Global Journal of Advanced Research
on Classical and Modern Geometries
ISSN: 2284-5569, Vol.15, (2026), Issue 1, pp.129-137

ANGLE ADDITION, THE EXTERIOR ANGLE THEOREM, AND THE EQUIVALENCE OF PLAYFAIR'S POSTULATE WITH THE EUCLIDEAN PARALLEL POSTULATE

JOHN DONNELLY

ABSTRACT. We answer a question of Brown, Castner, Davis, O'Shea, Seryozhenkov, and Vargas. In particular we prove that in a non-SAS geometry, if we assume Angle Addition (and to a limited amount, the Exterior Angle Theorem), then the classical Euclidean Parallel Postulate is equivalent to Playfair's Postulate. We prove that if we assume the classical Euclidean Parallel Postulate and Angle Addition, then we can prove the Exterior Angle Theorem as a consequence (without the use of Side-Angle-Side). We construct a model which satisfies Angle Addition, Playfair's Postulate, as well as all the axioms of absolute geometry (with the exception of SAS), but which does not satisfy either of the classical Euclidean Parallel Postulate or the Exterior Angle Theorem.

1. INTRODUCTION

In [1], Brown, Castner, Davis, O'Shea, Seryozhenkov, and Vargas construct a model of non-SAS geometry in which Playfair's Postulate holds, but in which the classical Euclidean Parallel Postulate fails. This model shows that without Side-Angle-Side, the equivalence between the classical Euclidean Parallel Postulate and Playfair's Postulate does not hold. They then ask for statements other than Side-Angle-Side which are strong enough (together with the other axioms of plane geometry) to ensure the equivalence of the classical Euclidean Parallel Property and Playfair's Postulate (as well as other traditionally equivalent statements). In particular, they ask if Side-Angle-Side is replaced with Angle-Addition, then can one prove the equivalence of the classical Euclidean Parallel Property and Playfair's Postulate. In this paper, we show that if we assume Angle-Addition and the classical Euclidean Parallel Property, then we can prove Playfair's Postulate. When showing this, we first prove that the Exterior Angle Theorem holds, and then use this fact to prove Playfair's Postulate.

We then show that if we assume Angle-Addition, Playfair's Postulate, and the Exterior Angle Theorem, then we can prove the classical Euclidean Parallel Property. We construct a model, which satisfies Angle Addition, Playfair's Postulate, as well as all the axioms of absolute geometry (with the exception of SAS), but which does not satisfy either of the classical Euclidean Parallel Postulate or the Exterior Angle Theorem. This model shows that even when assuming Angle-Addition and Playfair's Postulate in addition to

2010 *Mathematics Subject Classification.* 51P99.
Key words and phrases. Parallel Postulates, Exterior Angle Theorem, Non-SAS Geometry.

the other axioms, we cannot prove the Exterior Angle Theorem. This model also shows that it is not enough to assume Angle-Addition and Playfair's Postulate when trying to prove the classical Euclidean Parallel Postulate, we also need to assume the Exterior Angle Theorem as well.

2. The Axioms

In this section we state the axioms of absolute geometry as given in [5], with the exception of Side-Angle-Side. We note that as in [5], *betweenness* and *congruence* for segments and angles are taken here to be undefined notions.

The Incidence Axioms:

(I1) Given any two distinct points A and B, then there exists a unique line l containing A and B.

(I2) Every line contains at least two distinct points.

(I3) There exist three distinct noncollinear points.

The Betweenness Axioms:

(B1) If B is between A and C, (written $A - B - C$), then A, B, and C are three distinct collinear points, and also $C - B - A$.

(B2) Given any two distinct points A and B, then there exists a point C such that $A - B - C$.

(B3) Given three distinct points on a line, then one and only one of them is between the other two.

(B4) (PASCH) Let A, B, and C be three distinct noncollinear points, and let l be a line not containing any of A, B, or C. If l contains a point D such that $A - D - B$, then l also contains a point lying between A and C or else a point lying between B and C, but not both.

It is shown in [5] that PASCH is equivalent to the *Plane Separation Postulate*:
Let $\mathcal{P}$ denote the set of all points. For each line l, there exist two nonempty convex sets $\mathcal{H}_1$ and $\mathcal{H}_2$ such that
(1) $\mathcal{P} \backslash l = \mathcal{H}_1 \cup \mathcal{H}_2$
(2) If $P \in \mathcal{H}_1$, $Q \in \mathcal{H}_2$, and $P \neq Q$, then $\overline{PQ} \cap l \neq \varnothing$.

The sets $\mathcal{H}_1$ and $\mathcal{H}_2$ are called *halfplanes* (or *sides*) of l. If $A, B \in \mathcal{H}_i$ (i=1,2), then we say that A and B are on the *same side* of l. If $A \in \mathcal{H}_1$ and $B \in \mathcal{H}_2$, then we say that A and B are on *opposite sides* of l. Also, l is called an *edge* of each of the halfplanes H_1 and H_2. It can be proven that H_1 and H_2 are disjoint. When referring to the Plane Separation Postulate, we will abbreviate it as *PSP*.

Given two distinct points A and B, then we define the *line segment* $\overline{AB}$ to be the set consisting of A and B together with all points P such that $A - P - B$. Given two line segments $\overline{AB}$ and $\overline{CD}$, then we use the notation $\overline{AB} \cong \overline{CD}$ to denote when $\overline{AB}$ and $\overline{CD}$ are congruent to each other.

Given a segment $\overline{AB}$, then we define the *interior* of $\overline{AB}$ to be the set of all points C such that $A - C - B$. We denote the interior of $\overline{AB}$ by $int(\overline{AB})$.

Given two distinct points A and B, then we define the *ray* $\overrightarrow{AB}$ to be the set consisting of A and B, together with all points P such that exactly one of the following is true: $A - P - B$ or $A - B - P$. We refer to the point A as the *vertex* of *ray* $\overrightarrow{AB}$.

Given three distinct noncollinear points A, V, and B, then we define $\angle AVB = \overrightarrow{VA} \cup \overrightarrow{VB}$. We call $\angle AVB$ an *angle* with vertex V and sides $\overrightarrow{VA}$ and $\overrightarrow{VB}$. We note that since A, V, and B are not collinear, then we do not consider "zero angles" or "straight angles" in this paper. Given two angles $\angle ABC$ and $\angle DEF$, then we use the notation $\angle ABC \cong \angle DEF$ to denote when $\angle ABC$ and $\angle DEF$ are congruent to each other.
Given an angle $\angle ABC$, then we define the *interior* of $\angle ABC$ to be the intersection of the halfplane of line $\overleftrightarrow{AB}$ containing C together with the halfplane of line $\overleftrightarrow{AC}$ containing B. We denote the interior of $\angle ABC$ by $int(\angle ABC)$. If $G \in int(\angle ABC)$, then we say that ray $\overrightarrow{BG}$ is in the interior of $\angle ABC$. (We note that this is a slight abuse of terminology since the vertex point B is not in the interior of $\angle ABC$, but that all other points on ray $\overrightarrow{BG}$ are in the interior of $\angle ABC$).
Given three noncollinear points A, B, and C, then we define $\triangle ABC = \overline{AB} \cup \overline{BC} \cup \overline{CA}$. We call $\triangle ABC$ the *triangle* with vertices A, B, and C.
We next state the congruence axioms.

The Congruence Axioms:

(C1) Given a line segment $\overline{AB}$, and given a ray r originating at a point C, then there exists a unique point D on the ray r such that $\overline{AB} \cong \overline{CD}$.
(C2) If $\overline{AB} \cong \overline{CD}$ and $\overline{AB} \cong \overline{EF}$, then $\overline{CD} \cong \overline{EF}$. Every line segment is congruent to itself.
(C3) (Addition of Segments) Given three points A, B, and C satisfying $A - B - C$, and three other points D, E, and F satisfying $D - E - F$, if $\overline{AB} \cong \overline{DE}$ and $\overline{BC} \cong \overline{EF}$, then $\overline{AC} \cong \overline{DF}$.
(C4) Given an angle $\angle BAC$ and given a ray $\overrightarrow{DF}$, then there exists a unique ray $\overrightarrow{DE}$ on a given side of line $\overleftrightarrow{DF}$ such that $\angle BAC \cong \angle EDF$.
(C5) Given any three angles α, β, and γ, if $\alpha \cong \beta$ and $\alpha \cong \gamma$, then $\beta \cong \gamma$. Every angle is congruent to itself.

We do not include the sixth congruence axiom *Side-Angle-Side* given in [5] in our list of axioms. Instead we will assume the following new axiom below.

New Axiom:

(N1) (Addition of Angles) Assume that $\angle BAC$ is an angle and that ray $\overrightarrow{AD}$ is in the interior of angle $\angle BAC$. Suppose that $\angle D'A'C' \cong \angle DAC$, $\angle B'A'D' \cong \angle BAD$, and that rays $\overrightarrow{A'B'}$ and $\overrightarrow{A'C'}$ are on opposite sides of line $\overleftrightarrow{A'D'}$. Then rays $\overrightarrow{A'B'}$ and $\overrightarrow{A'C'}$ form an angle $\angle B'A'C'$ such that $\angle B'A'C' \cong \angle BAC$ and such that ray $\overrightarrow{A'D'}$ is in the interior of $\angle B'A'C'$.

As a consequence of axiom (C4) and Angle Addition, we have that the following is true: Given angles $\angle BAC$ and $\angle B'A'C'$, if ray $\overrightarrow{AD}$ is in the interior of angle $\angle BAC$, ray $\overrightarrow{A'D'}$ is in the interior of $\angle B'A'C'$, $\angle D'A'C' \cong \angle DAC$, and $\angle B'A'D' \cong \angle BAD$, then $\angle B'A'C' \cong \angle BAC$.
Let l be a line, and let A, B, C, and D be distinct points on l such that $A - B - C - D$. Let $\mathcal{H}$ denote a halfplane of l, and let P and Q be points in $\mathcal{H}$. Let ray $\overrightarrow{CT}$ be the unique ray

in $\mathcal{H}$ such that $\angle DBP \cong \angle DCT$. We say that angles $\angle CBP$ and $\angle BCQ$ are *less than two right angles* if $Q \in int(\angle BCT)$.
We now state the classical version of the *Euclidean Parallel Postulate*.

The Euclidean Parallel Postulate:
That, if a straight line falling on two straight lines make the interior angles on the same side less than two right angles, the two straight lines, if produced indefinitely, meet on that side on which are the angles less than the two right angles.

When referring to the classical Euclidean Parallel Postulate, we will abbreviate it as EPP. More precisely, EPP states that if two distinct lines l and q are cut by a transversal t such that two angles on the same side are less than two right angles, then the rays on that side formed by the two lines will intersect.
We now state *Playfair's Postulate*.

Playfair's Postulate:
Given a line l and a point P not on l, then there exists a unique line q passing through P such that q and l are parallel.

When referring to Playfair's Postulate, we will abbreviate it as PF.
We note that Axioms (I1)-(I3), (B1)-(B4), (C1)-(C5), (N1), EPP, PF and the Exterior Angle Theorem all hold in Taxicab Geometry, but that SAS does not hold in Taxicab Geometry. Thus, when assuming any of the above, we are not tacitly assuming SAS.
The following lemma is proven in [4]. The proof does not require the use of Side-Angle-Side, but only depends on Axioms Axioms (I1)-(I3), (B1)-(B4), (C1)-(C5), and (N1).

Lemma 2.1. *(Supplements of Congruent Angles) Given angle $\angle ABC$, let point P be such that $P - B - C$, so that $\angle ABC$ and $\angle ABP$ are supplementary angles. Similarly, given angle $\angle DEF$, let point Q be such that $Q - E - F$, so that $\angle DEF$ and $\angle DEQ$ are supplementary angles. If $\angle ABC \cong \angle DEF$, then $\angle ABP \cong \angle DEQ$. That is, supplementary angles of congruent angles are congruent.*

An immediate consequence of Lemma 2.1 is that vertical angles are congruent.

3. EPP IMPLIES PF

In this section, we show that if we assume Axioms (I1)-(I3), (B1)-(B4), (C1)-(C5), (N1), and also assume EPP, then we can prove that PF holds.
Given angles $\angle ABC$ and $\angle DEF$, we say that angle $\angle DEF$ is *less than* $\angle ABC$ if there exists a ray $\overrightarrow{BG}$ in the interior of $\angle ABC$ such that $\angle ABG \cong \angle DEF$. We denote this by $\angle DEF < \angle ABC$. Equivalently, we say that $\angle ABC$ is *larger than* $\angle DEF$, and denote this by $\angle ABC > \angle DEF$.
The following is proven in [4].

Lemma 3.1. *Let angles $\angle ABC$ and $\angle DEF$ be such that $\angle DEF < \angle ABC$. Then there exists a unique point $K \in int(\overline{AC})$ such that $\angle ABK \cong \angle DEF$.*

Theorem 3.1. *(The Exterior Angle Theorem) Assume that Axioms (I1)-(I3), (B1)-(B4), (C1)-(C5), and (N1) all hold. Also, assume that EPP holds. Then an exterior angle of a triangle is larger than either of its remote interior angles.*

Proof. Let $\triangle ABC$ and point D be such that $A - C - D$. Suppose that $\angle BAC \geq \angle BCD$. If $\angle BAC \cong \angle BCD$, then let $T = B$. If $\angle BAC > \angle BCD$, then let T be the unique point such that $B - T - C$ and $\angle TAC \cong \angle BCD$. In either case, we have triangle $\triangle TAC$ such that $\angle TAC \cong \angle TCD$.
Let F be a point on line $\overleftrightarrow{TA}$ such that $T - A - F$. Let Q be a point on ray $\overrightarrow{CT}$ such that $C - T - Q$. Let K be a point on line $\overleftrightarrow{QA}$ such that $Q - A - K$. Finally, let H be a point on line $\overleftrightarrow{QC}$ such that $Q - C - H$. Since $\angle TAC \cong \angle TCD$, and since supplements of congruent angles are congruent, then it follows that $\angle FAC \cong \angle QCA$. Since $\angle QCD$ and $\angle ACH$ are vertical angles, then $\angle QCD \cong \angle ACH$.
Since $Q - A - K$ and $T - A - F$, then it follows that $K \in int(\angle DAF)$. Thus, angles $\angle CAK$ and $\angle ACH$ are less than two right angles. Therefore, it follows by EPP that rays $\overrightarrow{AK}$ and $\overrightarrow{CH}$ intersect at a point I such that I is on the same side of line $\overleftrightarrow{AC}$ as the points F, K and H. Since $Q - C - H$, then Q and H are on opposite sides of $\overleftrightarrow{AC}$. Consequently, Q and I are on opposite sides of $\overleftrightarrow{AC}$, which implies that $Q \neq I$. But this implies that distinct lines $\overleftrightarrow{QA}$ and $\overleftrightarrow{QC}$ intersect at the two distinct points Q and I, a contradiction.
Hence, it must be the case that $\angle BAC < \angle BCD$. We can now use a similar argument together with the fact that vertical angles are congruent to show that $\angle ABC < \angle BCD$. □

Let $\overleftrightarrow{AB}$, $\overleftrightarrow{CD}$, and $\overleftrightarrow{FG}$ be three distinct lines such that $A - F - B$ and $C - G - D$, with A and C on the same side of $\overleftrightarrow{FG}$. We call $\overleftrightarrow{FG}$ a *transversal* of $\overleftrightarrow{AB}$ and $\overleftrightarrow{CD}$, and we say that lines $\overleftrightarrow{AB}$ and $\overleftrightarrow{CD}$ are *cut* by the transversal $\overleftrightarrow{FG}$. We call $\angle AFG$ and $\angle DGF$ *alternate interior angles*. We say that two lines l and q are *parallel* if they have no points in common. We denote this by the standard notation $l \parallel q$.
The following corollary follows immediately from Theorem 3.1.

Corollary 3.1. *If two lines are cut by a transversal such that a pair of alternate interior angles are congruent, then the two lines are parallel.*

Theorem 3.2. *Assume that Axioms (I1)-(I3), (B1)-(B4), (C1)-(C5), and (N1) all hold. Also, assume that EPP holds. Then PF also holds.*

Proof. Since Axioms (I1)-(I3), (B1)-(B4), (C1)-(C5), (N1), and EPP all hold, then it follows by Theorem 3.1 that the Exterior Angle Theorem holds as well. Let P be a point and let l be a line not passing through P. Let Q and K be two distinct points on l. Since P is not on l, then $P \neq Q$ and $P \neq K$. Let $t = \overleftrightarrow{PQ}$, and let D be a point on t such that $Q - P - D$. Let A be a point on the same side of t as K such that $\angle APD \cong \angle KQP$. It follows from Corollary 3.1 that lines $\overleftrightarrow{KQ}$ and $\overleftrightarrow{AP}$ are parallel.
Assume that h is a line through P such that $h \neq \overleftrightarrow{AP}$. Let I and J be distinct points on h such that $I - P - J$, and such that I, A and K are on the same side of t. Let B and C be points such that $K - Q - C$ and $A - P - B$. Thus, J, B and C are on the same side of t.
If K and I are on the same side of $\overleftrightarrow{AP}$, then it follows that $\angle KQP$ and $\angle IPQ$ are less than two right angles. Therefore, by EPP, rays $\overrightarrow{QK}$ and $\overrightarrow{PI}$ cross at a point X. In this case, l and h cross at X.
If K and I are on opposite sides of $\overleftrightarrow{AP}$, then it follows by PSP that J and C are on the same side of $\overleftrightarrow{AP}$. Since supplements of congruent angles are congruent, then it follows

that $\angle JPQ$ and $\angle CQP$ are less than two right angles. Thus, by EPP, rays $\overrightarrow{QC}$ and $\overrightarrow{PJ}$ cross at a point Y. In this case, l and h cross at Y.
In either case, we see that lines l and h are not parallel. Hence, $\overleftrightarrow{AP}$ is the unique line through P which is parallel to l, and it follows that PF holds. □

4. PF IMPLIES EPP

In this section, we show that if we assume Axioms (I1)-(I3), (B1)-(B4), (C1)-(C5), (N1), and also assume that the Exterior Angle Theorem and PF hold, then we can prove that EPP holds.

Theorem 4.1. *Assume that Axioms (I1)-(I3), (B1)-(B4), (C1)-(C5), and (N1) all hold. Also assume that the Exterior Angle Theorem and PF hold. Then EPP also holds.*

Proof. Let A and B be two distinct points, and let $l = \overleftrightarrow{AB}$. Let H and V be two distinct points on the same side of l such that angles $\angle HAB$ and $\angle VBA$ are less than two right angles. Let D, C, and E be such that $D - A - C - B - E$.
Since $\angle HAB$ and $\angle VBA$ are less than two right angles, then it follows by Axiom (C4) together with Angle Addition that there exists a unique ray $\overrightarrow{BF}$ on the same side of l as H such that $\angle FBE \cong \angle HAB < \angle VBE$. In particular, $\overleftrightarrow{VB} \neq \overleftrightarrow{FB}$. It follows by the Exterior Angle Theorem that lines $\overleftrightarrow{HA}$ and $\overleftrightarrow{FB}$ are parallel. Thus, it follows by PF that $\overleftrightarrow{FB}$ is the unique line through B which is parallel to $\overleftrightarrow{HA}$. This implies that $\overleftrightarrow{AH}$ and $\overleftrightarrow{BV}$ intersect at a point J.
We need to show that J is on the same side of line l as H and V, and consequently that J is the point of intersection of rays $\overrightarrow{AH}$ and $\overrightarrow{BV}$. Let K and W be points such that $H - A - K$ and $V - B - W$. Since $\overleftrightarrow{HA}$ and $\overleftrightarrow{FB}$ are parallel, then $J \neq B$. Moreover, since $\overleftrightarrow{HA}$ and $\overleftrightarrow{FB}$ are parallel, since $V \in int(\angle ABF)$, and since $V - B - W$, then it follows that W and K are on opposite sides of line $\overleftrightarrow{FB}$. More generally, all the points on ray $\overrightarrow{BW}$ (other than the vertex B) are on the opposite side of $\overleftrightarrow{FB}$ as line $\overleftrightarrow{HA}$. Thus, no point on ray $\overrightarrow{BW}$ is on line $\overleftrightarrow{HA}$. Therefore, since J is on both lines $\overleftrightarrow{AH}$ and $\overleftrightarrow{BV}$, then it must be the case that J is on rays $\overrightarrow{AH}$ and $\overrightarrow{BV}$.
Hence, rays $\overrightarrow{AH}$ and $\overrightarrow{BV}$ intersect, which implies that EPP holds. □

5. THE MODEL $\mathbb{A}$

In this section, we construct a non-continuous model $\mathbb{A}$ which satisfies Axioms (I1)-(I3), (B1)-(B4), (C1)-(C5), and (N1), as well as PF, but which does not satisfy either of EPP or the Exterior Angle Theorem. The model $\mathbb{A}$ is similar to the model $\mathbb{A}$ defined in [4], with the exception of one modification.
The set of points in the model $\mathbb{A}$ is the set $\mathbb{Q} \times \mathbb{Q}$ of points in $\mathbb{R}^2$ both of whose coordinates are rational numbers. Since $\mathbb{Q}^2$ is countably infinite, then we can enumerate the points $P_1, P_2, \ldots, P_j, \ldots$ in $\mathbb{Q}^2$. To define lines in the models $\mathbb{A}$, we let $\mathcal{L}$ be the set of all lines in $\mathbb{R}^2$ whose equations are $y = mx + b$ or $x = k$, where $m, b, k \in \mathbb{Q}$. We define a line in $\mathbb{A}$ to be the intersection of a line $l \in \mathcal{L}$ with $\mathbb{Q} \times \mathbb{Q}$. We will define distance and angle measure in the model $\mathbb{A}$ in a non-continuous way, and then use this distance and angle measure to determine when two segments or two angles are congruent. Since there is

an enumeration on the points $P_1, P_2, \ldots, P_j, \ldots$ in $\mathbb{Q}^2$, then when defining angle measure below, we treat the points in $\mathbb{Q}^2$ as a sequence (P_n). We denote the distance in $\mathbb{A}$ between two points P and Q by $t(P, Q)$. We denote the angle measure in $\mathbb{A}$ of an angle $\angle ABC$ by $m\angle ABC$.

We define distance in $\mathbb{A}$ using the taxicab metric t. More specifically, given two points $P_1 = (x_1, y_1)$ and $P_2 = (x_2, y_2)$, then we define $t(P_1, P_2) = |x_2 - x_1| + |y_2 - y_1|$. Since $x_1, x_2, y_1, y_2 \in \mathbb{Q}$, then $t(P_1, P_2) \in \mathbb{Q}$ as well.

It is proven in [2] and [3] that Axioms (I1)-(I3), (B1)-(B4), and (C1)-(C3) all hold in the model $\mathbb{A}$. It follows from the definition of points and lines in $\mathbb{A}$ that PF holds.

We next define angle measure in $\mathbb{A}$. Given a point $P_j = (a, b)$, where $a, b \in \mathbb{Q}$, let $E_j = (a+1, b)$, $W_j = (a-1, b)$, $N_j = (a, b+1)$, and $S_j = (a, b-1)$. If we are given a point $P_j = (a, b)$ in $\mathbb{A}$, where $a, b \in \mathbb{Q}$, then the lines $x = a$ and $y = b$ divide up the model $\mathbb{A}$ into four "quadrants" similar to the way that the x-axis and y-axis divide up the xy-plane into four quadrants. We will use a variation of this terminology when defining and altering angle measure to construct the model $\mathbb{A}$. We refer to the set $\{(x, y) \in \mathbb{A} \mid x > a \text{ and } y > b\}$ as P_j-quadrant I. We refer to the set $\{(x, y) \in \mathbb{A} \mid x < a \text{ and } y > b\}$ as P_j-quadrant II. We refer to the set $\{(x, y) \in \mathbb{A} \mid x < a \text{ and } y < b\}$ as P_j-quadrant III. Finally, we refer to the set $\{(x, y) \in \mathbb{A} \mid x > a \text{ and } y < b\}$ as P_j-quadrant IV.

Given a point $P_j \in \mathbb{Q}^2$, we define $m\angle E_jP_jN_j = m\angle E_jP_jS_j = m\angle W_jP_jN_j = m\angle W_jP_jS_j = \dfrac{\pi}{2}$.

Let (D_m) denote the subsequence of (P_n) consisting of points that are in P_j-quadrant I. Given a ray $\overrightarrow{P_jD_m}$ in P_j-quadrant I, we let $\hat{D}_m$ denote the point on $\overrightarrow{P_jD_m}$ which comes before all other points on $\overrightarrow{P_jD_m}$ with respect to the enumeration of (P_n). We note that $\overrightarrow{P_jD_m} = \overrightarrow{P_j\hat{D}_m}$. Let T_m be a point in $\mathbb{Q}^2$ such that $T_m - P_j - \hat{D}_m$. Thus, T_m is on the opposite ray of $\overrightarrow{P_j\hat{D}_m}$, and T_m is in P_j-quadrant III. We note that every point P_i in P_j-quadrant III is in the form T_m for some point $\hat{D}_m$ in P_j-quadrant I.

Let K_1 denote the first point in (D_m), and let K_2 denote the first point from (D_m) which is in the interior of angle $\angle K_1P_jN_j$. We define $m\angle E_jP_jK_1 = m\angle K_1P_jK_2 = m\angle K_2P_jN_j = \dfrac{1}{3}m\angle E_jP_jN_j = \left(\dfrac{1}{3}\right)\left(\dfrac{\pi}{2}\right)$.

Let K_3 denote the first point in (D_m) in the interior of angle $\angle E_jP_jK_1$, and let K_4 denote the first point from (D_m) which is in the interior of angle $\angle K_3P_jK_1$. We define $m\angle E_jP_jK_3 = m\angle K_3P_jK_4 = m\angle K_4P_jK_1 = \dfrac{1}{3}m\angle E_jP_jK_1 = \dfrac{1}{9}m\angle E_jP_jN_j = \left(\dfrac{1}{9}\right)\left(\dfrac{\pi}{2}\right)$.

Continuing on with this process, let K_5 denote the first point in (D_m) in the interior of angle $\angle E_jP_jK_3$, and let K_6 denote the first point from (D_m) which is in the interior of angle $\angle K_5P_jK_3$. We define $m\angle E_jP_jK_5 = m\angle K_5P_jK_6 = m\angle K_6P_jK_1 = \dfrac{1}{3}m\angle E_jP_jK_1 = \dfrac{1}{27}m\angle E_jP_jN_j = \left(\dfrac{1}{27}\right)\left(\dfrac{\pi}{2}\right)$.

We continue this process indefinitely. In particular, we continue with this process for every ray in P_j-quadrant I and for all sub-angles of $\angle E_jP_jN_j$. In this way, we construct infinitely many nested sequences of sub-angles. Each sub-angle in a sequence has as its

angle measure exactly $\dfrac{1}{3}$ the angle measure of the angle before it in the nested sequence. Thus, starting with the initial angle measure $m\angle E_jP_jN_j = \dfrac{\pi}{2}$ (and thinking of $\angle E_jP_jN_j$ as being the first angle in a sequence), then the angle measure of the m^{th} angle in a nested sequence is $\dfrac{1}{3^{m-1}} m\angle E_jP_jN_j = \left(\dfrac{1}{3^{m-1}}\right)\left(\dfrac{\pi}{2}\right)$.
Let P_{n_1} and P_{n_2} be two points in P_j-quadrant I such that P_j, P_{n_1}, and P_{n_2} are not collinear. To compute the measure of angle $\angle P_{n_1}P_jP_{n_2}$, we first use the method just described to compute the measure of all the angles in the form $\angle K_tP_jK_s$ for all points $K_{i_1}, K_{i_2}, \ldots, K_{i_b}$ which come before and include both P_{n_1} and P_{n_2} in (P_n). We then compute the measure of angle $\angle P_{n_1}P_jP_{n_2}$ using angle addition, adding up the measures of those sub-angles of $\angle P_{n_1}P_jP_{n_2}$ whose measures have already been computed. Since there are only a finite number of points in (P_n) that come before P_{n_1} and P_{n_2} in (P_n), then this is a finite process and is well-defined.
Let T_{m_1} and T_{m_2} be two points in P_j-quadrant III such that P_j, T_{m_1}, and T_{m_2} are not collinear. It follows by definition of T_{m_1} and T_{m_2} that P_j, D_{m_1}, and D_{m_2} are also not collinear. We define $m\angle T_{m_1}P_jT_{m_2} = m\angle D_{m_1}P_jD_{m_1}$. Since every point P_i in P_j-quadrant III is in the form T_m for some point $\hat{D}_m$ in P_j-quadrant I, and since for every ray the point $\hat{D}_m$ is unique, then $m\angle T_{m_1}P_jT_{m_2}$ is well defined. We define angle measure using a similar method for P_j-quadrant II and P_j-quadrant IV.
Given two points P_u and P_v such that the three points P_u, P_v, and P_j are distinct and non-collinear, then we define $m\angle P_uP_jP_v$ using the above method together with Axiom (N1) (i.e. angle addition) if P_u and P_v are in different P_j-quadrants.
We define two angles in $\mathbb{A}$ to be congruent if they have the same angle measure.
With these definitions, we see that angle addition holds in $\mathbb{A}$. We leave it to the reader to check that Axioms (C4) and (C5) hold in $\mathbb{A}$. Moreover, Side-Angle-Side does not hold in $\mathbb{A}$, and $\mathbb{A}$ is not a continuous model.
With these above definitions, we make the following specific definition to the points P_1, P_2, P_3.
We specifically define P_1=(0,0), P_2=(2,0), and P_3=(1,0).
We define the ray $\overrightarrow{r_1}$ to consist of all points in $\mathbb{Q}^2$ on the line $y = x$ whose $y-$coordinates are non-negative. We define the ray $\overrightarrow{r_2}$ to consist of all points in $\mathbb{Q}^2$ on the line $y = x - 2$ whose $y-$coordinates are non-negative. We define the ray $\overrightarrow{r_3}$ to consist of all points in $\mathbb{Q}^2$ on the line $y = x - 1$ whose $y-$coordinates are non-negative. We define the ray $\overrightarrow{r_4}$ to consist of all points in $\mathbb{Q}^2$ on the line $y = 2(x-1)$ whose $y-$coordinates are non-negative. We define the ray $\overrightarrow{r_5}$ to consist of all points in $\mathbb{Q}^2$ on the line $y = 4(x-1)$ whose $y-$coordinates are non-negative. Note that P_1 is the vertex of $\overrightarrow{r_1}$, P_2 is the vertex of $\overrightarrow{r_2}$, and P_3 is the vertex of all three of $\overrightarrow{r_3}$, $\overrightarrow{r_4}$, and $\overrightarrow{r_5}$.
We define the angle between the positive $x-$axis and each of the rays $\overrightarrow{r_1}$, $\overrightarrow{r_2}$, and $\overrightarrow{r_5}$ to be $\left(\frac{1}{3}\right)\left(\frac{\pi}{2}\right)$. We then define the angle between the positive $x-$axis and the ray $\overrightarrow{r_4}$ to be $\left(\frac{1}{9}\right)\left(\frac{\pi}{2}\right)$. Finally, we define the angle between the positive $x-$axis and the ray $\overrightarrow{r_3}$ to be $\left(\frac{1}{27}\right)\left(\frac{\pi}{2}\right)$.
Angle measure for the remaining angles in $\mathbb{A}$ are defined as stated above.
We see that the rays $\overrightarrow{r_1}$ and $\overrightarrow{r_4}$ intersect at the point Q= (2,2). However, this gives us $\triangle P_1P_3Q$ with exterior angle $\angle QP_3P_2$ which is smaller than its remote interior angle

$\angle QP_1P_3$. Moreover, the interior angles formed by the $x-$axis and rays $\overrightarrow{r_2}$ and $\overrightarrow{r_3}$ are less than two right angles. However, rays $\overrightarrow{r_2}$ and $\overrightarrow{r_3}$ do not intersect. Thus, we have that neither EPP nor the Exterior Angle Theorem hold in $\mathbb{A}$.

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Department of Mathematical Sciences
University of Southern Indiana
Evansville, Indiana 47712.
Email address: `jrdonnelly@usi.edu`