Global Journal of Advanced Research
on Classical and Modern Geometries
ISSN: 2284-5569, Vol.15, (2026), Issue 1, pp.138-151

BARYCENTRIC REPRESENTATION OF LINES WITH RESPECT TO A TRIANGLE

J C HO

ABSTRACT. We establish meaningful geometric interpretation of barycentric coordinates of lines with respect to a reference triangle. Such interpretation clarifies the duality observed between lines and points, and promotes study of lines independently in triangle geometry.

1. INTRODUCTION

In the classical trilinear and tripolar systems, triangle centers are represented as triples of signed distances to the sides and the vertices of the reference triangle respectively, as explained by Bottema [2] and Kimberling [5]. As we expand our exploration of affine and metrical geometries to disciplines such as chromogeometry, involving a three-fold symmetry between Euclidean and relativistic geometries, and geometry over finite fields with perhaps arbitrary symmetric bilinear forms as described by Wildberger in [9], [10] and [12], these two systems become increasingly problematic since distances involve square roots whose existence is a number theoretical issue.
On the other hand, homogeneous barycentric coordinates for triangle centers, as advocated by Yiu [13], use ratios of areas formed with pairs of vertices to form barycentric coordinates for any point on the plane with respect to a reference triangle. These barycentric coordinates can be expressed using affine combinations which connect directly also to centers of mass in physics, which has been further explored in [4].
Lines, on the other hand, are often treated as secondary to points, with the relationship between lines and a reference triangle often indirect and even obscure. It seems useful to bring lines into the geometry of the affine plane at an early stage and to investigate how to compute with them efficiently. In this paper, we derive a homogeneous barycentric coordinate system for lines with respect to the sides of the reference triangle. This not only provides a geometric interpretation of the line's coordinates, but will also facilitate the study of lines as interesting objects on their own.
While duality is not exactly an affine concept, belonging more properly to projective geometry, we do observe a close similarity in results between join of points and meet of lines. This is also exhibited in other coordinate systems (eg. trilinear), but in the

2010 *Mathematics Subject Classification.* Primary 14R99; Secondary 51N10.
Key words and phrases. Barycentric Coordinates, triangle geometry, lines.
This work is supported through an Australian Government Research Training Program Scholarship.

barycentric system we develop, such a similarity is further enhanced by an associated geometric significance. In addition, explicit and flexible tools for computation in the plane are developed. Finally, the close connection between barycentric coordinates and vector proportions is summarized in the Barycentric Ratio Theorem (18).

2. Affine Geometry

2.1. **Affine Planes over a Field.** We will be working in the **affine plane** $\mathbb{A}^2$ over a general field $\mathbb{F}$ consisting of **points**, written as $\mathbf{A} = [a_1, a_2]$ where $a_1, a_2 \in \mathbb{F}$ are called **components** of $\mathbf{A}$. Two points $\mathbf{A} = [a_1, a_2]$ and $\mathbf{B} = [b_1, b_2]$ are said to be distinct precisely when $a_1 \neq b_1$ or $a_2 \neq b_2$. The only valid operation on two points is affine combination.
First of all, an **affine pair** is a list of numbers $\langle \alpha, \beta \rangle$ satisfying $\alpha + \beta = 1$. More generally, for a natural number $n \geq 1$, an **affine n-tuple** is a list $\langle \alpha_1, \alpha_2, \cdots, \alpha_n \rangle$ of numbers such that $\alpha_1 + \alpha_2 + \cdots + \alpha_n = 1$.

Definition 1 (Affine Combination)**.** *For an affine pair $\langle \alpha, \beta \rangle$, the $\langle \alpha, \beta \rangle$-**affine combination** of points*
$\mathbf{A} = [a_1, a_2]$ *and* $\mathbf{B} = [b_1, b_2]$ *is defined as the point*

$$\alpha\mathbf{A} + \beta\mathbf{B} \equiv [\alpha a_1 + \beta b_1, \alpha a_2 + \beta b_2].$$

The associated **vector space** $\mathbb{V}^2$ is defined as displacements, or separations, between pairs of points, and written as $v = (v_1, v_2)$ where $v_1, v_2 \in \mathbb{F}$. The relationship between points and vectors can be defined below.

Definition 2 (Vector)**.** *If $\mathbf{A} = [a_1, a_2]$ and $\mathbf{B} = [b_1, b_2]$ are points, then the vector $v = \overrightarrow{\mathbf{AB}}$ will be denoted*

$$v = \overrightarrow{\mathbf{AB}} \equiv \mathbf{B} - \mathbf{A} \equiv (b_1 - a_1, b_2 - a_2) = (v_1, v_2),$$

*where v_1, v_2 are also known as the **components** of $\mathbf{v}$. A vector whose both components are zero is then known as a **zero vector**.*

Based on such definition, we further agree that we can add a point and a vector in that order, so that

$$\mathbf{B} = \mathbf{A} + \mathbf{v}.$$

However, $\mathbf{v} + \mathbf{A}$ has no meaning.
We note that, although both points and vectors consist of ordered pairs of numbers, the distinction between them is both logically and practically important. Vectors $(1, 3)$ and $(2, 6)$ are, as we shall show later, parallel and satisfy the relation $(2, 6) = 2(1, 3)$; but points $[1, 3]$ and $[2, 6]$ are simply distinct and, otherwise, unrelated.
The notion of affine combination and displacement of two points can be further generalized. For a natural number $n \geq 1$, if $\mathbf{A}_1, \mathbf{A}_2, \ldots, \mathbf{A}_n$ are points, then we may interpret a **1-combination** of points

$$\mathbf{B} = \sum_{k=1}^{n} \alpha_k \mathbf{A}_k \qquad \text{where } \alpha_k \in \mathbb{F} \text{ and } \sum_{k=1}^{n} \alpha_k = 1$$

as a point, and a **0-combination** of points

$$\mathbf{v} = \sum_{k=1}^{n} \gamma_k \mathbf{A}_k \qquad \text{where } \gamma_k \in \mathbb{F} \text{ and } \sum_{k=1}^{n} \gamma_k = 0$$

as a vector.

2.2. **Cross Product and Signed Area of a Triangle.** The two-dimensional version of the familiar three-dimensional operation is introduced in the *Algebraic Calculus One* course [11] and it underpins the notion of affine area in the plane. If $\mathbf{u} \equiv (u_1, u_2)$ and $\mathbf{v} \equiv (v_1, v_2)$ are vectors, then the **cross product** of $\mathbf{u}$ and $\mathbf{v}$ is the number

$$\mathbf{u} \times \mathbf{v} \equiv u_1 v_2 - u_2 v_1 = \det \begin{pmatrix} u_1 & u_2 \\ v_1 & v_2 \end{pmatrix}.$$

For any vectors $\mathbf{u}$ and $\mathbf{v}$ we have $\mathbf{u} \times \mathbf{v} = -\mathbf{v} \times \mathbf{u}$ and $\mathbf{u} \times \mathbf{u} = 0$. Non-zero vectors $\mathbf{u}$ and $\mathbf{v}$ are **parallel** precisely when $\mathbf{u} \times \mathbf{v} = 0$.

A **side** formed by two points $\mathbf{A}$ and $\mathbf{B}$ may be referred to as *oriented* (and denoted $\overrightarrow{\mathbf{AB}}$) or *unoriented* (and denoted simply as $\overline{\mathbf{AB}}$).

Definition 3 (Triangle). *If $\mathbf{A}_1, \mathbf{A}_2, \mathbf{A}_3$ are distinct points, then the* ***oriented triangle*** $\overrightarrow{\mathbf{A}_1\mathbf{A}_2\mathbf{A}_3}$ *is the cyclically ordered list $[\mathbf{A}_1, \mathbf{A}_2, \mathbf{A}_3]$, that is, $\overrightarrow{\mathbf{A}_1\mathbf{A}_2\mathbf{A}_3} = \overrightarrow{\mathbf{A}_2\mathbf{A}_3\mathbf{A}_1} = \overrightarrow{\mathbf{A}_3\mathbf{A}_1\mathbf{A}_2}$. We call $\overleftrightarrow{\mathbf{A}_1\mathbf{A}_2\mathbf{A}_3}$ an (**unoriented**) triangle when orientation is not considered, that is, $\overleftrightarrow{\mathbf{A}_1\mathbf{A}_2\mathbf{A}_3} = \overleftrightarrow{\mathbf{A}_1\mathbf{A}_3\mathbf{A}_2}$. $\overrightarrow{\mathbf{A}_1\mathbf{A}_2\mathbf{A}_3}$ is* ***degenerate*** *if $\mathbf{A}_1, \mathbf{A}_2, \mathbf{A}_3$ are collinear.*

Lemma 4 (Three points symmetry). *For any points $\mathbf{A}, \mathbf{B}$ and $\mathbf{C}$, we have*

$$(\mathbf{B} - \mathbf{A}) \times (\mathbf{C} - \mathbf{A}) = (\mathbf{C} - \mathbf{B}) \times (\mathbf{A} - \mathbf{B}) = (\mathbf{A} - \mathbf{C}) \times (\mathbf{B} - \mathbf{C}).$$

Proof. Let $\mathbf{A} \equiv [a_1, a_2], \mathbf{B} \equiv [b_1, b_2]$ and $\mathbf{C} \equiv [c_1, c_2]$, then

$$\begin{aligned} (\mathbf{B} - \mathbf{A}) \times (\mathbf{C} - \mathbf{A}) &= (b_1 - a_1, b_2 - a_2) \times (c_1 - a_1, c_2 - a_2) \\ &= (b_1 - a_1)(c_2 - a_2) - (b_2 - a_2)(c_1 - a_1) \\ &= a_1 b_2 - a_2 b_1 + b_1 c_2 - b_2 c_1 + c_1 a_2 - c_2 a_1 \end{aligned}$$

and this is symmetric under the cyclic rotation $\mathbf{A} \to \mathbf{B} \to \mathbf{C} \to \mathbf{A}$. □

We now define the signed area of the oriented triangle $\overrightarrow{\mathbf{ABC}}$ to be one-half of the common number in the previous lemma, and denote it $s\left(\overrightarrow{\mathbf{ABC}}\right)$.

Theorem 5 (Signed area formula). *If $\mathbf{A} \equiv [a_1, a_2], \mathbf{B} \equiv [b_1, b_2]$ and $\mathbf{C} \equiv [c_1, c_2]$, then*

$$s\left(\overrightarrow{\mathbf{ABC}}\right) = \frac{1}{2}\left(a_1 b_2 - a_2 b_1 + b_1 c_2 - b_2 c_1 + c_1 a_2 - c_2 a_1\right) = \frac{1}{2} \det \begin{bmatrix} 1 & a_1 & a_2 \\ 1 & b_1 & b_2 \\ 1 & c_1 & c_2 \end{bmatrix}.$$

Proof. By substitution. □

2.3. **Vector Proportions.** Many well-known results in affine geometry, such as the theorems of Ceva and Menelaus, are often expressed in terms of ratios of segment lengths [1, pp. 94 & 98] or signed distances; however these notions are properly part of metrical geometry and come with number theoretical challenges. The theorems themselves are nonetheless intrinsic to the more fundamental affine geometry, as they can be expressed purely in terms of proportions between parallel vectors.

A 2**-proportion**, or just **proportion**, of numbers a, b is an expression of the form $a : b$ and is defined provided that at least one of a or b is non-zero, with the notion of equality

$$a_1 : b_1 = a_2 : b_2$$

precisely when $a_1b_2 - a_2b_1 = 0$. This is equivalent to the condition that there is a non-zero number λ such that $a_2 = \lambda a_1$ and $b_2 = \lambda b_1$. For a natural number n, the concept can be extended to a more general n-proportion $a_1 : a_2 : \cdots : a_n$ provided at least one of a_i ($i \in \{1..n\}$) is non-zero, and with equality of two n-proportions

$$a_1 : a_2 : \cdots : a_n = b_1 : b_2 : \cdots : b_n$$

defined precisely when there is a non-zero scalar λ such that $a_i = \lambda b_i$ for all $i \in \{1..n\}$. We can then define a vector proportion based on parallel vectors. Non-zero vectors $\mathbf{u}$ and $\mathbf{v}$ are in the **proportion** $\alpha : \beta$ precisely when $\alpha\mathbf{v} - \beta\mathbf{u} = 0$, in which case we write $\mathbf{u} : \mathbf{v} = \alpha : \beta$. Clearly such vector proportions occur only between parallel vectors. We may also frame this in terms of collinear points by saying that the point $\mathbf{D}$ **divides** the oriented side $\overrightarrow{\mathbf{AB}}$ **in the ratio of** $\alpha : \beta$ precisely when

$$\overrightarrow{\mathbf{AD}} : \overrightarrow{\mathbf{DB}} = \alpha : \beta.$$

Proposition 6 (Proportion Point)**.** *The point* $\mathbf{D}$ *divides* $\overrightarrow{\mathbf{AB}}$ *in the ratio of* $\alpha : \beta$ *precisely when* $\mathbf{D}$ *can be expressed as the affine combination*

$$\mathbf{D} = \frac{\beta}{\alpha+\beta}\mathbf{A} + \frac{\alpha}{\alpha+\beta}\mathbf{B}.$$

Proof. If $\mathbf{D}$ divides $\overrightarrow{\mathbf{AB}}$ in the ratio of $\alpha : \beta$ then $(\mathbf{D} - \mathbf{A}) : (\mathbf{B} - \mathbf{D}) = \alpha : \beta$ and so $\beta(\mathbf{D} - \mathbf{A}) = \alpha(\mathbf{B} - \mathbf{D})$. Adding the vector $\alpha(\mathbf{D} - \mathbf{A})$ to both sides results in

$$(\alpha + \beta)(\mathbf{D} - \mathbf{A}) = \alpha(\mathbf{B} - \mathbf{D}) + \alpha(\mathbf{D} - \mathbf{A}) = \alpha(\mathbf{B} - \mathbf{A})$$

so that

$$\mathbf{D} = \mathbf{A} + \frac{\alpha}{\alpha+\beta}(\mathbf{B} - \mathbf{A}) = \frac{\beta}{\alpha+\beta}\mathbf{A} + \frac{\alpha}{\alpha+\beta}\mathbf{B}.$$

The converse is obtained by reversing the steps. □

2.4. **Lines.** The affine point $\mathbf{V}$ **lies on** the line $\mathbf{AB}$ precisely when $\mathbf{V}$ is an affine combination of $\mathbf{A}$ and $\mathbf{B}$. In this case we also say that $\mathbf{AB}$ **passes through** $\mathbf{V}$, or that $\mathbf{V}$ and $\mathbf{AB}$ are **incident**. If $\mathbf{V}$ is a variable point and $\mathbf{u} = \mathbf{B} - \mathbf{A}$, then both expressions $\mathbf{V} = \mathbf{A} + \lambda\mathbf{u}$ and $\mathbf{u} \times (\mathbf{V} - \mathbf{A}) = 0$ describe the same line $\mathbf{AB}$. We call the former the **parametric equation** form, and the latter the **point-direction** equation form, of the same line.

3. Affine Functionals

An **affine function** f from an affine space $\mathbb{A}$ to an affine space $\mathbb{A}'$ is a map such that for any points $\mathbf{A}$ and $\mathbf{B}$ of $\mathbb{A}$ and any affine pair $\langle\alpha, \beta\rangle$ we have

$$f(\alpha\mathbf{A} + \beta\mathbf{B}) = \alpha f(\mathbf{A}) + \beta f(\mathbf{B}).$$

Then an induction shows that for any affine n-tuple $\langle\alpha_1, \alpha_2, \cdots, \alpha_n\rangle$ and points $\mathbf{A}_1, \mathbf{A}_2, \cdots, \mathbf{A}_n$ of $\mathbb{A}$ we must have

$$f(\sum_{i=1}^{n} \alpha_i\mathbf{A}_i) = \sum_{i=1}^{n} \alpha_i f(\mathbf{A}_i).$$

We call this the **general affine property** of an affine function. We denote the space of affine functions from $\mathbb{A}$ to $\mathbb{A}'$ by $\mathrm{Aff}\left(\mathbb{A}, \mathbb{A}'\right)$.

If f is an affine function from an affine space $\mathbb{A}$ to an affine space $\mathbb{A}'$ then there is an **associated linear function** T_f from the associated vector space $\mathbb{V}$ of $\mathbb{A}$ to the associated vector space $\mathbb{V}'$ of $\mathbb{A}'$ given in terms of differences of points $\mathbf{A}$ and $\mathbf{B}$ of $\mathbb{A}$ by

$$T_f(\mathbf{B}-\mathbf{A}) \equiv f(\mathbf{B}) - f(\mathbf{A}).$$

An **affine functional** is an affine function from an affine space $\mathbb{A}$ to the space $\mathbb{F}$ of numbers, itself regarded as an affine space. As an example, if $\mathbb{A}$ is the space of affine points $\mathbf{A} \equiv [a_1, a_2]$ then in components an affine functional has the form

$$f([a_1, a_2]) = k - u_2 a_1 + u_1 a_2$$

for some numbers u_1, u_2 and k. A more invariant way is to express f in the form

$$f(\mathbf{V}) = \mathbf{u} \times (\mathbf{V} - \mathbf{A})$$

for some vector $\mathbf{u} \equiv (u_1, u_2)$ and some point $\mathbf{A} \equiv [a_1, a_2]$ satisfying $u_1 a_2 - u_2 a_1 = -k$. It follows that $f(\mathbf{V}) = 0$ precisely when $\mathbf{V}$ lies on the line

$$\ell : \mathbf{u} \times (\mathbf{V} - \mathbf{A}) = 0.$$

If we replace $\mathbf{u}$ with a multiple, then the functional will correspondingly change by that multiple. It is unchanged if we replace $\mathbf{A}$ with any point on the line. In short an affine functional determines a line, and a line determines an affine functional up to a non-zero multiple. We say that the line ℓ and the functional f **correspond** and write $l \equiv l_f$.
If we denote the space of affine functionals on $\mathbb{A}$ by $F(\mathbb{A})$, then this is actually a linear space over the field $\mathbb{F}$, as the sum of two affine functionals is also an affine functional, and any multiple of an affine functional is also an affine functional. In fact in the case of the affine plane, this is a three-dimensional space, as we shall now see explicitly.

4. Affine Functionals of an oriented triangle

Suppose that $\overrightarrow{\mathbf{ABC}}$ is a non-degenerate oriented triangle, so that twice its signed area is

$$\Delta \equiv 2s(\overrightarrow{\mathbf{ABC}}) = (\mathbf{B}-\mathbf{A}) \times (\mathbf{C}-\mathbf{A}) = (\mathbf{C}-\mathbf{B}) \times (\mathbf{A}-\mathbf{B}) = (\mathbf{A}-\mathbf{C}) \times (\mathbf{B}-\mathbf{C}) \neq 0.$$

Now we may use the three points consistently to define canonical affine functionals for each side:

$$f_{\mathbf{AB:C}}(\mathbf{V}) \equiv \frac{1}{\Delta}(\mathbf{B}-\mathbf{A}) \times (\mathbf{V}-\mathbf{A}) = \frac{s(\overrightarrow{\mathbf{ABV}})}{s(\overrightarrow{\mathbf{ABC}})}$$
$$f_{\mathbf{CA:B}}(\mathbf{V}) \equiv \frac{1}{\Delta}(\mathbf{A}-\mathbf{C}) \times (\mathbf{V}-\mathbf{C}) = \frac{s(\overrightarrow{\mathbf{CAV}})}{s(\overrightarrow{\mathbf{ABC}})}$$
$$f_{\mathbf{BC:A}}(\mathbf{V}) \equiv \frac{1}{\Delta}(\mathbf{C}-\mathbf{B}) \times (\mathbf{V}-\mathbf{B}) = \frac{s(\overrightarrow{\mathbf{BCV}})}{s(\overrightarrow{\mathbf{ABC}})}.$$

If the three points $\mathbf{A}$, $\mathbf{B}$ and $\mathbf{C}$ are fixed, then we will compress the notation and just write

$$f_{\mathbf{C}} \equiv f_{\mathbf{AB:C}}$$

and similarly for the other vertices. Then $f_{\mathbf{C}}(\mathbf{A}) = f_{\mathbf{C}}(\mathbf{B}) = 0$ while $f_{\mathbf{C}}(\mathbf{C}) = 1$ so that $f_{\mathbf{C}}$ corresponds to the line $\mathbf{AB}$, and similarly for the other functionals. These three functionals thus represent *delta functions* on the vertices $\mathbf{A}, \mathbf{B}, \mathbf{C}$. So naturally any functional

h can be written uniquely as a linear combination of the three functionals $f_{\mathbf{A}}, f_{\mathbf{B}}$ and $f_{\mathbf{C}}$. In fact

$$h = h(\mathbf{A})f_{\mathbf{A}} + h(\mathbf{B})f_{\mathbf{B}} + h(\mathbf{C})f_{\mathbf{C}}.$$

So the space of affine functionals is actually a three-dimensional vector space with $\{f_{\mathbf{A}}, f_{\mathbf{B}}, f_{\mathbf{C}}\}$ being an ordered basis.

We will then say that the **affine coordinates** of such an affine functional h are respectively $h(\mathbf{A}), h(\mathbf{B})$ and $h(\mathbf{C})$. For $k \neq 0$, the affine coordinates $kh(\mathbf{A}), kh(\mathbf{B})$ and $kh(\mathbf{C})$ of kh are those of h but multiplied by the factor k; yet h and kh correspond to the same line $l_h = l_{kh}$. We will therefore introduce the homogeneous notation

$$l_h \equiv (h(\mathbf{A}) : h(\mathbf{B}) : h(\mathbf{C}))$$

to represent the line corresponding to h, or to any non-zero multiple of it with respect to the oriented triangle $\overrightarrow{\mathbf{ABC}}$.

This now means that, given a reference triangle $\overrightarrow{\mathbf{ABC}}$, we have a way of representing any line in the plane in terms of such **line barycentric coordinates**. We'll see that such notation is closely connected to the more usual (point) barycentric coordinates for points $\mathbf{P} = [\alpha : \beta : \gamma]$.

Let's define a functional h to be **normalized** (with respect to the reference oriented triangle $\overrightarrow{\mathbf{ABC}}$) precisely when

$$h(\mathbf{A}) + h(\mathbf{B}) + h(\mathbf{C}) = 1.$$

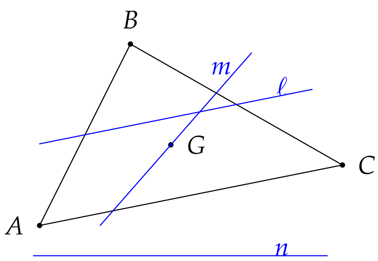
Then $f_{\mathbf{A}}, f_{\mathbf{B}}$ and $f_{\mathbf{C}}$ are clearly normalized, and $\alpha f_{\mathbf{A}} + \beta f_{\mathbf{B}} + \gamma f_{\mathbf{C}}$ is normalized precisely when $\alpha + \beta + \gamma = 1$. The space of normalized functionals is therefore an affine space (closed under affine combinations).

Let's also define a functional h to be **null** (with respect to the oriented triangle $\overrightarrow{\mathbf{ABC}}$) precisely when

$$h(\mathbf{A}) + h(\mathbf{B}) + h(\mathbf{C}) = 0.$$

Then the space of null functionals is not just an affine space, closed under affine combinations, but is in fact a two-dimensional linear space closed under more general linear combinations. A line ℓ in the plane is **centroidal** (with respect to the oriented triangle $\overrightarrow{\mathbf{ABC}}$) precisely when it corresponds to a null functional. An indication of why this terminology is appropriate is seen below.

Example 7. *The figure on the right shows an oriented triangle* $\overrightarrow{\mathbf{ABC}}$ *together with some lines in the plane with line barycentric coordinates:* $\ell = (1 : -1 : 1), m = (8 : 17 : -25)$ *and* $n = (1 : 7 : 3)$. *Here* $\mathbf{G}$ *is the centroid of* $\overrightarrow{\mathbf{ABC}}$*, and notice that* m *is centroidal.*



Proposition 8. *A line* l *is centroidal (with respect to the oriented triangle* $\overrightarrow{\mathbf{ABC}}$*) precisely when it passes through the centroid* $\mathbf{G} \equiv [1 : 1 : 1]$ *of the triangle* $\overrightarrow{\mathbf{ABC}}$*.*

Proof. Suppose ℓ is a line which passes through $\mathbf{G}$. Then we can write ℓ as $\mathbf{u} \times (\mathbf{V} - \mathbf{G}) = 0$ where $\mathbf{V}$ is a variable point and $\mathbf{u}$ a direction vector of ℓ. This means $h = \mathbf{u} \times (\mathbf{V} - \mathbf{G})$ and

$$\begin{aligned} h(\mathbf{A}) + h(\mathbf{B}) + h(\mathbf{C}) &= \mathbf{u} \times (\mathbf{A} - \mathbf{G}) + \mathbf{u} \times (\mathbf{B} - \mathbf{G}) + \mathbf{u} \times (\mathbf{C} - \mathbf{G}) \\ &= \mathbf{u} \times (\mathbf{A} + \mathbf{B} + \mathbf{C} - 3\mathbf{G}) = 0 \end{aligned}$$

as $\mathbf{G} = \frac{1}{3}\mathbf{A} + \frac{\mathbf{1}}{\mathbf{3}}\mathbf{B} + \frac{\mathbf{1}}{\mathbf{3}}\mathbf{C}$.
Conversely, if $h = \mathbf{u} \times (\mathbf{V} - \mathbf{P})$ and $h(\mathbf{A}) + h(\mathbf{B}) + h(\mathbf{C}) = 0$, then $\mathbf{u} \times (\mathbf{A} + \mathbf{B} + \mathbf{C} - 3\mathbf{P}) = 0$. Since $\mathbf{u} \neq \mathbf{0}$, $\mathbf{P} = \frac{1}{3}\mathbf{A} + \frac{1}{3}\mathbf{B} + \frac{1}{3}\mathbf{C} = \mathbf{G}$. □

So centroidal lines correspond to null affine functionals, up to a multiple, while non-centroidal lines correspond to normalized functionals.

Lemma 9 (Cartesian Correspondence)**.** *Given a reference triangle $\overrightarrow{\mathbf{ABC}}$ where $\mathbf{A} \equiv [a_1, a_2]$, $\mathbf{B} \equiv [b_1, b_2]$ and $\mathbf{C} \equiv [c_1, c_2]$, then*
(a) a line in Cartesian form $rx + sy + t = 0$ has barycentric coordinates

$$(ra_1 + sa_2 + t : rb_1 + sb_2 + t : rc_1 + sc_2 + t)$$

(b) a line with barycentric coordinates $(l : m : n)$ has a Cartesian equation

$$\begin{aligned} &\big((b_2 - c_2)l + (c_2 - a_2)m + (a_2 - b_2)n\big)\, x \\ -&\big((b_1 - c_1)l + (c_1 - a_1)m + (a_1 - b_1)n\big)\, y \\ +&\big((b_1c_2 - b_2c_1)l + (c_1a_2 - c_2a_1)m + (a_1b_2 - a_2b_1)n\big) = 0 \end{aligned}$$

Proof. (a) By applying the line's corresponding functional $h = rx + sy + t$ to the definition.
(b) By definition, the line has a corresponding functional $h = lf_{\mathbf{A}} + mf_{\mathbf{B}} + nf_{\mathbf{C}}$. Let $\mathbf{V} \equiv [x, y]$ be the variable point and substitute into $f_{\mathbf{A}}, f_{\mathbf{B}}$ and $f_{\mathbf{C}}$, and get the result by collecting terms.

□

Theorem 10 (Barycentric Incidence)**.** *Given an oriented triangle $\overrightarrow{\mathbf{ABC}}$ and a line ℓ with line barycentric coordinates $(l : m : n)$ and a point $\mathbf{P}$ with barycentric coordinates $[\alpha : \beta : \gamma]$, then ℓ is incident with $\mathbf{P}$ precisely when*

$$\alpha l + \beta m + \gamma n = 0.$$

Proof. Let ℓ be represented by the normalized functional f_ℓ. Then $f_\ell = lf_{\mathbf{A}} + mf_{\mathbf{B}} + nf_{\mathbf{C}}$. Without loss of generality, suppose that $\mathbf{P} = \alpha\mathbf{A} + \beta\mathbf{B} + \gamma\mathbf{C}$ with $\alpha + \beta + \gamma = 1$. Now ℓ is incident with $\mathbf{P}$ precisely when $f_\ell(\mathbf{P}) = 0$. Since

$$\begin{aligned} f_{\mathbf{A}}(\alpha\mathbf{A}) &= \alpha f_{\mathbf{A}}(\mathbf{A}) = \alpha \quad &\text{and}\quad f_{\mathbf{A}}(\beta\mathbf{B}) &= \beta f_{\mathbf{A}}(\mathbf{B}) = 0 = f_{\mathbf{A}}(\gamma\mathbf{C}) \\ f_B(\beta\mathbf{B}) &= \beta f_{\mathbf{B}}(\mathbf{B}) = \beta \quad &\text{and}\quad f_{\mathbf{B}}(\gamma\mathbf{C}) &= \gamma f_{\mathbf{B}}(\mathbf{C}) = 0 = f_{\mathbf{B}}(\alpha\mathbf{A}) \\ f_{\mathbf{C}}(\gamma\mathbf{C}) &= \gamma f_{\mathbf{C}}(\mathbf{C}) = \gamma \quad &\text{and}\quad f_{\mathbf{C}}(\alpha\mathbf{A}) &= \alpha f_{\mathbf{C}}(\mathbf{A}) = 0 = f_{\mathbf{C}}(\beta\mathbf{B}) \end{aligned}$$

this incidence occurs precisely when

$$\begin{aligned} 0 &= f_\ell(\alpha\mathbf{A} + \beta\mathbf{B} + \gamma\mathbf{C}) \\ &= (lf_{\mathbf{A}} + mf_{\mathbf{B}} + nf_{\mathbf{C}})(\alpha\mathbf{A} + \beta\mathbf{B} + \gamma\mathbf{C}) \\ &= \alpha l f_{\mathbf{A}}(\mathbf{A}) + \beta m f_{\mathbf{B}}(\mathbf{B}) + \gamma n f_{\mathbf{C}}(\mathbf{C}) \\ &= \alpha l + \beta m + \gamma n. \end{aligned}$$

□

4.1. **Matrix Form for Barycentric Coordinates of Points and Lines.** Since the homogeneous barycentric coordinates of points and lines are 3-proportions, we may adopt, in the same spirit as in projective coordinates, a matrix form for these coordinates. To be consistent, with respect to a given oriented triangle $\overrightarrow{\mathbf{ABC}}$, we will associate to the barycentric coordinates $[\alpha : \beta : \gamma]$ of an affine point $\mathbf{P}$ the 1×3 matrix $\begin{bmatrix}\alpha & \beta & \gamma\end{bmatrix}$ and to the barycentric coordinates $(l : m : n)$ of an affine line ℓ the 3×1 matrix $\begin{bmatrix} l \\ m \\ n \end{bmatrix}$. The square brackets here indicative that the matrix **projective**, meaning that it is considered to be invariant under a non-zero scaling of all its entries. Incidence is then also captured the same way: the point $\mathbf{P}$ and the line ℓ above are incident precisely when

$$\begin{bmatrix}\alpha & \beta & \gamma\end{bmatrix}\begin{bmatrix} l \\ m \\ n \end{bmatrix} = \begin{bmatrix}0\end{bmatrix}.$$

Note that this condition is well-defined. The reason to represent points as rows and lines as columns, apart from being visually distinctive, is to capture point-line incidence in this matrix form.

5. Properties of Points and Lines in Barycentric Coordinates

Proposition 11. *Given an oriented triangle $\overrightarrow{\mathbf{ABC}}$, a point $\mathbf{P} = [\alpha : \beta : \gamma]$ lies on one of the sides precisely when $\alpha\beta\gamma = 0$.*

Proof. Without loss of generality, we can assume $\alpha + \beta + \gamma = 1$ so that $\mathbf{P} = \alpha\mathbf{A} + \beta\mathbf{B} + \gamma\mathbf{C}$. Suppose $\mathbf{P}$ lies on $\mathbf{AB}$, then $\mathbf{P}$ can be expressed as an affine combination of $\mathbf{A}$ and $\mathbf{B}$. Since $\mathbf{C}$ does not lie on $\mathbf{AB}$, we conclude $\gamma = 0$. Similarly, if $\mathbf{P}$ lies on $\mathbf{BC}$ or $\mathbf{CA}$, then, respectively, $\alpha = 0$ or $\beta = 0$.
Conversely, suppose $\alpha\beta\gamma = 0$. We know that α, β, γ cannot all be zero as that implies the signed areas of triangles $\overrightarrow{\mathbf{ABP}}, \overrightarrow{\mathbf{BCP}}$ and $\overrightarrow{\mathbf{CAP}}$ are all zero. If $\alpha = 0$, then $\mathbf{P} = \beta\mathbf{B} + \gamma\mathbf{C}$ is an affine combination of $\mathbf{B}$ and $\mathbf{C}$ so $\mathbf{P}$ lies on $\mathbf{BC}$. Likewise, if $\beta = 0$ or $\gamma = 0$, then, respectively, P lies on $\mathbf{CA}$ or $\mathbf{AB}$. □

Proposition 12. *Given an oriented triangle $\overrightarrow{\mathbf{ABC}}$, a line $\ell = (l : m : n)$ passes through one of the vertices precisely when $lmn = 0$.*

Proof. Let h be a functional corresponding to ℓ. If ℓ passes through $\mathbf{A}$, then $h(\mathbf{A}) = 0$. Likewise, if ℓ passes through $\mathbf{B}$ or $\mathbf{C}$, then, respectively, $m = 0$ or $n = 0$.
Conversely, suppose ℓ corresponds to a functional h where $l = h(\mathbf{A})$, $m = h(\mathbf{B})$, and $n = h(\mathbf{C})$. Then it is clear that if l, m, n equals 0, then ℓ passes through $\mathbf{A}, \mathbf{B}$ or $\mathbf{C}$. □

Proposition 13 (Barycentric Meet of Lines). *Two lines with barycentric coordinates with respect to an oriented triangle* $\overrightarrow{\mathbf{ABC}}$

$$\ell_1 = (l_1 : m_1 : n_1) \qquad \ell_2 = (l_2 : m_2 : n_2)$$

meet at a point with associated barycentric coordinates

$$\mathbf{P} = [m_1 n_2 - m_2 n_1 : n_1 l_2 - n_2 l_1 : l_1 m_2 - l_2 m_1].$$

Proof. This follows from the Barycentric Incidence Theorem (10) since

$$(m_1 n_2 - m_2 n_1)\, l_1 + (n_1 l_2 - n_2 l_1)\, m_1 + (l_1 m_2 - l_2 m_1)\, n_1 = 0 \qquad \text{and}$$
$$(m_1 n_2 - m_2 n_1)\, l_2 + (n_1 l_2 - n_2 l_1)\, m_2 + (l_1 m_2 - l_2 m_1)\, n_2 = 0.$$

□

Corollary 14 (Barycentric Concurrence). *Three lines with barycentric coordinates with respect to an oriented triangle* $\overrightarrow{\mathbf{ABC}}$

$$\ell_1 = (l_1 : m_1 : n_1) \qquad \ell_2 = (l_2 : m_2 : n_2) \qquad \ell_3 = (l_3 : m_3 : n_3)$$

concur precisely when

$$\begin{aligned} \Delta &= \det \begin{bmatrix} l_1 & l_2 & l_3 \\ m_1 & m_2 & m_3 \\ n_1 & n_2 & n_3 \end{bmatrix} \\ &= l_1 m_2 n_3 - l_1 m_3 n_2 - l_2 m_1 n_3 + l_2 m_3 n_1 + l_3 m_1 n_2 - l_3 m_2 n_1 = 0 \end{aligned}$$

Proof. By the previous proposition, ℓ_1, ℓ_2 meet at the point

$$[m_1 n_2 - m_2 n_1 : n_1 l_2 - n_2 l_1 : l_1 m_2 - l_2 m_1]$$

which lies on ℓ_3 precisely when

$$(m_1 n_2 - m_2 n_1)\, l_3 + (n_1 l_2 - n_2 l_1)\, m_3 + (l_1 m_2 - l_2 m_1)\, n_3 = \Delta = 0$$

□

Proposition 15 (Barycentric Join of Points). *The join of two points with barycentric coordinates with respect to an oriented triangle* $\overrightarrow{\mathbf{ABC}}$

$$\mathbf{P}_1 = [\alpha_1 : \beta_1 : \gamma_1] \qquad \mathbf{P}_2 = [\alpha_2 : \beta_2 : \gamma_2]$$

is the line with associated barycentric coordinates

$$\ell = (\beta_1 \gamma_2 - \beta_2 \gamma_1 : \gamma_1 \alpha_2 - \gamma_2 \alpha_1 : \alpha_1 \beta_2 - \alpha_2 \beta_1).$$

Proof. This follows from the Barycentric Incidence Theorem (10) since

$$\alpha_1 (\beta_1 \gamma_2 - \beta_2 \gamma_1) + \beta_1 (\gamma_1 \alpha_2 - \gamma_2 \alpha_1) + \gamma_1 (\alpha_1 \beta_2 - \alpha_2 \beta_1) = 0 \qquad \text{and}$$
$$\alpha_2 (\beta_1 \gamma_2 - \beta_2 \gamma_1) + \beta_2 (\gamma_1 \alpha_2 - \gamma_2 \alpha_1) + \gamma_2 (\alpha_1 \beta_2 - \alpha_2 \beta_1) = 0.$$

□

Corollary 16 (Barycentric Collinearity)**.** *Three points with barycentric coordinates with respect to an oriented triangle* $\overrightarrow{\mathbf{ABC}}$

$$\mathbf{P}_1 = [\alpha_1 : \beta_1 : \gamma_1] \qquad \mathbf{P}_2 = [\alpha_2 : \beta_2 : \gamma_2] \qquad \mathbf{P}_3 = [\alpha_3 : \beta_3 : \gamma_3]$$

are collinear precisely when

$$\Delta = \det \begin{bmatrix} \alpha_1 & \beta_1 & \gamma_1 \\ \alpha_2 & \beta_2 & \gamma_2 \\ \alpha_3 & \beta_3 & \gamma_3 \end{bmatrix} = 0$$

Proof. This is the same situation as the Barycentric Concurrence Corollary (14). □

Theorem 17. *Given an oriented triangle* $\overrightarrow{\mathbf{ABC}}$*, no line has barycentric coordinates* $(1 : 1 : 1)$.

Proof. Suppose for contradiction that a line $\ell = (1 : 1 : 1)$ where ℓ is represented by $h = \mathbf{u} \times (\mathbf{V} - \mathbf{P})$ where $\mathbf{u}, \mathbf{P}$ are, respectively, a direction vector and a base point of ℓ and $\mathbf{V}$ a variable point. Since $h(\mathbf{A}) = h(\mathbf{B})$, $\mathbf{u} \times (\mathbf{A} - \mathbf{P}) = \mathbf{u} \times (\mathbf{B} - \mathbf{P})$ and

$$\begin{aligned} \mathbf{u} \times (\mathbf{A} - \mathbf{P}) - \mathbf{u} \times (\mathbf{B} - \mathbf{P}) &= \mathbf{u} \times ((\mathbf{A} - \mathbf{P}) - (\mathbf{B} - \mathbf{P})) \\ &= \mathbf{u} \times (\mathbf{A} - \mathbf{B}) = 0 \end{aligned}$$

so $\mathbf{u} \parallel \overrightarrow{\mathbf{AB}}$. By the same reasoning, we can also show $\mathbf{u} \parallel \overrightarrow{\mathbf{BC}}$ and $\mathbf{u} \parallel \overrightarrow{\mathbf{CA}}$. Since $\mathbf{A}, \mathbf{B}, \mathbf{C}$ are not collinear, there is a contradiction. □

6. The Barycentric Ratio Theorem

Given an oriented triangle $\overrightarrow{\mathbf{ABC}}$ and a point $\mathbf{P}$ not on the sides, the lines $\mathbf{PC}, \mathbf{PA}, \mathbf{PB}$ are known as **Cevians** through $\mathbf{P}$. We say a line ℓ **divides** an oriented side $\overrightarrow{\mathbf{AB}}$ in the ratio of $\alpha : \beta$ precisely when the meet of the two lines, $\mathbf{D} \equiv (\ell)(\mathbf{AB})$, divides $\overrightarrow{\mathbf{AB}}$ in the ratio of $\alpha : \beta$.

Theorem 18 (Barycentric Ratio)**.** *Given an oriented triangle* $\overrightarrow{\mathbf{ABC}}$*,*
(a) the cevians through a point $\mathbf{P} = [\alpha : \beta : \gamma]$*, which is not on any side, divides* $\overrightarrow{\mathbf{AB}}, \overrightarrow{\mathbf{BC}}$ *and* $\overrightarrow{\mathbf{CA}}$ *respectively into ratios*

$$\beta : \alpha \qquad \gamma : \beta \qquad \text{and} \qquad \alpha : \gamma$$

(b) a transversal line $\ell = (l : m : n)$*, which does not pass any vertex, divides* $\overrightarrow{\mathbf{AB}}, \overrightarrow{\mathbf{BC}}$ *and* $\overrightarrow{\mathbf{CA}}$ *respectively into ratios*

$$-l : m \qquad -m : n \qquad \text{and} \qquad -n : l$$

Proof. (a) Since $\mathbf{P}$ does not lie on any side, $\alpha\beta\gamma \neq 0$ so the ratios are well-defined. In terms of barycentric coordinates, we have $\mathbf{P} = [\alpha : \beta : \gamma]$, $\mathbf{C} = [0 : 0 : 1]$, $\mathbf{AB} = (0 : 0 : 1)$, so by the Barycentric Join of Points Proposition (15), $\mathbf{PC} = (\beta : -\alpha : 0)$ and, by the Barycentric Meet of Lines Proposition (13), $\mathbf{D} = (\mathbf{PC})(\mathbf{AB}) = [-\alpha : -\beta : 0]$. Since $\mathbf{D} = \frac{\alpha}{\alpha+\beta}\mathbf{A} + \frac{\beta}{\alpha+\beta}\mathbf{B}$, by the Proportion Point Proposition (6)

$$\overrightarrow{\mathbf{AD}} : \overrightarrow{\mathbf{DB}} = \beta : \alpha.$$

The other two ratios can be obtained similarly.

(b) Since ℓ does not pass through any vertex, $lmn \neq 0$ so the ratios are well-defined. Taking a similar approach to (a), suppose ℓ and $\mathbf{AB}$ meet at $\mathbf{D}'$. Then, by the Barycentric Meet of Lines Proposition (13), $\mathbf{D}' = (\ell)(\mathbf{AB}) = [m : -l : 0] = \frac{m}{m-l}\mathbf{A} - \frac{l}{m-l}\mathbf{B}$ and

$$\overrightarrow{\mathbf{AD}'} : \overrightarrow{\mathbf{D}'\mathbf{B}} = -l : m$$

by the Proportion Point Proposition (6). The other two ratios are obtained similarly. □

7. Conversion from Trilinear Form in the Euclidean Triangle Geometry

In the literature on (Euclidean) triangle geometry [5, p.25-27], trilinear coordinates are used to describe points and lines. With respect to a reference oriented triangle $\overrightarrow{\mathbf{ABC}}$ where a, b and c are the lengths of sides $\overline{\mathbf{BC}}, \overline{\mathbf{CA}}$ and $\overline{\mathbf{AB}}$, the line $\mathbf{BC}$ separates the plane into two **half-planes**. The half-plane containing vertex $\mathbf{A}$ is considered the **positive** side of $\mathbf{BC}$, and the other half-plane is the **negative** side of $\mathbf{BC}$. If $\mathbf{P}$ is any point in the plane, then its **signed distance** from the line is $\delta\alpha$, where α is the usual Euclidean distance from $\mathbf{P}$ to $\mathbf{BC}$, and $\delta = -1$ if $\mathbf{P}$ lies on the negative side of $\mathbf{BC}$ and $\delta = 1$ otherwise. Similarly, we can speak of signed distances of $\mathbf{P}$ from $\mathbf{CA}$ and $\mathbf{AB}$. If α, β, γ are the signed distances of $\mathbf{P}$ from the sides $\overline{\mathbf{BC}}, \overline{\mathbf{CA}}$ and $\overline{\mathbf{AB}}$, then the 3-proportion $\alpha : \beta : \gamma$ is called the **homogeneous trilinear coordinates** of the point $\mathbf{P}$. In order to distinguish from the barycentric coordinates, we will denote such coordinates as

$$\mathbf{P} = \lceil \alpha : \beta : \gamma \rceil .$$

Proposition 19 (Point Conversion)**.** [13, p.570] *Given an oriented triangle* $\overrightarrow{\mathbf{ABC}}$ *with side lengths a, b and c, a point $\mathbf{P} = \lceil \alpha : \beta : \gamma \rceil$ has equivalent barycentric coordinates $[a\alpha : b\beta : c\gamma]$.*

By the Barycentric Incidence Theorem (10), such a point $\mathbf{P}$ lies on a line with barycentric coordinates $\ell = (r : s : t)$ precisely when

$$(a\alpha)r + (b\beta)s + (c\gamma)t = 0 = \alpha(ar) + \beta(bs) + \gamma(ct).$$

This means, we may consider the 3-proportion $ar : bs : ct$ as the **trilinear coordinates** of the line ℓ as defined in [5, p.27], and denote as $\lfloor ar : bs : ct \rfloor$ to differentiate from the barycentric coordinates. This also allow us to capture the conversion between the two coordinate systems below.

Proposition 20 (Line Conversion)**.** *Given an oriented triangle* $\overrightarrow{\mathbf{ABC}}$ *with side lengths a, b and c, a line with trilinear coordinates $\lfloor l : m : n \rfloor$ has equivalent barycentric coordinates $\left(\frac{l}{a} : \frac{m}{b} : \frac{n}{c}\right)$, or equivalently, $(bcl : cam : abn)$ since $abc \neq 0$.*

8. Examples of Lines in Barycentric Coordinates

We can now apply the framework of barycentric coordinates for lines in triangle geometry and look at some famous lines in that subject.

8.1. **An Affine Example - Medians.** Given an oriented triangle $\overrightarrow{\mathbf{ABC}}$, the mid-points of the sides are, respectively,

$$\mathbf{M}_{AB} = [1:1:0] \qquad \mathbf{M}_{BC} = [0:1:1] \qquad \mathbf{M}_{CA} = [1:0:1].$$

Hence, by the Barycentric Join of Points Proposition (15), the medians are

$$\mathbf{AM}_{BC} = (0:1:-1) \qquad \mathbf{BM}_{CA} = (1:0:-1) \qquad \mathbf{CM}_{AB} = (1:-1:0).$$

8.2. **Examples in Euclidean Geometry.** The separation of two points in Euclidean geometry is usually measured by **distance**, which is based on the classical definition of the **inner product** or **dot product** of two vectors. A dot product of two vectors $\mathbf{u} \equiv (u_1, u_2)$ and $\mathbf{v} \equiv (v_1, v_2)$ is defined as [8, p.182]

$$\mathbf{u} \cdot \mathbf{v} \equiv u_1 v_1 + u_2 v_2.$$

The distance between the points $\mathbf{P} \equiv [p_1, p_2]$ and $\mathbf{Q} \equiv [q_1, q_2]$ (or, equivalently, the **length** of the line segment $\overline{\mathbf{PQ}}$) is then defined as the non-negative square root of the expression

$$(q_1 - p_1)^2 + (q_2 - p_2)^2$$

as described by Meserve [7, p.199], Brannan, Esplen and Gray [1, p.7] and Kiselev [6, p.176]. Since such definition requires a prior notion of positivity and square root, we will instead adopt, for a general field, the rational measurement of **quadrance** as introduced by Wildberger [9, p.59]: the quadrance between two points $\mathbf{P} \equiv [p_1, p_2]$ and $\mathbf{Q} \equiv [q_1, q_2]$ is defined as

$$\mathcal{Q}(\mathbf{P}, \mathbf{Q}) \equiv (q_1 - p_1)^2 + (q_2 - p_2)^2.$$

Given an oriented triangle $\overrightarrow{\mathbf{ABC}}$ with side lengths a, b and c where

$$a \equiv |\overline{\mathbf{BC}}| \qquad b \equiv |\overline{\mathbf{CA}}| \qquad \text{and} \qquad c \equiv |\overline{\mathbf{AB}}|,$$

we may adopt the corresponding rational measurements Q_a, Q_b and Q_c where

$$Q_a \equiv \mathcal{Q}(\mathbf{B}, \mathbf{C}) = a^2 \qquad Q_b \equiv \mathcal{Q}(\mathbf{C}, \mathbf{A}) = b^2 \qquad \text{and} \qquad Q_c \equiv \mathcal{Q}(\mathbf{A}, \mathbf{B}) = c^2.$$

Many lines are well studied in the Euclidean setting and have been named before 1900 [5, p. 150]. By combining the use of quadrances and the Line Conversion Proposition (20), we are able to identify many of these lines with rational barycentric coordinates. Note that notations such as $L(2,3)$ and L_2 are also due to Kimberling [5].

8.2.1. *Euler Line* $L(2,3)$. The Euler line is classically known as the line joining the triangle's centroid $\mathbf{G}$ and circumcenter $\mathbf{O}$ [5, p. 150]. In barycentric coordinates, these centers are

$$\begin{aligned}
\mathbf{G} &= [1:1:1] \\
\mathbf{O} &= [Q_a(Q_b + Q_c - Q_a) : Q_b(Q_c + Q_a - Q_b) : Q_c(Q_a + Q_b - Q_c)]
\end{aligned}$$

so, by the Barycentric Join of Points Proposition (15), the Euler line has barycentric coordinates

$$\begin{gathered}
((Q_b - Q_c)(Q_b - Q_a + Q_c) : \\
(Q_c - Q_a)(Q_a - Q_b + Q_c) : \\
(Q_a - Q_b)(Q_a + Q_b - Q_c)).
\end{gathered}$$

8.2.2. *Brocard Axis* $L(3,6)$. Although the Brocard axis is known in the Euclidean setting as the join of the two isodynamic points, it may be re-defined as the radical axis of the Apollonian circles of the given triangle [3, p.1247-1248] to accommodate different metric structures. From the equations of Apollonian circles and radical axis in the same paper, we have (with $\mathbf{V}$ as the variable point)

$$\ell_B : \mathbf{z} \cdot (\mathbf{V} - \mathbf{S}) = 0$$

where

$$\begin{aligned}
\mathbf{z} &= Q_a (Q_b - Q_c) \mathbf{A} + Q_b (Q_c - Q_a) \mathbf{B} + Q_c (Q_a - Q_b) \mathbf{C} \\
\mathbf{S} &= \frac{Q_a (2Q_a - Q_b - Q_c)}{2T} \mathbf{A} + \frac{Q_b (2Q_b - Q_a - Q_c)}{2T} \mathbf{B} \\
&\quad + \frac{Q_c (2Q_c - Q_b - Q_a)}{2T} \mathbf{C} \\
T &= Q_a^2 + Q_b^2 + Q_c^2 - Q_a Q_b - Q_b Q_c - Q_c Q_a.
\end{aligned}$$

Note that the Brocard Axis is expressed in terms of dot product where the Lemoine Axis (below) is expressed in terms of cross product; highlighting that these two axes are perpendicular to each other, where perpendicularity is established by the definition of dot product imposed by the additional metrical structure. By evaluating $\mathbf{z} \cdot (\mathbf{V} - \mathbf{S})$ at $\mathbf{A}, \mathbf{B}, \mathbf{C}$ and simplifying resultant expressions, the Brocard axis has barycentric coordinates

$$\left(\frac{Q_b - Q_c}{Q_a} : \frac{Q_c - Q_a}{Q_b} : \frac{Q_a - Q_b}{Q_c} \right).$$

8.2.3. *Lemoine Axis* L_2. The Lemoine axis is the join of the centers of the Apollonian circles of the given triangle [3, p.1247-1248] and can be expressed as (with $\mathbf{V}$ as the variable point)

$$\ell_L : \mathbf{z} \times (\mathbf{V} - \mathbf{S}) = 0$$

where $\mathbf{z}$ and $\mathbf{S}$ are given in the previous paragraph under the Brocard axis. Through a similar process to the above, the Lemoine axis has barycentric coordinates

$$(Q_b Q_c : Q_c Q_a : Q_a Q_b).$$

The remaining lines are obtained from [5, p. 150] and converted from their trilinear coordinates listed in the same book.

8.2.4. *Line at Infinity* L_6. The Line at Infinity is, strictly speaking, not part of Euclidean geometry, but it still plays a role in the projective version. It has trilinear coordinates $\lfloor a : b : c \rfloor$ or, by the Line Conversion Proposition (20), barycentric coordinates

$$(1 : 1 : 1).$$

8.2.5. *Orthic Axis* L_3. The Orthic axis has trilinear coordinates

$$\left\lfloor a \left(b^2 + c^2 - a^2\right) : b \left(c^2 + a^2 - b^2\right) : c \left(a^2 + b^2 - c^2\right) \right\rfloor$$

or, by the Line Conversion Proposition (20), barycentric coordinates

$$(Q_b + Q_c - Q_a : Q_c + Q_a - Q_b : Q_a + Q_b - Q_c).$$

8.2.6. *Antiorthic Axis L_1.* The Antiorthic axis has trilinear coordinates $\lfloor 1:1:1 \rfloor$ or, by the Line Conversion Proposition (20), barycentric coordinates

$$\left(\frac{1}{a}:\frac{1}{b}:\frac{1}{c}\right).$$

Note that these coordinates suggest that the Antiorthic axis may not be defined for a general triangle over a general field.

8.2.7. *DeLongchamp Axis L_{32}.* The DeLongchamp axis has trilinear coordinates $\left\lfloor a^3 : b^3 : c^3 \right\rfloor$ or, by the Line Conversion Proposition (20), barycentric coordinates

$$\left(Q_a : Q_b : Q_c\right).$$

9. Conclusion

The framework of barycentric coordinates for lines brings points and lines in triangle geometry on an equal footing by providing a geometric interpretation to the coordinates. Lines whose barycentric coordinates consist of only quadrances already provide a convenient starting point for exploration in geometries where the metrical structures are defined differently. Example of such investigations include the Euler line by Wildberger [12] and the Lemoine and Brocard axes by Ho and Wildberger [3].
In [5], Kimberling defines a line with trilinear coordinates $\lfloor l : m : n \rfloor$ as a **central line** if the point $\lceil l : m : n \rceil$ is a **center**. We may establish a similar correspondence between a point and a line having the same barycentric coordinates. In fact, one may even suspect that the barycentric system exhibits a closer relationship. For example, both the centroid (the only affine triangle center) and the line at infinity (a fundamental object in projective geometry) share the same coordinates $1:1:1$.

References

[1] D. A. Brannan, M. F. Esplen, and J. J. Gray, *Geometry*, 2nd Ed., Cambridge University Press, 2012

[2] O. Bottema, On the Distances of a Point to the Vertices of a Triangle, *Crux Math.* **10** (1984), 242-246

[3] J. C. Ho & N. J. Wildberger, A Case Study in Universal Geometry: Extending Apollonian Circles to Relativistic Geometry and Finite Fields, *Sib. Èlektron. Mat. Izv.*, **17** (2020), 1227–1257

[4] J. C. Ho & N. J. Wildberger, A Physical Archimedean Approach to Affine Geometry and the Remarkable 13 (Mixed) Configuration, *Journal for Geometry and graphics*, **26** (1) (2022), 65-80

[5] C. Kimberling, *Triangle Centers and Central Triangles*, Congressus Numerantium, 1998

[6] A. P. Kiselev, *Geometry, Book I: Planimetry*, Sumizdat, 2006

[7] B. E. Meserve, *Fundamental Concepts of Geometry*, Dover Publications, Inc., 1983

[8] S. Roman, Advanced Linear Algebra, 2nd ed., Springer, 2005

[9] N. J. Wildberger, *Divine Proportions: Rational Trigonometry to Universal Geometry*, Wild Egg Books, 2005

[10] N. J. Wildberger, Affine and Projective Universal Geometry, *arXiv:math/0612499v1*

[11] N. J. Wildberger, *Algebraic Calculus One*, https://www.openlearning.com/courses/algebraic-calculus-one/

[12] N. J. Wildberger, Chromogeometry, *Mathematical Intelligencer* **32** (1) (2010), 26-32

[13] P. Yiu, The Uses of Homogeneous Barycentric Coordinates in Plane Euclidean Geometry, *International Journal of Mathematical Education in Science and Technology*, **31** (4) (2000), 569-578

School of Mathematics and Statistics, UNSW Sydney
Email address: joshua.ho@unsw.edu.au