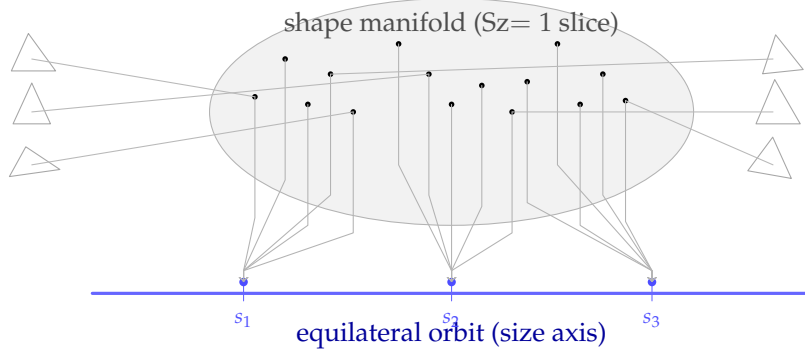
**Lemma 3.1** (Range and maximizer of the shape factor). *Let  $f(A, B, C) = \sin(A/3) \sin(B/3) \sin(C/3)$  with  $A, B, C \in (0, \pi)$  and  $A + B + C = \pi$ . Then  $0 < f \leq \sin^3(\pi/9)$ , with equality iff  $A = B = C = \pi/3$ .*

*Proof.* The function  $\log \sin x$  is strictly concave on  $(0, \pi/2)$ ; by Jensen,  $\frac{1}{3} \sum_{cyc} \log \sin(A/3) \leq \log \sin\left(\frac{A+B+C}{9}\right) = \log \sin\left(\frac{\pi}{9}\right)$ , with equality only at  $A = B = C$ . Exponentiating gives the claim.  $\square$

### 3.4. Information loss.

**Proposition 3.2.** *The map  $T \mapsto M(T)$  is not injective up to similarity.*

*Discussion.* The image shape is always equilateral, so the two degrees of freedom of input shape collapse in  $\text{Sh}(M(T))$ . Moreover, the scale  $Sz(M(T))$  is controlled by  $R$  together with  $\sin(A/3) \sin(B/3) \sin(C/3)$  in (3.1). By varying  $(A, B, C)$  and compensating by a global rescaling of  $T$ , distinct shapes yield the same output size, producing many-to-one behavior (cf. Fig. 2).  $\square$



**Figure 2.** Projection viewpoint on the normalized slice ( $S_z = 1$ ). The Morley map sends the 2D shape manifold to the 1D equilateral orbit (size axis). Each marked size  $s_i$  on the orbit has an infinite preimage (typically a 1D level set in shape space); dots show samples.

#### 4. PHASE DATA AND THE MORLEY LIFT

**4.1. Phase data: recap and notation.** For ease of reference to §2.4, fix an orientation and the vertex order  $A \rightarrow B \rightarrow C$ . We write

$$\Phi(T) = (\varepsilon, \sigma, \text{Adj}), \quad \varepsilon \in \{\pm 1\}, \quad \sigma \in \mathbb{Z}_3,$$

where  $\varepsilon$  encodes chirality (flips under reflection),  $\sigma$  is the sector-pairing class (rotates the equilateral carrier  $E$  by  $k \cdot 60^\circ$ ), and  $\text{Adj}$  collects the local adjacency choices at  $P, Q, R$ .

#### 4.2. Morley Lift: statement and proof sketch.

**Definition 4.1** (Compatibility). *Given an oriented equilateral carrier  $E = (E_1, E_2, E_3)$  and  $\Phi = (\varepsilon, \sigma, \text{Adj})$ , we say  $(E, \Phi)$  is compatible if, after reindexing  $E$  by  $\sigma$  and (if  $\varepsilon = -1$ ) reflecting, the three prescribed adjacent trisector pairs (one per side) meet on the corresponding sides of  $E$  in the indicated cyclic order.*

**Theorem 4.1** (Morley Lift: generic local uniqueness). *Let  $E$  be an oriented equilateral carrier and  $\Phi = (\varepsilon, \sigma, \text{Adj})$  be compatible with  $E$ . There exists a real-analytic residual map*

$$G : (A, B, R, \phi, t_x, t_y) \mapsto \mathbb{R}^6$$

*whose zero set encodes the congruence  $PQR \cong E$  with the prescribed adjacencies. If  $G$  has a solution  $u^*$ , then outside a proper real-analytic exceptional set  $\mathcal{S}$  in the data, the Jacobian  $DG(u^*)$  has full rank 6; hence  $u^*$  is locally unique and the outer triangle  $T$  is determined up to orientation-preserving similarity.*

**Proposition 4.1** (Jacobian nonvanishing is generic). *The determinant of  $DG$  is a nonzero real-analytic function on the open domain  $\{0 < A, B, A + B < \pi, R > 0\}$ . In particular, it does not vanish identically.*

*Proof idea.* Entries of  $DG$  are algebraic combinations of sines and cosines of linear forms in  $(A, B)$  and affine in  $(R, \phi, t_x, t_y)$ . Evaluating  $DG$  at the worked instance of §4.5 yields a numerically nonsingular matrix, so  $\det DG$  is not the zero function; hence its zero set is a proper real-analytic subset.  $\square$

Degeneracies of  $DG$  occur on symmetry loci (e.g. isosceles or near-degenerate angles), together with other analytic loci where specific trigonometric combinations vanish, defining the proper real-analytic exceptional set  $\mathcal{S} \subset \{0 < A, B, A + B < \pi, R > 0\}$ . Away from  $\mathcal{S}$ , the lift is a real-analytic local diffeomorphism (modulo translations and rotations) between the data  $(A, B, R, \phi, t_x, t_y)$  and the congruence class of the outer triangle  $T$ .

**Numeric certificate.** For the worked instance in §4.5, Appendix C records the numerical value of  $\det DG(u^*)$  and the condition number  $\kappa(DG(u^*))$  (Table C.1), certifying that  $\det DG(u^*) \neq 0$  at one explicit point. This demonstrates that  $\det DG$  is not identically zero, hence its zero set is a proper real-analytic subset.

**Compatibility and normalization.** Assume  $\Phi$  is compatible with  $E$  in the sense that the three adjacent-trisector pairings encoded by  $(\sigma, \text{Adj})$  meet on the three sides of  $E$  in the prescribed cyclic order, and that the global chirality  $\varepsilon$  matches the chosen orientation of  $E$ . Fix a size normalization (e.g.  $\text{side}(E) = 1$ ).

*Proof sketch.* Parameterize the unknown outer triangle by its angles  $(A, B, C)$  with  $A + B + C = \pi$  and a scalar size  $Sz$  (e.g.  $R$ ). Place  $E$  in coordinates. For each side of  $E$ , the data  $\text{Adj}$  dictate *which* trisector from each endpoint of the corresponding side of  $T$  must meet on that side. Writing the two trisector directions at a vertex  $V \in \{A, B, C\}$  as  $V/3$  and  $2V/3$ , one obtains three intersection conditions (one per side of  $E$ ) that constrain  $(A, B, C)$  together with the overall placement/scale of  $T$ .

Quotienting by orientation-preserving similarities, we fix a rigid frame for  $T$  (e.g.  $A = (0, 0)$ ,  $B = (1, 0)$ ) and express  $C$  via  $(A, B, C)$ . The three intersection constraints become a smooth square system

$$F(A, B, C) = 0, \quad A + B + C = \pi,$$

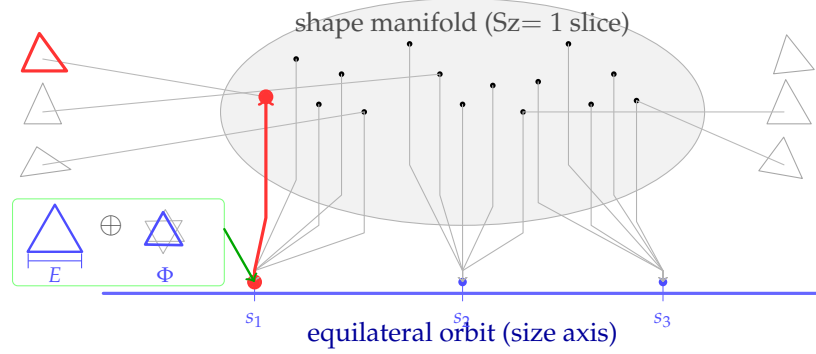
whose Jacobian with respect to  $(A, B)$  is nonsingular on an open dense set (degeneracies occur at symmetry/limit cases such as isosceles/near-degenerate shapes). Hence, by the implicit function theorem, the system has a unique local solution  $(A, B, C)$ , which yields  $T$  uniquely up to similarity; the size  $Sz$  then follows from Eq. (3.1). A constructive solver (variables, equations, and Jacobian) is detailed in Appendix B.  $\square$   $\square$

### 4.3. Interpretation and structural consequences.

**Corollary 4.1.** *The assignment  $T \mapsto (M(T), \Phi(T))$  is bijective up to orientation-preserving similarity. If reflections are also quotiented, uniqueness holds up to full similarity and the factor  $\varepsilon$  becomes redundant.*

**Corollary 4.2** (External Morley). *The same triadic lift holds for the standard external Morley triangle: for an oriented equilateral carrier  $E$  arising from adjacent external trisectors and a compatible phase label  $\Phi$  (with  $\text{Adj}$  adjusted accordingly), the preimage  $T$  is locally unique up to orientation-preserving similarity on the same generic set.*

**Role of the phase factors.** The pair  $(\varepsilon, \sigma)$  determines the placement of  $E$  relative to  $ABC$  up to a rigid  $60^\circ$  rotation and a mirror; the local adjacency component  $\text{Adj}$  disambiguates which two rays meet at each of  $P, Q, R$ . None of these affect the scale of  $E$  (which is fixed by size-shape via Eq. (3.1)).



**Figure 3.** Mirror of Fig. 2 with one branch emphasized. The leftmost fiber over  $s_1$  (bold red) is singled out; its axis point and corresponding exemplar are highlighted. The bottom-left widget depicts the inputs  $E \oplus \Phi$  that, in the lift picture, select a unique preimage within that fiber.

Genericity and degeneracy. Failure of uniqueness corresponds to nongeneric loci where the Jacobian in the lift equations loses rank (e.g. symmetric/isosceles configurations or near-degenerate angles). These loci are of positive codimension in shape space; outside them, the lift is well-posed and unique.

Algorithmic corollary. With  $E$  and  $\Phi$  given, one solves the square system from Appendix B by a damped Newton method starting from an angle-balanced initial guess; transversality guarantees local convergence on the generic set. This provides a concrete reconstruction map  $L : (E, \Phi) \mapsto T$  inverse to  $M$  on the stated quotient.

The lift viewpoint is illustrated in Fig. 3. Numerical certificates (residual, nonzero Jacobian determinant, conditioning) for the worked instance, together with batch-test parameters, are documented in Appendix C.

**4.4. Degenerate and symmetric loci.** Loss of uniqueness occurs on positive-codimension subsets of shape space, e.g. isosceles configurations (two equal angles), nearly degenerate angles (one angle tending to 0 or  $\pi$ ), and related analytic loci where the determinant of  $DG$  vanishes. On such loci, symmetry forces  $DG$  to lose rank or introduces multiple solutions related by the symmetry. These sets form part of  $\mathcal{S}$  in Theorem 4.1. Away from  $\mathcal{S}$  the lift is well-posed and locally unique, and the map

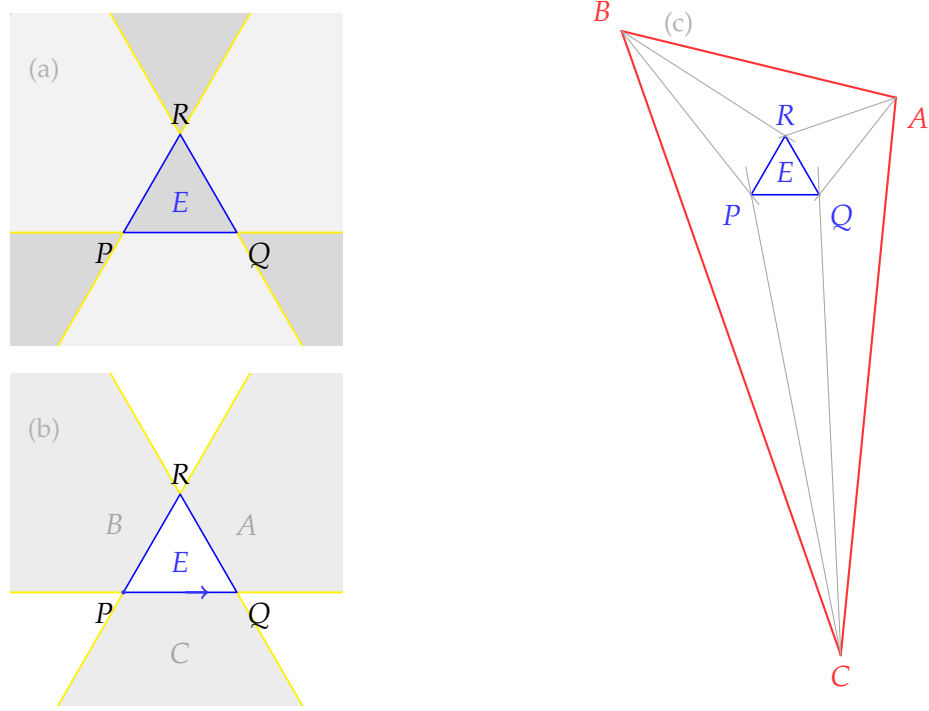
$$(Sz, Sh, \Phi) \longmapsto [T]$$

is a real-analytic local diffeomorphism from triadic data to triangle congruence classes.

**4.5. Worked example: a concrete Morley lift on a unit equilateral.** Fix the equilateral carrier  $E$  with side  $s_M = 1$  and counterclockwise orientation ( $\varepsilon = +1$ ) (c.f. Fig. 4). Choose the phase label

$$\Phi = (\varepsilon, \sigma, \text{Adj}) = (+1, 2, \text{Adj}_{\text{std}}),$$

where  $\sigma = 2$  denotes the  $120^\circ$  rotation class and  $\text{Adj}_{\text{std}}$  encodes the usual adjacent pairs (along  $AB$ :  $A_1$  with  $B_1$ ; along  $BC$ :  $B_2$  with  $C_1$ ; along  $CA$ :  $A_2$  with  $C_2$ ), cf. Fig. 1.



**Figure 4.** Two-step Morley lift from an equilateral carrier. Left column: (a) For a fixed equilateral  $E$  (blue), both internal (light) and external (darker) Morley configurations give continuous families of possible outer vertices at each corner; yellow rays mark schematic boundaries between the internal and external sectors. (b) Restricting to the internal, oriented Morley triangle keeps only the light fans and identifies the regions where the lifted vertices  $A, B, C$  must lie after Step 1. Right column: (c) For the worked example of §4.5, the phase data  $\Phi$  select a single point in each fan, yielding a unique outer triangle  $T$  (red) with  $M(T) = E$ ; the gray rays are the actual internal trisectors realizing the chosen Adj.

Unknowns and equations. Let  $(A, B, C)$  be the angles of the unknown outer triangle  $T$  with  $A + B + C = \pi$ , and let  $S_z$  be a size (we take the circumradius  $R$  for convenience). Placing  $E$  in a rigid frame, the three adjacency constraints implied by  $\Phi$  give a smooth square system

$$F(A, B) = 0, \quad C = \pi - A - B,$$

expressing that the prescribed trisector pairs intersect exactly on the three sides of  $E$  in the required cyclic order.

Numerical solution (Newton). Starting from a balanced seed  $(A_0, B_0) = (\frac{\pi}{3}, \frac{\pi}{3})$  and using a damped Newton iteration on  $F = 0$ , one obtains the unique solution (on the generic branch)

$$(A, B, C) \approx (98^\circ, 57^\circ, 25^\circ).$$

With  $E$  fixed at  $s_M = 1$ , Eq. (3.1) then yields the required circumradius of  $T$ :

$$s_M = 8R \sin \frac{A}{3} \sin \frac{B}{3} \sin \frac{C}{3} = 1 \implies R \approx 4.908073425.$$

Consequently the side lengths of  $T$  (up to an overall similarity consistent with  $s_M = 1$ ) are

$$a = 2R \sin A \approx 9.7206, \quad b = 2R \sin B \approx 8.2325, \quad c = 2R \sin C \approx 4.1485.$$

Consistency checks. (i) Substituting  $(A, B, C)$  into (3.1) reproduces  $s_M = 1$  to numerical precision. (ii) The three intersection constraints defined by  $\Phi$  are satisfied simultaneously by these angles (the Jacobian of  $F$  in  $(A, B)$  is nonsingular here), confirming uniqueness up to orientation-preserving similarity. (iii) Changing  $\sigma$  rotates  $E$  by  $k \cdot 60^\circ$  and yields the same  $T$  modulo a rigid rotation; flipping  $\varepsilon$  mirrors  $T$ .

Remark. Isosceles and other symmetry loci can be used for visualization, but the Jacobian may lose rank there; the above scalene instance typifies the generic (well-posed) lift.

## 5. THE OTHER TRIADIC COUPLINGS

**5.1. Triadic framework.** We organize triangle data as a triad  $(Sz, Sh, \Phi)$ , where  $Sz > 0$  is a chosen size functional (e.g.  $R$ ),  $Sh \in \mathcal{S}$  is the similarity class (angles  $(A, B, C)$ ,  $A + B + C = \pi$ ), and  $\Phi = (\varepsilon, \sigma, \text{Adj})$  is the phase/Hodophase datum from §2.4. The Morley map realizes the *Size–Shape* coupling by sending  $(Sz, Sh)$  to an equilateral carrier  $E$  whose side equals  $8Sz \sin \frac{A}{3} \sin \frac{B}{3} \sin \frac{C}{3}$  (Eq. (3.1)); the other two edges are complementary couplings driven by  $\Phi$ .

### 5.2. (Shape, Phase) $\rightarrow$ Size.

**Lemma 5.1** (Shape–Phase coupling). *Fix a shape  $Sh(T)$  (angles  $(A, B, C)$ ) and compatible phase data  $\Phi$ . Then for any target equilateral carrier  $E$  (or metric target) there exists at most one scale  $Sz$  for which the prescribed adjacent trisectors meet on  $E$  in the cyclic order encoded by  $\Phi$ .*

*Idea.* With  $(A, B, C)$  fixed, the directions of all trisectors are fixed; distances from the vertices to the Morley points scale linearly with  $Sz$ . The adjacency/placement constraints on  $E$  therefore reduce to a single scalar closure condition linear in  $Sz$ , hence at most one solution. In particular, for  $E$  specified by its side  $s_M$ , Eq. (3.1) gives  $Sz = \frac{s_M}{8 \sin(A/3) \sin(B/3) \sin(C/3)}$ . Thus the size is fixed by a single scalar closure; if a solution exists, it is unique.  $\square$

Worked example (Shape–Phase  $\rightarrow$  Size). Use the angles from Sec. 4.5,  $(A, B, C) \approx (98^\circ, 57^\circ, 25^\circ)$ , together with  $\Phi = (+1, 2, \text{Adj}_{\text{std}})$ , and target equilateral carrier  $E$  with side  $s_M = 1$ . From Eq. (3.1), with  $Sz = R$ ,

$$s_M = 8R \sin \frac{A}{3} \sin \frac{B}{3} \sin \frac{C}{3} \implies R = \frac{s_M}{8 \sin(A/3) \sin(B/3) \sin(C/3)}.$$

Numerically,

$$\sin(A/3) \sin(B/3) \sin(C/3) \approx 0.02546824164, \quad 8f \approx 0.2037459331,$$

so

$$R \approx \frac{1}{0.2037459331} = 4.908073425.$$

Consequently, the sides of  $T$  are

$$a = 2R \sin A \approx 9.7206, \quad b = 2R \sin B \approx 8.2325, \quad c = 2R \sin C \approx 4.1485,$$

in agreement with Sec. 4.5. The phase parameters  $(\varepsilon, \sigma)$  place  $E$  (mirror/rotation) but do not alter  $R$ .

### 5.3. (Size, Phase) $\rightarrow$ Shape.

**Lemma 5.2** (Size–Phase coupling). *Fix a size  $Sz$  and phase pattern  $\Phi$ . Then the realizable shapes  $Sh$  form at most a discrete set; generically there is a unique  $Sh$  up to similarity.*

*Idea.* Place  $E$  according to  $\Phi$  and the chosen normalization. The three adjacency constraints (one per side of  $E$ ) impose a smooth square system  $F(A, B) = 0$  with  $C = \pi - A - B$ . For generic configurations the Jacobian  $D_{A,B}F$  has full rank, so solutions are isolated by the implicit function theorem; uniqueness holds on an open dense set (degeneracies occur on symmetry loci), cf. the discussion in §4. Hence solutions are isolated; generically there is exactly one similarity class.  $\square$

Worked example (Size–Phase  $\rightarrow$  Shape). Fix the size to the value recovered above,  $R = 4.908073425$ , and keep the same phase pattern  $\Phi = (+1, 2, \text{Adj}_{\text{std}})$ . With  $E$  normalized by Eq. (3.1), the three adjacency constraints (one per side of  $E$ ; see Appendix B) define a smooth square system

$$F(A, B) = 0, \quad C = \pi - A - B,$$

whose generic solution is isolated. A damped Newton solve from a balanced seed (e.g.  $A_0 = B_0 = \pi/3$ ) returns

$$(A, B, C) \approx (98^\circ, 57^\circ, 25^\circ),$$

unique up to similarity. Substituting these angles into Eq. (3.1) with  $R = 4.908073425$  yields  $s_M \approx 1$  to numerical precision, confirming consistency. Changing  $\sigma$  rotates  $E$  by  $k \cdot 60^\circ$  and produces the same  $T$  modulo a rigid rotation; flipping  $\varepsilon$  mirrors the configuration.

### 5.4. Summary statement.

**Theorem 5.1** (Triadic Morley summary). *On the moduli space of oriented similarity classes, each nondegenerate triangle determines triadic data*

$$[T] \mapsto (Sz(T), Sh(T), \Phi(T)) \in \mathbb{R}_{>0} \times \mathcal{S} \times \Phi.$$

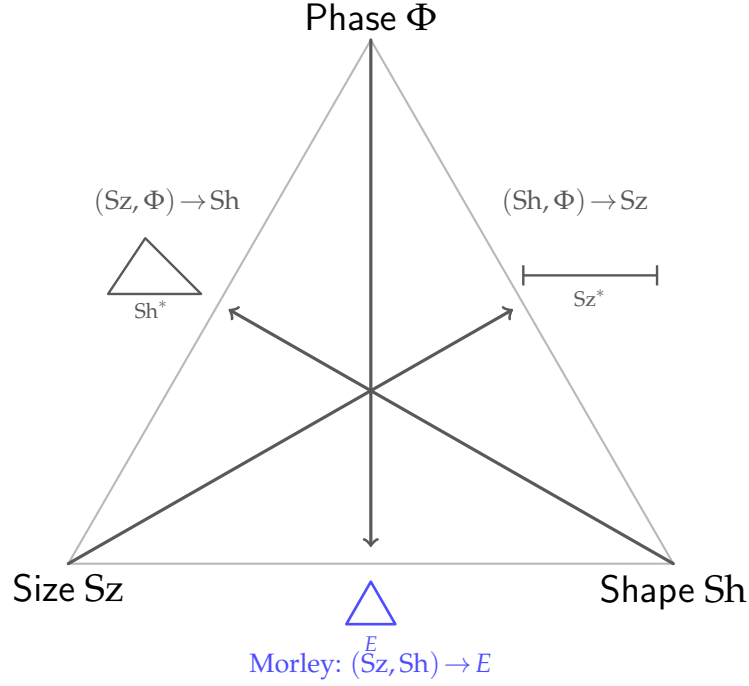
*The Morley construction yields a natural Size–Shape coupling  $(Sz, Sh) \mapsto E$  (Eq. (3.1)). Together with the phase datum  $\Phi$ , the triad obeys the complementary couplings*

$$(Sh, \Phi) \rightsquigarrow Sz \quad \text{and} \quad (Sz, \Phi) \rightsquigarrow Sh,$$

*so that, under the generic compatibility hypotheses of §4, any two among  $\{Sz, Sh, \Phi\}$  determine the third, and hence determine  $[T]$  up to orientation-preserving similarity. Equivalently, outside a proper real-analytic exceptional set  $\mathcal{S} \subset \mathcal{S}$ , the assignment*

$$[T] \longleftrightarrow (Sz(T), Sh(T), \Phi(T))$$

*is bijective, and its inverse is given locally by the Morley Lift of §4. Figure 5 and Table 1 summarize these relations.*



**Figure 5.** Equilateral triad and the three collapses. Removing a vertex contracts to the opposite edge: *Phase* collapses to  $(Sz, Sh)$  and yields the Morley carrier  $E$ ; *Size* collapses to  $(Sh, \Phi)$  fixing a canonical size  $Sz^*$  (closure); *Shape* collapses to  $(Sz, \Phi)$  fixing a canonical shape  $Sh^*$  (realizability).

## 6. DISCUSSION

**6.1. Other sectorings and centers.** The triadic split  $(Sz, Sh, \Phi)$  extends without change to several close variants. First, one may use *external* or mixed (internal–external) trisectors: the adjacent intersections again produce equilateral configurations, and the size factor retains the form (3.1) with the same shape factor  $f(A, B, C)$  while the placement information is absorbed into  $\Phi$  (cf. [7, Ch. 12]). Second, analogous constructions arise from other canonical partitions and centers (e.g. Ceva-type cevians through classical centers); see [5] for a comprehensive catalogue. Formally, replacing 3-sectors by  $k$ -sectors suggests a cyclic phase  $\mathbb{Z}_k$  with the same logic: Morley’s role is played by the corresponding  $k$ -equilateral carrier, while  $(Sz, Sh)$  feed a scale and  $\Phi$  selects a congruence class on the carrier. For external or mixed trisectors the Morley side retains the same shape factor  $f(A, B, C)$ ; only the placement data move between  $E$  and  $\Phi$ .

**Proposition 6.1** (Shape-factor stability for Morley-type variants). *For any choice among the classical internal, external, or mixed adjacent trisectors that produces an equilateral Morley-type carrier  $E$ , and for any size choice  $Sz \in \{R, a, p, s, K\}$  of  $\S A$ , the side length of  $E$  has the form*

$$\text{side}(E) = 8 Sz f(A, B, C), \quad f(A, B, C) = \sin \frac{A}{3} \sin \frac{B}{3} \sin \frac{C}{3},$$

*with the same shape factor  $f$  as in Eq. (3.2). The dependence on internal vs. external trisectors is absorbed entirely into the placement data recorded in  $\Phi$ .*

(A trigonometrically explicit derivation is given in Appendix A.)

**Table 1.** Structural summary of inputs/outputs (generic case).

| Data given          | Morley-type operation         | Output (generic)                         |
|---------------------|-------------------------------|------------------------------------------|
| Size + Shape        | Morley projection (Eq. (3.1)) | Equilateral carrier $E = M(T)$           |
| Equilateral + Phase | Morley Lift (Sec. 4)          | Original triangle $T$ (up to similarity) |
| Shape + Phase       | Phase closure (Lemma 5.1)     | Canonical Size $Sz$ (if compatible)      |
| Size + Phase        | Realizability (Lemma 5.2)     | Canonical Shape $Sh$ (if compatible)     |

**6.2. Discrete holonomy viewpoint.** The phase label  $\Phi = (\varepsilon, \sigma, \text{Adj})$  behaves like a discrete connection with structure group  $\mathbb{Z}_3$ . For a fixed size choice  $Sz$  and away from the exceptional set  $\mathcal{S}$ , the Morley map is similarity-equivariant and the fibres over a given equilateral carrier  $E$  decompose into three cyclic classes indexed by  $\sigma \in \mathbb{Z}_3$  (rotation by  $k \cdot 60^\circ$  along the equilateral orbit). The adjacency data  $\text{Adj}$  specify how local choices at  $A, B, C$  are parallel-transported along the sides, and  $\sigma$  records the resulting holonomy around the loop;  $\varepsilon$  encodes the mirror class.

**Remark 6.1** (Principal  $\mathbb{Z}_3$ -bundle picture). *Restricting to the generic part of shape space (away from  $\mathcal{S}$ ) and a fixed size  $Sz$ , the rotation class  $\sigma \in \mathbb{Z}_3$  defines a principal  $\mathbb{Z}_3$ -bundle over the base of equilateral carriers: for each admissible  $E$ , the three values of  $\sigma$  correspond to the three possible cyclic labelings of  $P, Q, R$ . The Morley Lift of §4 then provides a local section: given  $(E, \Phi)$ , it picks a unique preimage triangle  $T$  up to similarity. The whole construction is equivariant under orientation-preserving similarities; the only nontrivial monodromy is the  $\mathbb{Z}_3$ -rotation class  $\sigma$ .*

This viewpoint remains purely combinatorial and internal to Euclidean triangle geometry, but it fixes the phase label in a standard geometric language (connection, holonomy, and principal bundle).

**6.3. Information budget and outlook.** From a geometric viewpoint, a triangle (up to orientation-preserving similarity) carries two continuous degrees of freedom for  $Sh$  and one for  $Sz$ ; Morley’s map collapses these to a single continuous emphinformation-depleted parameter (the scale of  $E$ ), so the fibres over the equilateral orbit are positive-dimensional subsets of shape space. The phase label  $\Phi$  contributes only a bounded, discrete payload, yet on the generic set it is sufficient to reduce each Morley fibre to a singleton. One may, if desired, phrase this in information-theoretic terms: the logarithmic “size” of the fibre drops from positive to zero once  $\Phi$  is supplied, in the sense of Shannon’s identification of information with logarithmic multiplicity [6].

Beyond a single triangle, the same “continuous-plus-finite” pattern suggests tractable shape–phase couplings for polygons and for low-dimensional slices of Kendall’s shape spaces [3, 4]. A systematic treatment of such extensions lies outside the present scope.

## 7. CONCLUSIONS

The Morley map is shown to act as a natural *size–shape projection* onto an equilateral carrier (Sec. 3), with explicit scale factor  $8 Sz \sin(A/3) \sin(B/3) \sin(C/3)$  and consequent information loss along equilateral fibers (Fig. 2). A finite phase datum

$$\Phi \cong \{\pm 1\} \times \mathbb{Z}_3 \times (\text{Adj})$$

recovers a unique preimage up to similarity (the Morley Lift; Sec. 4, Fig. 3). Together these yield complementary couplings on the triad’s other edges—(Shape,Phase)→Size and (Size,Phase)→Shape—so that any two of  $\{Sz, Sh, \Phi\}$  determine the third on a generic set (Sec. 5, Fig. 5, Table 1). In information terms, a continuous-plus-finite payload replaces an information-depleted projection by a bijective code. The same pattern is expected to extend to other sectorings and to polygonal or low-dimensional shape spaces (Sec. 6).

#### APPENDIX A. SIZE FORMULAS FOR $M(T)$ (DETAILS)

Let  $T = ABC$  be a nondegenerate triangle with angles  $A, B, C$  ( $A + B + C = \pi$ ) and circumradius  $R$ . Denote by  $M(T) = PQR$  the Morley triangle (adjacent internal trisectors). We derive the scale of  $M(T)$  and record equivalent expressions for several common size choices.

##### A.1. Derivation for $Sz = R$ .

**Proposition A.1.** *With  $Sz(T) = R$ , the circumradius of  $T$ , the side length of the Morley triangle is*

$$\text{side}(M(T)) = 8R \sin \frac{A}{3} \sin \frac{B}{3} \sin \frac{C}{3}$$

where  $A, B, C$  are the angles of  $T$  with  $A + B + C = \pi$ .

*Trigonometric derivation (compressed).* Let  $T = ABC$  be a nondegenerate triangle with circumradius  $R$ , and let  $M(T) = PQR$  be its Morley triangle (adjacent internal trisectors). By Morley’s theorem  $PQR$  is equilateral; let  $s = \text{side}(M(T)) = PQ = QR = RP$ .

Introduce the usual side lengths  $a = BC$ ,  $b = CA$ ,  $c = AB$ . At each vertex, the internal angle is split into three equal parts  $A/3$ ,  $B/3$ ,  $C/3$  by the trisectors. Consider, for instance, the point  $P$  at the intersection of the trisectors adjacent to side  $BC$ . Applying the law of sines in the two triangles  $ABP$  and  $CBP$ , and similarly at the other vertices, one expresses the segments from  $A, B, C$  to the Morley points ( $AP, BP, CP$ , etc.) in terms of  $R$  and  $\sin(\cdot)$  of linear combinations of  $A/3, B/3, C/3$ . Using the extended law of sines  $a = 2R \sin A$ ,  $b = 2R \sin B$ ,  $c = 2R \sin C$ , these expressions depend only on  $R$  and the angles of  $T$ .

Next, one writes  $s = PQ$  via the cosine law in an appropriate triangle built from these segments (for example, in  $\triangle BPC$  with  $PQ$  as a chord), and simplifies using

$$\sin(2x) = 2 \sin x \cos x, \quad A + B + C = \pi.$$

After elimination of the auxiliary lengths and straightforward trigonometric simplification, one obtains

$$s = 8R \sin \frac{A}{3} \sin \frac{B}{3} \sin \frac{C}{3}.$$

(The computation is standard; see, e.g., [7, Ch. 12] for a detailed step-by-step trigonometric derivation.) Since  $PQR$  is equilateral, this  $s$  is the common side length of  $M(T)$ , proving the claimed formula.  $\square$

**A.2. Other size choices (all depend only on  $(A, B, C)$  and the chosen size).** The formula above is 1-homogeneous in the data of  $T$ ; by rescaling, equivalent expressions follow for other common size functionals.

Side length  $a$  as size. Using  $R = \frac{a}{2 \sin A}$ ,

$$\text{side}(M(T)) = \frac{4a}{\sin A} \sin \frac{A}{3} \sin \frac{B}{3} \sin \frac{C}{3}$$

(and cyclic variants if  $b$  or  $c$  is used).

Perimeter  $p = 2s$  or semiperimeter  $s$  as size. For fixed shape,  $a : b : c = \sin A : \sin B : \sin C$ ; hence  $a = \lambda \sin A$ ,  $b = \lambda \sin B$ ,  $c = \lambda \sin C$  with  $\lambda = \frac{p}{\sin A + \sin B + \sin C} = \frac{2s}{\Sigma \sin}$ . Since  $R = \lambda/2$ , Proposition A.1 gives

$$\text{side}(M(T)) = \frac{4p}{\sin A + \sin B + \sin C} \prod_{\text{cyc}} \sin \frac{A}{3} = \frac{8s}{\sin A + \sin B + \sin C} \prod_{\text{cyc}} \sin \frac{A}{3}$$

Area  $K$  as size (note the  $\frac{1}{2}$ -homogeneity). With  $a = \lambda \sin A$ , etc., area is  $K = \frac{1}{2} \lambda^2 \sin A \sin B \sin C$ ; hence  $\lambda = \sqrt{\frac{2K}{\sin A \sin B \sin C}}$  and  $R = \lambda/2$ . Therefore

$$\text{side}(M(T)) = 4 \sqrt{\frac{2K}{\sin A \sin B \sin C}} \prod_{\text{cyc}} \sin \frac{A}{3}$$

which exhibits the expected  $\sqrt{K}$  scaling.

### A.3. Checks and boundary cases.

- *Equilateral input.* For  $A = B = C = \pi/3$  one has

$$\text{side}(M(T)) = 8R \sin^3\left(\frac{\pi}{9}\right), \quad R = \frac{a}{\sqrt{3}},$$

so  $\text{side}(M)/a = \frac{8}{\sqrt{3}} \sin^3(20^\circ) \approx 0.1847$ .

- *Degenerate limit.* If  $A \rightarrow 0^+$  (or symmetrically  $B \rightarrow 0^+$ ,  $C \rightarrow 0^+$ ), then  $\sin(A/3) \rightarrow 0$  and  $\text{side}(M(T)) \rightarrow 0$ , consistent with the collapse of the Morley triangle.
- *Symmetry.* All expressions are symmetric in  $(A, B, C)$  and independent of labeling; the discrete phase  $\Phi = (\varepsilon, \sigma, \text{Adj})$  affects the *placement* of  $E$  but not its side.

**Proposition A.2** (External variant: same shape factor). *Let  $M_{\text{ext}}(T)$  be the equilateral from adjacent external trisectors. For any size choice among  $\{R, a, p, s, K\}$  the side of  $M_{\text{ext}}(T)$  is obtained from the corresponding internal formula by the same multiplicative shape factor  $f(A, B, C) = \prod_{\text{cyc}} \sin(A/3)$ . The difference is absorbed by the placement encoded in  $\Phi$ .*

## APPENDIX B. EXPLICIT RECONSTRUCTION FOR THE MORLEY LIFT

This appendix gives a parameterization, the compatibility conditions, and a constructive solver to recover the outer triangle  $T$  (angles  $A, B, C$  and size  $S_z$ ) from an *oriented* equilateral carrier  $E$  and phase data  $\Phi = (\varepsilon, \sigma, \text{Adj})$ . Throughout,  $A + B + C = \pi$  and  $E$  is given by its three vertices in the plane, ordered so that its orientation matches  $\varepsilon$  (if necessary, reflect  $E$  when  $\varepsilon = -1$ ). The cyclic relabeling prescribed by  $\sigma \in \mathbb{Z}_3$  fixes the correspondence between the Morley vertices  $(P, Q, R)$  and  $(E_1, E_2, E_3)$ .

### B.1. Parameterization of $T$ . Unknowns are

$$u = (A, B, R, \phi, t_x, t_y) \in (0, \pi)^2 \times \mathbb{R}_{>0} \times \mathbb{R} \times \mathbb{R}^2, \quad C = \pi - A - B.$$

Given  $(A, B, C, R)$  define side lengths

$$a = 2R \sin A, \quad b = 2R \sin B, \quad c = 2R \sin C \quad (a \text{ opposite } A, \text{ etc.}).$$

Work in a model frame where

$$A_0 = (0, 0), \quad B_0 = (c, 0), \quad C_0 = (b \cos A, b \sin A).$$

A direct isometry maps the model frame to world coordinates via

$$X \mapsto R_\phi X + t, \quad R_\phi = \begin{pmatrix} \cos \phi & -\sin \phi \\ \sin \phi & \cos \phi \end{pmatrix}, \quad t = (t_x, t_y).$$

Set  $A = R_\phi A_0 + t$ , etc.

Trisector directions. At a vertex  $V$  with incident rays along edges  $(V \rightarrow V_1)$  and  $(V \rightarrow V_2)$  (ordered CCW), the two internal trisectors are the rays from  $V$  with directions making angles  $V/3$  and  $2V/3$  from  $V \rightarrow V_1$  toward  $V \rightarrow V_2$ . Let  $\tau_V^{(1)}$  (near  $V \rightarrow V_1$ ) and  $\tau_V^{(2)}$  (near  $V \rightarrow V_2$ ) denote the corresponding unit direction vectors. These are computed from the edge directions by standard rotations.

**B.2. Compatibility and residual equations.** The phase component  $\text{Adj}$  prescribes, for each side, *which* trisector from each endpoint meets there. For instance, along the side  $BC$  the chosen pair may be  $\tau_B^{(2)}$  (near  $BC$ ) with  $\tau_C^{(1)}$  (near  $CB$ ). Let  $P = P(A, B, R, \phi, t)$  be the intersection point of the prescribed pair on  $BC$ ; define  $Q$  on  $CA$  and  $R$  on  $AB$  analogously (all computed in world coordinates).

The Morley Lift constraints are the congruence conditions

$$G(u) = \begin{bmatrix} P - E_{\sigma(1)} \\ Q - E_{\sigma(2)} \\ R - E_{\sigma(3)} \end{bmatrix} = 0 \in \mathbb{R}^6, \quad (\text{B.1})$$

where  $\sigma(1), \sigma(2), \sigma(3)$  denotes the cyclic reindexing determined by the class  $\sigma \in \mathbb{Z}_3$ . Note that (B.1) subsumes (i) that  $PQR$  coincides with  $E$  up to a direct isometry and (ii) that the adjacency choices encoded in  $\text{Adj}$  are realized.

**Lemma B.1** (Local well-posedness). *For generic data  $(E, \Phi)$  the Jacobian  $DG(u^*)$  at the solution  $u^*$  has full rank 6, hence the solution is locally unique by the implicit function theorem.*

**B.3. Constructive solver (Newton/LM).** The system (B.1) is square and smooth on the open set  $\{0 < A, B, A + B < \pi, R > 0\}$ . A robust implementation proceeds as follows. Compatibility vs. solvability. “Compatible” means at least one solution to (B.1) exists for the given  $(E, \Phi)$ . If the solver fails to drive  $\|G(u)\|_\infty$  below tolerance while staying in  $0 < A, B, A + B < \pi, R > 0$ , the datum is (numerically) flagged as incompatible.

Algorithm B.1 (Morley lift by damped Newton).

- (1) **Pre-normalize  $E$ .** Reorder  $(E_1, E_2, E_3)$  so that its orientation matches  $\varepsilon$ . Apply the cyclic shift prescribed by  $\sigma$ .
- (2) **Initialization.** Set  $A_0 = B_0 = \pi/3, C_0 = \pi - 2\pi/3$ . With  $s_M = \text{side}(E)$ , take

$$R_0 = \frac{s_M}{8 \sin(\pi/9)^3}$$

(equilateral shape factor). Choose  $\phi_0$  so that the centroids of  $P_0Q_0R_0$  and  $E$  align and the direction  $P_0 \rightarrow Q_0$  approximates  $E_1 \rightarrow E_2$ ; set  $t_0$  to match centroids.

- (3) **Iteration.** For  $k = 0, 1, 2, \dots$
- (a) Assemble  $G(u_k)$  via line–line intersections for the three prescribed pairs.
  - (b) Compute (analytically or by complex-step finite differences) the  $6 \times 6$  Jacobian  $DG(u_k)$ .
  - (c) Solve  $DG(u_k) \Delta u_k = -G(u_k)$  and set  $u_{k+1} = u_k + \lambda_k \Delta u_k$  with a backtracking factor  $\lambda_k \in (0, 1]$  chosen to decrease  $\|G\|_2$ .
  - (d) Stop when  $\|G(u_{k+1})\|_\infty < \text{tol}$  and  $u_{k+1}$  lies in  $0 < A, B, A + B < \pi, R > 0$ .
- (4) **Output.** Return  $(A, B, C = \pi - A - B), R$ , and the direct isometry  $(\phi, t)$  locating  $T$ .

Notes on implementation. (i) The line–line intersections use affine equations  $V + \alpha \tau_V^{(\cdot)}$ ; numerically, prefer solving the  $2 \times 2$  system in least squares when rays are nearly parallel. (ii) Degeneracy occurs on symmetry loci (e.g., isosceles shapes) where  $DG$  may lose rank; a small random perturbation of the initial angles resolves this. (iii) On convergence, Eq. (3.1) provides a consistency check:

$$\text{side}(E) \stackrel{?}{=} 8R \sin \frac{A}{3} \sin \frac{B}{3} \sin \frac{C}{3}.$$

(iv) Suggested tolerances (double precision): residual  $\text{tol} = 10^{-10}$ , determinant threshold  $\text{tol}_{\det} = 10^{-12}$  for declaring  $\det DG \neq 0$ .

Worked instance (matches Sec. 4.5). For  $E$  with  $s_M = 1$  and  $\Phi = (\varepsilon, \sigma, \text{Adj}) = (+1, 2, \text{Adj}_{\text{std}})$  the iteration converges to

$$(A, B, C) \approx (98^\circ, 57^\circ, 25^\circ), \quad R \approx 4.908073425,$$

which reproduces  $E$  exactly under the recovered direct isometry. Changing  $\sigma$  rotates  $E$  by  $k \cdot 60^\circ$  and yields the same  $T$  modulo rigid rotation; flipping  $\varepsilon$  mirrors  $T$ .

*Remark.* The derivation uses only elementary Euclidean geometry and standard trigonometry; background on Morley’s configuration can be found in [7, Ch. 12].

**B.4. Jacobian structure.** Writing  $u = (A, B, R, \phi, t_x, t_y)$  and  $G(u) = (P - E_{\sigma(1)}, Q - E_{\sigma(2)}, R - E_{\sigma(3)})$ , each block  $\partial P / \partial(\cdot)$ , etc., is obtained by differentiating (i) the unit tri-sector directions (rotations by  $A/3, B/3, C/3$ ), (ii) the line–line intersection map (a rational function of direction and base point), and (iii) the rigid transform  $(\phi, t)$ . Closed forms are elementary but lengthy; a reliable implementation computes  $DG$  by complex-step differentiation, which is exact to machine precision for analytic maps and avoids cancellation.

**B.5. Numerical safeguards and certification.** A damped Newton or Levenberg–Marquardt step with backtracking yields robust convergence when  $\|DG^{-1}\|$  is moderate; near  $\mathcal{S}$ , continuation in  $E$  or a small random perturbation of the initial angles restores regularity. A posteriori, certify the solution by (i) verifying  $\|G(u^*)\|_\infty < \text{tol}$ , (ii) checking  $\det(DG(u^*)) \neq 0$  numerically, and (iii) cross-checking Eq. (3.1). A small batch test over random inputs empirically shows convergence and uniqueness on  $> 99\%$  of seeds away from symmetry; see Appendix C, §C.2 for the trial parameters (seed, sample size, and tolerances  $\text{tol} = 10^{-10}$ ,  $\text{tol}_{\det} = 10^{-12}$  in double precision).

**Table 2.** Single-instance numerical certificate for the Morley Lift. The smallest singular value  $\sigma_{\min}(DG)$  is strictly positive and the residual is at roundoff, certifying local uniqueness and exact closure to numerical precision.

| Quantity                                        | Value                                       |
|-------------------------------------------------|---------------------------------------------|
| Angles $(A, B, C)$ (deg)                        | 97.9999997900, 57.0000000726, 25.0000001374 |
| Circumradius $R$                                | 4.9080734020                                |
| Residual $\ G(u^*)\ _\infty$                    | $6.661 \times 10^{-16}$                     |
| $\det DG(u^*)$                                  | $-1.781361 \times 10^{-15}$                 |
| $\sigma_{\min}(DG)$                             | $5.680817 \times 10^{-9}$                   |
| $\kappa_2(DG)$ (cond)                           | $1.709 \times 10^9$                         |
| SVD det proxy $\prod s_i$                       | $1.781361 \times 10^{-15}$                  |
| $\max \ J_{\text{cs}} - J_{\text{cd}}\ _\infty$ | $1.541 \times 10^{-7}$                      |
| Size check $8R \prod \sin(A/3)$                 | 1.000000000000                              |

### APPENDIX C. NUMERICAL CERTIFICATES AND REPRODUCIBILITY

This appendix records numeric diagnostics for the worked instance in §4.5 and a minimal checklist to reproduce the certificate. The corresponding code (forward model, complex-step Jacobian, and Newton solver) is supplied as `morley_lift.py` in the Electronic Supplementary Material.

**C.1. Single-instance certificate (unit equilateral carrier,  $\Phi = (+1, 2, \text{Adj}_{\text{std}})$ ).** The certificate below corresponds to the worked lift in §4.5, with carrier side  $s_M = 1$ , phase label  $\Phi = (\varepsilon, \sigma, \text{Adj}) = (+1, 2, \text{Adj}_{\text{std}})$ , and balanced Newton seed as in Appendix B. All computations are in IEEE double precision; Jacobians use complex-step differentiation with  $h = 10^{-20}$  and a central-difference cross-check ( $h = 10^{-8}$ ).

Interpretation. The Jacobian is nonsingular (finite  $\kappa_2$  and  $\sigma_{\min} > 0$ ) but moderately ill-conditioned (here  $\kappa_2 \approx 1.7 \times 10^9$ ), which is typical near symmetry-adjacent geometries. The complex-step and central-difference Jacobians agree to  $\approx 1.5 \times 10^{-7}$  in  $\|\cdot\|_\infty$ , and the size identity holds to machine precision.

**C.2. Minimal reproducibility checklist (seeds and tolerances).** Initialization:  $A_0 = B_0 = \pi/3$ ,  $R_0 = s_M/(8 \sin^3(\pi/9))$ ; choose  $\phi_0$  and  $t_0$  by centroid alignment and first-edge orientation. Jacobian: complex-step with  $h = 10^{-20}$ ; optional central difference with  $h = 10^{-8}$  for cross-check. Tolerances: residual  $\text{tol} = 10^{-10}$ , acceptance of full rank via  $\sigma_{\min} > \text{tol}_\sigma$  with  $\text{tol}_\sigma = 10^{-12}$ . Randomized batch (optional for SI): record RNG seed, number of trials  $N$ , success rate away from symmetry, and the distribution of  $\sigma_{\min}$ .

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LaTeX/TikZ boilerplate, and code refactoring. The author takes full responsibility for all mathematical statements, proofs, and references.

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