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CHARACTERIZATION OF LIGHTLIKE HYPERSURFACES IN THE LORENTZ-MINKOWSKI SPACE $\mathbb{R}^4_1$ WITH ITS STRUCTURE EQUATIONS

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ABSTRACT. In this paper, we introduce the notion of structure equations on lightlike hypersurfaces and we give a characterization of lightlike hypersurfaces in the Lorentz-Minkowski space $\mathbb{R}^4_1$ satisfies the equation $\Delta H = \lambda H$, where H and Δ are the mean curvature of M and Laplace operator respectivelly.

1. INTRODUCTION

Theory of hypersurfaces is one of the fundamental theories of submanifolds. In semi-Riemannian manifold, there are three types of hypersurfaces : Riemannian; semi-Riemannian and lightlike. The geometry of lightlike hypersurfaces is different and rather difficult since its normal vector bundle intersects with the tangent bundle. Duggal and Bejancu [7] published in 1996 a book on general theory of lightlike submanifolds of semi-Riemannian manifolds. They introduced a non-degenerate screen distribution to construct a lightlike transversal vector bundle which is non-intersecting to its lightlike tangent bundle. We know that the shape operator plays a key role in the study geometry of submanifolds. In this paper, we will study a class of lightlike hypersurfaces of a Lorentz-Minkowski space $\mathbb{R}^4_1$, whose shape operators are conformal to shape operators of their corresponding screen distributions.

The main purpose of this paper is to introduce the structure equations and give a characterization of lightlike hypersurfaces satisfying the equation $\Delta H = \lambda H$. In section 2, we give useful preliminaries for study the geometry of lightlike hypersurfaces. In section 3, we will give the structure equations on the lightlike hypersurfaces of the Lorentz-Minkowski space $\mathbb{R}^4_1$.

Let $x : M \longrightarrow \mathbb{R}^4_1$ be an isometric immersion of a lightlike hypersurface into the Lorentz-Minkowski space $\mathbb{R}^4_1$. Denoted by H and Δ the mean curvature of M and Laplace operator respectivelly. The equation $\Delta H = \lambda H$, for some real constant λ, is a natural generalization of the biharmonique equation $\Delta H = 0$. In the case of a non-degenerate hypersurface, Ferrandez and Lucas originate in [8] the study of hypersurfaces in $\mathbb{R}^3_1$

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satisfying $\Delta H = \lambda H$, where they proved that the hypersurfaces in $\mathbb{R}^3_1$ have constant mean curvature and classified such hypersurfaces. Moreover, A.Arvanitoyeorgos and G.Kaimakamis made a conjecture in [1] saying that any hypersurface satisfying $\Delta H = \lambda H$ in pseudo-Euclidean space $\mathbb{E}^{n+1}_s$ has constant mean curvature, A.Arvanitoyeorgos and Al. proved in [2] that this conjecture is true for hypersurfaces M^3_r of $\mathbb{E}^4_s$ whose shape operator is diagonalizable. As a matter of course, it is interesting to study this conjecture for the lightlike hypersurfaces of $\mathbb{R}^4_1$. In this paper, we will give some conditions on the screen distribution for which the lightlike hypersurface satisfy the equation $\Delta H = \lambda H$ and has constant mean curvature in section 4. Finally, we take the lightlike cone of $\mathbb{R}^3_1$ as an application example of a lightlike hypersurface of $\mathbb{R}^4_1$ satisfying the equation $\Delta H = \lambda H$ in section 5.

2. Preliminaries on lightlike hypersurfaces

2.1. **Geometry of lightlike hypersurface.** Let M be a hypersurface of a $(m+2)-$dimensional semi-Riemannian manifold $(\bar{M}, \bar{g})$. Denote by g the tensor field induced by $\bar{g}$ on M. We say that M is a lightlike hypersurface if $rank(g) = m$.Thus the normal vector bundle $TM^{\perp}$ intersects the tangent bundle along a nonzero distribution which called the radical distribution of M and denoted by $Rad(TM)$ where

$$Rad(TM) : p \mapsto Rad(T_pM) = T_pM \cap T_pM^{\perp}. \tag{2.1}$$

A non-degenerate vector bundle complementary to $TM^{\perp}$ on M is called a screen distribution denoted by $S(TM)$. A lightlike hypersurface endowed with a screen distribution is denoted by the triple $(M, g, S(TM))$.
The following result has an important role in the study of the geometry of the lightlike hypersurfaces.

Theorem 2.1. *([7]) Let $(M, g, S(TM))$ be a lightlike hypersurface of $(\bar{M}, \bar{g})$.Then there exists a unique vector bundle $tr(TM)$ of rank1 over M, such that for any non zero section ξ of $TM^{\perp}$ on a coordinate neighborhood $\mathcal{U} \subset M$, there exists a unique section N of $tr(TM)$ on $\mathcal{U}$ satisfying*

$$\bar{g}(N, \xi) = 1 \text{ and } \bar{g}(N, N) = \bar{g}(N, W) = 0, \ \forall W \in \Gamma(S(TM)). \tag{2.2}$$

By this theorem we may write the following decomposition

$$T\overline{M}_{|M} = S(TM) \perp (TM^{\perp} \oplus tr(TM)) = TM \oplus tr(TM), \tag{2.3}$$

where $\perp$ denotes an orthogonal direct sum and $\oplus$ a direct sum.
Let $(M, g, S(TM))$ be a lightlike hypersurface of a semi$-$Riemannian manifold $(\overline{M}, \bar{g})$, $\overline{\nabla}$ be the Levi$-$Civita connexion of $\overline{M}$ and ∇ the induced connexion on (M, g). Gauss and Weigarten formulas provide the following relations (for details see [7])

$$\overline{\nabla}_X Y = \nabla_X Y + h(X, Y), \tag{2.4}$$

$$\overline{\nabla}_X V = -A_V X + \nabla^t_X V, \tag{2.5}$$

for all $X, Y \in \Gamma(TM)$ and $V \in tr(TM)$, where $\nabla_X Y$ and $A_V X$ belong to $\Gamma(TM)$ while h is a $\Gamma(tr(TM))-$valued symmetric $C^{\infty}(M)-$bilinear form on $\Gamma(TM)$ and ∇^t is a linear connection on $tr(TM)$ which is a torsion-free.
Difine a $C^{\infty}(M)-$bilinear form B and a $1-$form τ on the coordinate neighborhood $\mathcal{U} \subset M$ by

$$B(X, Y) = \bar{g}(h(X, Y), \xi), \tag{2.6}$$

$$\tau(X) = \bar{g}(\nabla^t_X N, \xi), \tag{2.7}$$

for all $X, Y \in \Gamma(TM_{|\mathcal{U}})$. Thus, equations (2.4) and (2.5) become on $\mathcal{U}$

$$\overline{\nabla}_X Y = \nabla_X Y + B(X,Y)N, \tag{2.8}$$

$$\overline{\nabla}_X V = -A_V X + \tau(X)N, \tag{2.9}$$

respectively. B is called the local second fundamental form and does not depend on the choice of the screen distribution, and satisfies

$$B(X, \xi) = 0, \tag{2.10}$$

for all $X \in \Gamma(TM_{|\mathcal{U}})$. Let P be the projection morphism of TM to $S(TM)$ with respect to the decomposition (2.3). Then

$$\nabla_X PY = \overset{*}{\nabla}_X PY + \overset{*}{h}(X, PY), \tag{2.11}$$

$$\nabla_X U = -\overset{*}{A}_U X + \overset{*}{\nabla}{}^t{}_X U, \tag{2.12}$$

where $\overset{*}{\nabla}_X PY$ and $\overset{*}{A}_U X$ belong to $\Gamma(S(TM))$, $\overset{*}{\nabla}$ and $\overset{*}{\nabla}{}^t$ are linear connections on $\Gamma(S(TM))$ and $\Gamma(TM^{\perp})$ respectively, $\overset{*}{h}$ is a $C^\infty(M)-$bilinear form on $\Gamma(TM) \times \Gamma(S(TM))$, $\overset{*}{A}_U$ is a $\Gamma(S(TM))-$valued $C^\infty(M)-$linear operator on $\Gamma(S(TM))$. $\overset{*}{h}$ and $\overset{*}{A}_U$ are the second fundamental form and the shape operator of the screen distribution $S(TM)$ respectively. On $\mathcal{U}$ difine the following relations

$$C(X, PY) = \bar{g}(\overset{*}{h}(X, PY), N), \tag{2.13}$$

$$\epsilon(X) = \bar{g}(\overset{*}{\nabla}{}^t{}_X \xi, N). \tag{2.14}$$

It is easy to show that $\epsilon(X) = -\tau(X)$. Thus, equations (2.11) et (2.12) become

$$\nabla_X PY = \overset{*}{\nabla}_X PY + C(X, PY)\xi, \tag{2.15}$$

$$\nabla_X U = -\overset{*}{A}_\xi X - \tau(X)\xi, \tag{2.16}$$

respectively. The linear connection $\overset{*}{\nabla}$ is a metric connection on $\Gamma(S(TM))$.
In general, the induced connection ∇ on M is not metric and we have

$$(\nabla_X g)(Y, Z) = B(X,Y)\eta(Z) + B(X,Z)\eta(Z), \tag{2.17}$$

for all $X, Y \in \Gamma(TM_{|\mathcal{U}})$, where

$$\eta(X) = \bar{g}(X, N), \text{ for all} X \in \Gamma(TM_{|\mathcal{U}}). \tag{2.18}$$

It is easy to verify that for all $X, Y \in \Gamma(TM_{|\mathcal{U}})$

$$B(X,Y) = g(\overset{*}{A}_\xi X, Y), \; g(A_N X, N) = 0, \tag{2.19}$$

$$C(X, PY) = g(A_N X, Y), \; \overset{*}{A}_\xi \xi = 0, \tag{2.20}$$

Let $\overline{R}$ et R the curvature tensor associated with $\overline{\nabla}$ et ∇. Then we have the following relations [7]

$$\overline{R}(X,Y)Z = R(X,Y)Z + A_{h(X,Z)}Y - A_{h(Y,Z)}X + (\nabla_X h)(Y,Z) - (\nabla_Y h)(X,Z), \tag{2.21}$$

$$\bar{g}(\overline{R}(X,Y)\xi, N) = C(Y, \overset{*}{A}_\xi X) - C(X, \overset{*}{A}_\xi Y) - 2d\tau(X,Y). \tag{2.22}$$

Next, we give some characteristics of the lightlike hypersurfaces.

2.2. **Quasi-conformal screen Lightlike hypersurfaces.**

Definition 2.1. *(*[10]*) A lightlike hypersurface $(M, g, S(TM))$ of a semi−Riemannian manifold is locally screen quasi-conformal if the shape operators A_N and $\overset{*}{A}_\xi$ of M and $S(TM)$ respectively satisfy*

$$A_N = \varphi \overset{*}{A}_\xi + \psi P, \tag{2.23}$$

where φ and ψ are non-vanishing smooth function on a neighborhood $\mathcal{U}$ in M and P be the natural projection morphism of TM to $S(TM)$. If the above property holds everywhere on M, we say that $(M, g, S(TM))$ is globally screen quasi-conformal.

It easy to see that (2.23) is equivalent to

$$C(X, PY) = \varphi B(X, PY) + \psi g(X, PY) \tag{2.24}$$

for any $X, Y \in \Gamma(TM)$. If $\psi \equiv 0$, we have :

Definition 2.2. *(*[4]*) A lightlike hypersurface $(M, g, S(TM))$ of a semi-Riemannian manifold is said to be locally screen (resp. globally) conformal if on any coordinate neighborhood $\mathcal{U}$ (resp. $\mathcal{U} = M$), the shape operators A_N and $\overset{*}{A}_\xi$ satisfy*

$$A_N = \varphi \overset{*}{A}_\xi, \tag{2.25}$$

where φ is a non-vanishing smooth function on $\mathcal{U}$ (resp. $\mathcal{U} = M$).
In particular, M is screen homothetic if φ is non-zero constant.

Next, we show how to induce a Riemannian metric on a lightlike hypersurface (see for details [9]).

Definition 2.3. *Let M be lightlike hypersurface in a Lorentzian manifold. A rigging for M is a vector field ζ defined on some open set containing M such that $\zeta_p \notin T_pM$ for each $p \in M$.*

Given a rigging ζ in a neighborhood of M in $(\overline{M}, \bar{g})$, denoted by α the $1-$form $\bar{g}-$metrically equivalent to ζ, *i.e.* $\alpha = \bar{g}(\zeta, .)$. Take $\omega = i^*\alpha$, being $i : M \hookrightarrow \bar{M}$ the canonical inclusion. Consider the tensors

$$\breve{g} = g + \alpha \otimes \alpha \text{ and } \tilde{g} = i^*\breve{g} = i^*g + \omega \otimes \omega. \tag{2.26}$$

It is easy to show that $\tilde{g}$ defines a Riemannian metric on the hypersurface M.The rigged vector field of ζ is the $\tilde{g}-$metrically equivalent vector field to the $1-$form ω and it is denoted by ξ. In fact, the rigged vector field ξ is the unique lightlike vector field in M, such that $\bar{g}(\zeta, \xi) = 1$. Moreover ξ is $\tilde{g}-$unitary. We can consider the screen distribution given by $TM \cap \zeta^\perp$ which denoted by $\mathcal{S}(\zeta)$ and it is $\tilde{g}-$orthogonal subspace to ξ. The corresponding lightlike transverse vector field to $\mathcal{S}(\zeta)$ is as follow

$$N = \zeta - \frac{1}{2}\bar{g}(\zeta, \zeta)\xi. \tag{2.27}$$

3. Structure equations on Lightlike hypersurfaces

Let $(M, g, S(TM))$ be a lightlike hypersurface of Lorentz-Minkoswki space $\mathbb{R}^4_1$, ∇ and $\overline{\nabla}$ be connection on M and Levi−Civita connection of $\mathbb{R}^4_1$ respectively. Consider $\{E_1, E_2\}$ an orthonormal frame field of $S(TM)$ and set $\{E_1, E_2, \xi\} = \{E_1, E_2, E_3\}$ an orthonormal basis relative to the metric $\tilde{g}$.
Put $\omega^i = \tilde{g}(E_i, .)$, for $i = 1, 2, 3$. Then $\{\omega^i\}_{i=1,2,3}$ forms the dual basis of $\{E_i\}_{i=1,2,3}$. We have $X = \sum_{i=1}^{3} \tilde{g}(X, E_i)E_i = \sum_{i=1}^{3} \omega^i(X)E_i$ for any $X \in \Gamma(TM)$. The connection forms are defined as follow :

$$\omega_i^j(X) = \tilde{g}(\nabla_X E_i, E_j), \quad 1 \leq i, j \leq 3, \tag{3.1}$$

with

$$\nabla_X E_i = \sum_j \tilde{g}(\nabla_X E_i, E_j)E_j = \sum_j \omega_i^j(X)E_j. \tag{3.2}$$

We have the following result, which gives the structural equations on the lightlike hypersurfaces.

Proposition 3.1. *Let $(M, g, S(TM))$ be a lightlike hypersurface of Lorentz-Minkoswki space $\mathbb{R}^4_1$, ∇ and $\overline{\nabla}$ be connection on M and Levi−Civita connection of $\mathbb{R}^4_1$ respectively, $\{\omega^i\}_{i=1,2,3}$ the dual basis of $\{E_i\}_{i=1,2,3}$ and $\{\omega_i^j\}_{1\leq i,j\leq 3}$ the connection forms. We have the following result*

i)

$$\omega_i^j(X) + \omega_j^i(X) = \begin{cases} 0 \text{ if } i, j \in \{1, 2\} \\ -2\tau(X) \text{ if } i = j = 3 \\ g(A_N X - \overset{*}{A}_\xi X, E_i) \text{ si } i \in \{1, 2\} \text{ and } j = 3. \end{cases}$$

ii)

$$d\omega^i(E_j, E_k) = \omega_j^i(E_k) - \omega_k^i(E_j).$$

iii)

$$\begin{aligned} d\omega_i^j(E_k, E_l) =& B(E_k, \nabla_{E_l} E_i)\omega(E_j) + B(E_k, E_j)\omega_i^3(E_l) - B(E_l, \nabla_{E_k} E_i)\omega(E_j) \\ & - B(E_l, E_j)\omega_i^3(E_k) + B(E_l, E_i)g(A_N E_k, E_j) \\ & - B(E_k, E_i)g(A_N E_l, E_j) - \sum_{t=1}^{2} \omega_i^t \wedge \omega_j^t(E_k, E_l) \\ & + \{E_k(\omega_i^3(E_l)) - E_l(\omega_i^3(E_k)) - \omega_i^3([E_k, E_l])\}\omega(E_j). \end{aligned}$$

Proof. i) For $i, j \in \{1, 2, 3\}$, we have $\tilde{g}(E_i, E_j) = \delta_{ij}$. Then for all $X \in \Gamma(TM)$, $X.\tilde{g}(E_i, E_j) = 0$

$$\begin{aligned} &\Leftrightarrow (\nabla_X \tilde{g})(E_i, E_j) + \tilde{g}(\nabla_X E_i, E_j) + \tilde{g}(E_i, \nabla_X E_j) = 0 \\ &\Leftrightarrow (\nabla_X \tilde{g})(E_i, E_j) + \omega_i^j(X) + \omega_j^i(X) = 0 \\ &\Rightarrow \omega_i^j(X) + \omega_j^i(X) = -(\nabla_X \tilde{g})(E_i, E_j). \end{aligned}$$

and for all $Y, Z \in \Gamma(TM)$, we have :

$$(\nabla_X \tilde{g})(Y, Z) = X.\tilde{g}(Y, Z) - \tilde{g}(\nabla_X Y, Z) - \tilde{g}(Y, \nabla_X Z). \tag{3.3}$$

But $\tilde{g} = i^*g + \omega \otimes \omega$ with $\omega = \tilde{g}(\xi, .)$.

Thus $\tilde{g}(Y,Z) = g(Y,Z) + \omega(Y)\omega(Z)$. And $X.\tilde{g}(Y,Z) = X.g(Y,Z) + X(\omega(Y))\omega(Z) + \omega(Y)X(\omega(Z)$.

$\tilde{g}(\nabla_X Y, Z) = g(\nabla_X Y, Z) + \omega(\nabla_X Y)\omega(Z)$

$\tilde{g}(Y, \nabla_X Z) = g(Y, \nabla_X Z) + \omega(Y)\omega(\nabla_X Z)$.

Then, we have:

$$\begin{aligned}(\nabla_X \tilde{g})(Y,Z) =& (\nabla_X g)(Y,Z) + \{X(\omega(Y)) - \omega(\nabla_X Y\}\omega(Z) \\ &+ \{X(\omega(Z)) - \omega(\nabla_X Z)\}\omega(Y).\end{aligned}$$

But $(\nabla_X g) = B(X,Y)\eta(Z) + B(X,Z)\eta(Y)$ where $\eta(Y) = g(N,Y)$ and since $N = \zeta - \frac{1}{2}g(\zeta,\zeta)\xi$ then $\eta(X) = \omega(X), \quad \forall X \in \Gamma(TM)$. So $(\nabla_X g) = B(X,Y)\omega(Z) + B(X,Z)\omega(Y)$. Therefore,

$$\begin{aligned}(\nabla_X \tilde{g})(Y,Z) =& B(X,Y)\omega(Z) + B(X,Z)\omega(Y) + \{X(\omega(Y)) - \omega(\nabla_X Y\}\omega(Z) \\ &+ \{X(\omega(Z)) - \omega(\nabla_X Z)\}\omega(Y).\end{aligned}$$

Assume, $Y = E_i, Z = E_j$, for any $i,j \in \{1,2,3\}$, we obtain,

$$\begin{aligned}(\nabla_X \tilde{g})(E_i,E_j) =& B(X,E_i)\omega(E_j) + B(X,E_j)\omega(E_i) + \{X(\omega(E_i)) - \omega(\nabla_X E_i)\}\omega(E_j) \\ &+ \{X(\omega(E_j)) - \omega(\nabla_X E_j)\}\omega(E_i).\end{aligned}$$

$\forall i,j \in \{1,2,3\}, \quad X(\omega(E_i)) = X(\omega(E_j)) = 0$.

So,

$$\begin{aligned}(\nabla_X \tilde{g})(E_i,E_j) =& B(X,E_i)\omega(E_j) + B(X,E_j)\omega(E_i) - \omega(\nabla_X E_i)\omega(E_j) \\ &- \omega(\nabla_X E_j)\omega(E_i).\end{aligned}$$

In particular, we have

- if $i,j \in \{1,2\}$, we have $\omega(E_i) = \omega(E_j) = 0$, so $(\nabla_X \tilde{g})(E_i,E_j) = 0$. So, $\omega_i^j + \omega_j^i = 0$ and $\omega_j^j = 0$.
- if $i \in \{1,2\}$ and $j = 0$, we have $\omega(E_j) = 1$, $\omega(E_i) = 0$ and $B(X,E_j) = 0$. In this case,$(\nabla_X \tilde{g})(E_i,E_j) = B(X,E_i) - \omega(\nabla_X E_i)$.

  But $\nabla_X \xi = \sum_j \omega_3^j(X)E_j$ et $\nabla_X E_i = \sum_j \omega_i^j(X)E_j$.

  So,

$$\begin{aligned}\omega(\nabla_X \xi) =& \sum_j \omega_3^j(X)\omega(E_j) \\ =& \omega_3^3(X)\omega(\xi) \text{ car } \forall j \in \{1,2\}, \omega(E_j) = 0 \\ =& \omega_3^3(X) \text{ beacause} \omega(\xi) = 1 \\ =& \tilde{g}(\nabla_X \xi, \xi) \\ =& \tilde{g}(-\overset{*}{A}_\xi X - \tau(X)\xi, \xi) \\ =& -\tau(X).\end{aligned}$$

  In the same way, we show

$$\omega(\nabla_X E_i) = C(X,E_i), \quad \forall i \in \{1,2\}.$$

  So, $(\nabla_X \tilde{g})(E_i,E_j) = B(X,E_i) - C(X,E_i)$.

- if $i = j = 3$, we have $\omega(E_i) = \omega(E_j) = 1$ et $B(X, E_i) = 0$.
  therefore, $(\nabla_X \tilde{g})(E_i, E_j) = -2\omega(\nabla_X E_3) = -2\tau(X)$.
  Hence $i)$.

ii) For all $i, j, k \in \{1, 2, 3\}$, we have:

$$d\omega^i(E_j, E_k) = E_j(\omega^i(E_k)) - E_k(\omega^i(E_j)) - \omega^i([E_j, E_k]).$$

$E_j(\omega^i(E_k)) = E_j(\tilde{g}(E_i, E_k)) = E_j(g(E_i, E_k) + \omega(E_i)\omega(E_k)) = 0$.
Similarly, we have $E_k(\omega^i(E_j)) = 0$.
Then,

$$\begin{aligned} d\omega^i(E_j, E_k) &= -\omega^i([E_j, E_k]) \\ &= -\tilde{g}(E_i, [E_j, E_k]) \\ &= \tilde{g}(E_i, \nabla_{E_k} E_j) - \tilde{g}(E_i, \nabla_{E_j} E_k) \\ &= g(E_i, \nabla_{E_k} E_j) - g(E_i, \nabla_{E_j} E_k) + \omega(E_i)\{\omega(\nabla_{E_k} E_j) - \omega(\nabla_{E_j} E_k)\}. \end{aligned}$$

But $\nabla_{E_k} E_j = \sum_t \omega_j^t(E_k) E_t$, so

$$\begin{aligned} \omega(\nabla_{E_k} E_j) &= \sum_t \omega_j^t(E_k)\omega(E_t) \\ &= \omega_j^3(E_k) \quad \text{beacause } \omega(E_1) = \omega(E_2) = 0. \end{aligned}$$

Similarly $\omega(\nabla_{E_j} E_k) = \omega_k^3(E_j)$.
So

$$d\omega^i(E_j, E_k) = g(E_i, \nabla_{E_k} E_j) - g(E_i, \nabla_{E_j} E_k) + \omega(E_i)\{\omega_j^3(E_k) - \omega_k(E_j)\}.$$

Hence $ii)$.

iii) For all $i, j, k, l \in \{1, 2, 3\}$, we have:

$$d\omega_i^j(E_k, E_l) = E_k(\omega_i^j(E_l)) - E_l(\omega_i^j(E_k)) - \omega_i^j([E_k, E_l]).$$

with

$$\begin{aligned} \omega_i^j(E_l) &= \tilde{g}(\nabla_{E_l} E_i, E_j) \\ &= g(\nabla_{E_l} E_i, E_j) + \omega(\nabla_{E_l} E_i)\omega(E_j) \\ \Rightarrow E_k(\omega_i^j(E_l)) &= E_k(g(\nabla_{E_l} E_i, E_j)) + E_k(\omega(\nabla_{E_l} E_i)\omega(E_j)). \end{aligned}$$

Similarly

$$E_l(\omega_i^j(E_k)) = E_l(g(\nabla_{E_k} E_i, E_j)) + E_l(\omega(\nabla_{E_k} E_i)\omega(E_j)).$$

$$E_k(g(\nabla_{E_l} E_i, E_j)) = (\nabla_{E_k} g)(\nabla_{E_l} E_i, E_j) + g(\nabla_{E_k} \nabla_{E_l} E_i, E_j) + g(\nabla_{E_l} E_i, \nabla_{E_k} E_j).$$

$$E_l(g(\nabla_{E_k} E_i, E_j) = (\nabla_{E_l} g)(\nabla_{E_k} E_i, E_j) + g(\nabla_{E_l} \nabla_{E_k} E_i, E_j) + g(\nabla_{E_k} E_i, \nabla_{E_l} E_j).$$

$$\omega_i^j([E_k, E_j]) = g(\nabla_{[E_k, E_l]} E_i, E_j) + \omega(\nabla_{[E_k, E_l]} E_i)\omega(E_j).$$

$$E_k(\omega(\nabla_{E_l} E_i)\omega(E_j)) = E_k(\omega(\nabla_{E_l} E_i))\omega(E_j).$$

$$E_l(\omega(\nabla_{E_k} E_i)\omega(E_j)) = E_l(\omega(\nabla_{E_k} E_i))\omega(E_j).$$

Then

$$\begin{aligned} d\omega_i^j(E_k,E_l) =&(\nabla_{E_k}g)(\nabla_{E_l}E_i,E_j) + g(\nabla_{E_k}\nabla_{E_l}E_i,E_j) + g(\nabla_{E_l}E_i,\nabla_{E_k}E_j) \\ &+ E_k(\omega(\nabla_{E_l}E_i))\omega(E_j) - (\nabla_{E_l}g)(\nabla_{E_k},E_j) - g(\nabla_{E_l}\nabla_{E_k}E_i,E_j) \\ &- g(\nabla_{E_k}E_i,\nabla_{E_l}E_j) - E_l(\omega(\nabla_{E_k}E_i))\omega(E_j) - g(\nabla_{[E_k,E_l]}E_i,E_j) \\ &- \omega(\nabla_{[E_k,E_l]}E_i)\omega(E_j). \end{aligned}$$

$$\begin{aligned} d\omega_i^j(E_k,E_l) =&(\nabla_{E_k}g)(\nabla_{E_l}E_i,E_j) - (\nabla_{E_l}g)(\nabla_{E_k},E_j) \\ &+ g(\nabla_{E_k}\nabla_{E_l}E_i - \nabla_{E_l}\nabla_{E_k}E_i - \nabla_{[E_k,E_l]}E_i,E_j) \\ &+ g(\nabla_{E_l}E_i,\nabla_{E_k}E_j) - g(\nabla_{E_k}E_i,\nabla_{E_l}E_j) + E_k(\omega(\nabla_{E_l}E_i))\omega(E_j) \\ &- E_l(\omega(\nabla_{E_k}E_i))\omega(E_j) - \omega(\nabla_{[E_k,E_l]}E_i)\omega(E_j). \end{aligned}$$

$$\begin{aligned} d\omega_i^j(E_k,E_l) =&(\nabla_{E_k}g)(\nabla_{E_l}E_i,E_j) - (\nabla_{E_l}g)(\nabla_{E_k},E_j) \\ &+ g(R(E_k,E_l)E_i,E_j) + g(\nabla_{E_l}E_i,\nabla_{E_k}E_j) - g(\nabla_{E_k}E_i,\nabla_{E_l}E_j) \\ &+ \{E_k(\omega(\nabla_{E_l}E_i)) - E_l(\omega(\nabla_{E_k}E_i)) - \omega(\nabla_{[E_k,E_l]}E_i)\}\omega(E_j). \end{aligned}$$

By Gauss-Codazzi equations, we have:

$$\begin{aligned} 0 = g(\bar{R}(E_k,E_l)E_i,E_j) =&g(R(E_k,E_l)E_i,E_j) + B(E_k,E_i)g(A_N E_l,E_j) \\ &- B(E_l,E_i)g(A_N E_k,E_j). \end{aligned}$$

$$\Rightarrow g(R(E_k,E_l)E_i,E_j) =B(E_l,E_i)g(A_N E_k,E_j) - B(E_k,E_i)g(A_N E_l,E_j).$$

But

$$\begin{aligned} (\nabla_{E_k}g)(\nabla_{E_l}E_i,E_j) =&B(E_k,\nabla_{E_l}E_i)\omega(E_j) + B(E_k,E_j)\omega(\nabla_{E_l}E_i). \\ (\nabla_{E_l}g)(\nabla_{E_k}E_i,E_j) =&B(E_l,\nabla_{E_k}E_i)\omega(E_j) + B(E_l,E_j)\omega(\nabla_{E_k}E_i). \end{aligned}$$

$$\begin{aligned} g(\nabla_{E_l}E_i,\nabla_{E_k}E_j) - g(\nabla_{E_k}E_i,\nabla_{E_l}E_j) &= \sum_t\{\omega_i^t(E_l)\omega_j^t(E_k) - \omega_i^t(E_k)\omega_j^t(E_l)\} \\ &= -\sum_t \omega_i^t \wedge \omega_j^t(E_k,E_l). \end{aligned}$$

Since

$$\omega(\nabla_{E_k}E_i) = \omega_i^3(E_k),\ \omega(\nabla_{E_l}E_i) = \omega_i^3(E_l)$$

and

$$\omega(\nabla_{[E_k,E_l]}E_i) = \omega_i^3([E_k,E_l]),$$

we obtain:

$$\begin{aligned} d\omega_i^j(E_k,E_l) =&B(E_k,\nabla_{E_l}E_i)\omega(E_j) + B(E_k,E_j)\omega_i^3(E_l) - B(E_l,\nabla_{E_k}E_i)\omega(E_j) \\ &- B(E_l,E_j)\omega_i^3(E_k) + B(E_l,E_i)g(A_N E_k,E_j) \\ &- B(E_k,E_i)g(A_N E_l,E_j) - \sum_t \omega_i^t \wedge \omega_j^t(E_k,E_l) \\ &+ \{E_k(\omega_i^3(E_k)) - E_l(\omega_i^3(E_k)) - \omega_i^3([E_k,E_l])\}\omega(E_j). \end{aligned}$$

Hence *iii*).

□

Corollary 3.1. *If the lightlike hypersurface M is quasi-conformal screen, in other words* $A_N = \varphi \overset{*}{A}_\xi + \psi P$, *then we have :*

$$\omega_i^j(X) + \omega_j^i(X) = \begin{cases} 0 \text{ if } i,j \in \{1,2\} \\ -2\tau(X) \text{ if } i = j = 3 \\ (\varphi - 1)g(\overset{*}{A}_\xi X, E_i) + \psi g(PX, E_i) \text{ if } i \in \{1,2\} \text{ and } j = 3. \end{cases} \tag{3.4}$$

Since the operator $\overset{*}{A}_\xi$ is diagonalizable, we can take the basis $\{E_i\}_{i=1,2,3}$ as a basis of eigenvectors associated with the eigenvalues of the matrix associated with $\overset{*}{A}_\xi$ *i.e,* $\overset{*}{A}_\xi E_i = \lambda_i E_i$.
We have,

Corollary 3.2. *By proposition* (3.1)*, we have*

(1)

$$\omega_i^j(X) + \omega_j^i(X) = \begin{cases} 0 \text{ if } i,j \in \{1,2\} \\ -2\tau(X) \text{ if } i = j = 3 \\ (\varphi - 1)\lambda_i \omega^i(X) \text{ if } i \in \{1,2\} \text{ and } j = 3. \end{cases}$$

(2)

$$\begin{aligned} d\omega_i^j(E_k, E_l) = &\{\lambda_k \omega_i^k(E_l) - \lambda_l \omega_i^l(E_k) + d\omega_i^3(E_k, E_l)\}\omega(E_j) + \lambda_k \omega_i^3(E_l) \\ &- \lambda_l \omega_i^3(E_k) - \sum_{t=1}^{2} \omega_i^t \wedge \omega_j^t(E_k, E_l). \end{aligned}$$

If in addition, the lightlike hypersurface M is quasi-conformal screen, we obtain

$$\omega_i^j(X) + \omega_j^i(X) = \begin{cases} 0 \text{ if } i,j \in \{1,2\} \\ -2\tau(X) \text{ if } i = j = 3 \\ \{(\varphi - 1)\lambda_i + \psi\}\omega^i(X) \text{ if } i \in \{1,2\} \text{ and } j = 3. \end{cases}$$

4. Null mean curvature field and its Laplacian of lightlike hypersurface

Let $(M, g, S(TM))$ be a lightlike hypersurface of Lorentz-Minkoswki space $\mathbb{R}_1^4$. The mean curvature of the lightlike hypersurface M, denoted α, is defined by

$$\alpha = tr(\overset{*}{A}_\xi) \tag{4.1}$$

Thus, with respect to the orthonormal basis $\{E_i\}_{i=1,2}$, we have

$$\alpha = \sum_i \tilde{g}(\overset{*}{A}_\xi E_i, E_i) = \sum_i g(\overset{*}{A}_\xi E_i, E_i) = \sum_i B(E_i, E_i) = tr(B).$$

4.1. **The Laplacian operator** Δ. Let $x : M \longrightarrow \mathbb{R}_1^4$ be an isometric immersion.
The Laplacian operator is defined by

$$\Delta x = -\sum_i \{\bar{\nabla}_{E_i} \bar{\nabla}_{E_i} x - \bar{\nabla}_{\nabla_{E_i} E_i} x\}. \tag{4.2}$$

x being a position vector in $\mathbb{R}_1^4$, then $\bar{\nabla}_{E_i} x = E_i$ and $\bar{\nabla}_{\nabla_{E_i} E_i} x = \nabla_{E_i} E_i$.
So,

$$\Delta x = -\sum_i \{\bar{\nabla}_{E_i} E_i - \nabla_{E_i} E_i\}.$$

But $\bar{\nabla}_{E_i} E_i - \nabla_{E_i} E_i = B(E_i, E_i) N$, thus equation (4.2) become

$$\Delta x = -\sum_i B(E_i, E_i) N = -tr(B) N = -\alpha N. \tag{4.3}$$

4.2. **Laplacian of the mean curvature field H.** We denote by S the shape operator of the hypersurface M and H the mean curvature field, *i.e* $H = \alpha N$.
Then for any $X \in \Gamma(TM)$, we have $\bar{\nabla}_X H = \bar{\nabla}_X (\alpha N) = X(\alpha) N + \alpha \bar{\nabla}_X N$. But $\bar{\nabla}_X N = -SX + \tau(X) N$, so $\bar{\nabla}_X H = (X(\alpha) + \alpha \tau(X)) N - \alpha S X$.
Thus, with respect to the orthonormal basis $\{E_i\}_{i=1,2}$, we obtain the Laplacian of H is given by:

Lemma 4.1.

$$\begin{aligned} \Delta H =& (\Delta \alpha) N + 2S(\nabla \alpha) + \alpha \{\sum_i (\nabla_{E_i} S)(E_i) - \sum_i (\nabla_{E_i} \tau)(E_i)\} + \alpha \sum_i B(E_i, SE_i) N \\ & - 2\sum_i E_i(\alpha) \tau(E_i) N + \alpha \sum_i \{\tau(E_i) SE_i - \tau^2(E_i) N\}. \end{aligned} \tag{4.4}$$

,

Proof. let $\{E_i\}_{i=1,2}$ the orthonormal basis. We have, $\bar{\nabla}_{E_i} H = (E_i(\alpha) + \alpha \tau(E_i)) N - \alpha S E_i$.
So,

$$\begin{aligned} \bar{\nabla}_{E_i} \bar{\nabla}_{E_i} H =& \left(E_i E_i(\alpha) + E_i(\alpha)\tau(E_i) + \alpha E_i(\tau(E_i))\right) N - E_i(\alpha) S E_i - \alpha \bar{\nabla}_{E_i} S E_i \\ & + \left(E_i(\alpha) + \alpha \tau(E_i)\right) \left(-SE_i + \tau(E_i) N\right) \\ =& \left(E_i E_i(\alpha) + E_i(\alpha)\tau(E_i) + \alpha E_i(\tau(E_i))\right) N - E_i(\alpha) S E_i - \alpha \bar{\nabla}_{E_i} S E_i \\ & - E_i(\alpha) S E_i + E_i(\alpha) \tau(E_i) N - \alpha \tau(E_i) S E_i + \alpha \tau^2(E_i) N \\ =& \left(E_i E_i(\alpha) + E_i(\alpha)\tau(E_i) + \alpha E_i(\tau(E_i))\right) N - 2E_i(\alpha) S E_i - \alpha \bar{\nabla}_{E_i} S E_i \\ & + E_i(\alpha) \tau(E_i) N - \alpha \tau(E_i) S E_i + \alpha \tau^2(E_i) N. \end{aligned}$$

and

$$\bar{\nabla}_{\nabla_{E_i} E_i} H = (\nabla_{E_i} E_i(\alpha) + \alpha \tau(\nabla_{E_i} E_i)) N - \alpha S \nabla_{E_i} E_i.$$

The Laplacian is defined as follows :

$$
\begin{aligned}
\Delta H =& -\sum_i \{\bar{\nabla}_{E_i}\bar{\nabla}_{E_i}H - \bar{\nabla}_{\nabla_{E_i}E_i}H\} \\
=& -\sum_i \{(E_iE_i(\alpha) + E_i(\alpha)\tau(E_i) + \alpha E_i(\tau(E_i)))\,N - 2E_i(\alpha)SE_i - \alpha\bar{\nabla}_{E_i}SE_i \\
& + E_i(\alpha)\tau(E_i)N - \alpha\tau(E_i)SE_i + \alpha\tau^2(E_i)N - (\nabla_{E_i}E_i(\alpha) + \alpha\tau(\nabla_{E_i}E_i))N + \alpha S\nabla_{E_i}E_i\} \\
=& -\sum_i\{E_iE_i(\alpha) - \nabla_{E_i}E_i(\alpha)\}N + 2\sum_i E_i(\alpha)SE_i + \alpha\sum_i(\bar{\nabla}_{E_i}SE_i - S\nabla_{E_i}E_i) \\
& + \alpha\sum_i\tau(E_i)SE_i + \sum_i\{-2E_i(\alpha)\tau(E_i) + \alpha(-E_i(\tau(E_i) - \tau^2(E_i) + \tau(\nabla_{E_i}E_i)\}N \\
=& \Delta\alpha N + 2S(\sum_i E_i(\alpha)E_i) + \alpha\sum_i\tau(E_i)SE_i + \alpha\sum_i(\bar{\nabla}_{E_i}SE_i - S\nabla_{E_i}E_i) \\
& + \sum_i\{-2E_i(\alpha)\tau(E_i) + \alpha(-E_i(\tau(E_i) - \tau^2(E_i) + \tau(\nabla_{E_i}E_i)\}N \\
=& \Delta\alpha N + 2S(\nabla\alpha) + \alpha\sum_i\tau(E_i)SE_i + \alpha\sum_i(\nabla_{E_i}SE_i + B(E_i, SE_i)N - S(\nabla_{E_i}E_i)) \\
& + \sum_i\{-2E_i(\alpha)\tau(E_i) + \alpha(-E_i(\tau(E_i) - \tau^2(E_i) + \tau(\nabla_{E_i}E_i)\}N \\
=& \Delta\alpha N + 2S(\nabla\alpha) + \alpha\sum_i\tau(E_i)SE_i + \alpha\sum_i\{\nabla_{E_i}SE_i - S(\nabla_{E_i}E_i)\} + \alpha\sum_i B(E_i, SE_i)N \\
& + \sum_i\{-2E_i(\alpha)\tau(E_i) + \alpha(-E_i(\tau(E_i) - \tau^2(E_i) + \tau(\nabla_{E_i}E_i)\}N
\end{aligned}
$$

But $\nabla_{E_i}SE_i - S(\nabla_{E_i}E_i) = (\nabla_{E_i}S)E_i$ and $(\nabla_{E_i}\tau)(E_i) = E_i(\tau(E_i)) - \tau(\nabla_{E_i}E_i)$.
Hence the result. □

Corollary 4.1. *If $(M, S(TM), g)$ is quasi-conformal screen then we have*

$$
\begin{aligned}
(\nabla_{E_i}S)(E_i) =& \nabla_{E_i}(SE_i) - S(\nabla_{E_i}E_i) \\
=& \nabla_{E_i}(\varphi \overset{*}{A}_\xi E_i + \psi E_i) - [\varphi\overset{*}{A}_\xi + \psi P](\nabla_{E_i}E_i) \\
=& \nabla_{E_i}(\varphi\overset{*}{A}_\xi E_i) + \nabla_{E_i}(\psi E_i) - \varphi\overset{*}{A}_\xi(\nabla_{E_i}E_i) - \psi(\nabla_{E_i}E_i) \\
=& E_i(\varphi)\overset{*}{A}_\xi E_i + E_i(\psi)E_i + \varphi(\nabla_{E_i}\overset{*}{A}_\xi)(E_i).
\end{aligned}
$$

Moreover, if the operator $\overset{*}{A}_\xi$ is parallel, and the functions φ and ψ are constant, we have

$$\Delta H = (\Delta\alpha)N + \varphi\sum_i B(E_i, \overset{*}{A}_\xi E_i)H + \alpha\psi H + 2\{\varphi\overset{*}{A}_\xi(\nabla\alpha) + \psi(\nabla\alpha)\}.$$

Remark 4.1. *If $(M, S(TM), g)$ is conformal screen (i.e.$\psi = 0$), we have*

$$\Delta H = (\Delta\alpha)N + \varphi\sum_i B(E_i, \overset{*}{A}_\xi E_i)H + 2\varphi\overset{*}{A}_\xi(\nabla\alpha).$$

4.3. Lightlike hypersurfaces verifying the equation $\Delta H = \lambda H$.

Theorem 4.1. *Let $(M, g, S(TM))$ be a screen homothetic lightlike hypersurface of Lorentz-Minkoswki space $\mathbb{R}^4_1$ such that the shape operator $\overset{*}{A}_\xi$ has the maximal rank2 and is parallel,* *then the mean curvature vector field H satisfies the equation $\Delta H = \lambda H$ with λ a real constant if and only if the mean curvature α is constant along the screen distribution.*

Proof. $\Rightarrow$ Assuming that the mean curvature vector field H satisfies the equation $\Delta H = \lambda H$.
Under the above assumptions, we have

$$\Delta H = (\Delta\alpha)N + \varphi\sum_i B(E_i, \overset{*}{A}_\xi E_i)H + 2\varphi\overset{*}{A}_\xi(\nabla\alpha). \tag{4.5}$$

Thus $\Delta H = \lambda H$ is equivalent to :

$$2\varphi\overset{*}{A}_\xi(\nabla\alpha) = 0, \tag{4.6}$$

and

$$\frac{1}{\alpha}\Delta\alpha + \varphi\sum_i B(E_i, \overset{*}{A}_\xi E_i) = \lambda. \tag{4.7}$$

By (4.6) we have $\overset{*}{A}_\xi(\nabla\alpha) = 0$, which implies the existence of a smooth function γ such that $\nabla\alpha = \gamma\xi$.With respect to the orthonormal basis E of the screen, we have $\nabla\alpha = E_1(\alpha)E_1 + E_2(\alpha)E_2$, so $\nabla\alpha = \gamma\xi$ is equivalent to $E_1(\alpha)E_1 + E_2(\alpha)E_2 = \gamma\xi$ and this implies that $E_1(\alpha) = E_2(\alpha) = \gamma = 0$, because (E_1, E_2, ξ) forms an orthonormal basis of the hypersurface M.
$E_1(\alpha) = E_2(\alpha) = 0$ implies that the mean curvature α is constant.
$\Leftarrow$ Assuming that the mean curvature α is constant.
Then

$$\Delta H = \varphi\sum_i B(E_i, \overset{*}{A}_\xi E_i)H. \tag{4.8}$$

So, $\Delta H = \lambda H \Leftrightarrow \lambda = \varphi\sum_i B(E_i, \overset{*}{A}_\xi E_i)$. It only remains to show that λ is constant.
Then we have

$$B(E_i, \overset{*}{A}_\xi E_i) = g(\overset{*}{A}_\xi E_i, \overset{*}{A}_\xi E_i) = g(\lambda_i E_i, \lambda_i E_i) = \lambda_i^2,$$

where λ_i are the eigenvalues of matrix associated to $\overset{*}{A}_\xi$. So, $\lambda = \varphi\sum_i \lambda_i^2$, this implies that λ is constant.□ □

Example 4.1. *Let $\mathbb{R}^4_1$ be the space $\mathbb{R}^4$ endowed with the semi-euclidean metric given by*

$$\bar{g}(u, v) = -xx' + yy' + zz' + tt',$$

where $u = (x, y, z, t)$ and $v = (x', y', z', t')$. Let Λ^3_0 the lightlike cone of $\mathbb{R}^4_1$ given by

$$\Lambda^3_0 = \{(x, y, z, t) \in \mathbb{R}^4_1; -x^2 + y^2 + z^2 + t^2 = 0\} \tag{4.9}$$

with $(x, y, z, t) \neq (0, 0, 0, 0)$.
It is shown in [4] *that the lightlike cone Λ^3_0 is conformal screen. The radical distribution $Rad(\Lambda^3_0)$ and screen distribution $ST(\Lambda^3_0)$ are generated respectively by(see* [5] *)*

$$Rad(\Lambda^3_0) = <\xi> \tag{4.10}$$

with $\xi = x\partial_x + y\partial_y + z\partial_z + t\partial_t$. And

$$ST(\Lambda^3_0) = <E_1, E_2> \tag{4.11}$$

with

$$E_1 = \left(\frac{t^2+y^2}{t^2}\right)^{\frac{1}{2}} \left(\partial_y - \frac{y}{t}\partial_t\right)$$

and

$$E_2 = \left(\frac{t^2+y^2}{x^2}\right)^{\frac{1}{2}} \left(-\frac{yz}{t^2+y^2}\partial_y - \frac{zt}{t^2+y^2}\partial_t\right).$$

It is easy to show that (E_1, E_2) is an orthonormal basis of screen distribution.

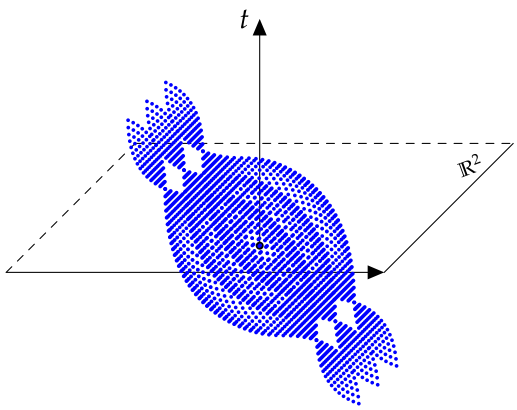

Figure 1. Projection of M in $\mathbb{R}^3$ for $x=-1$, $x=0$ and $x=1$

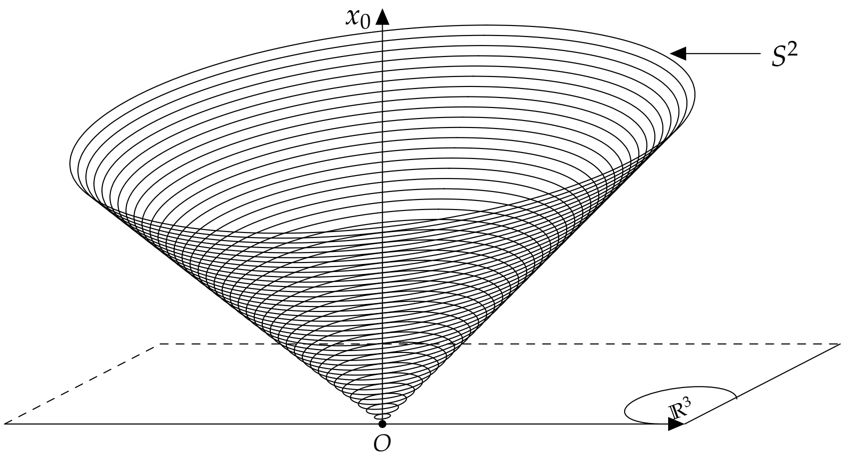

Figure 2. The lightcone Λ_0^3 of $\mathbb{R}_1^4$ is a stacking of spheres $S^2(x_0)$ of $\mathbb{R}^3$

ξ being a position vector in $\mathbb{R}_1^4$, then we have $\overline{\nabla}_{E_i}\xi = \nabla_{E_i}\xi = E_i$, so by Weingarten equation we have $\nabla_{E_i}\xi = -\overset{*}{A}_\xi E_i - \tau(E_i)\xi$. Then $\overset{*}{A}_\xi E_i - \tau(E_i)\xi = E_i$. So, $\overset{*}{A}_\xi E_i = -E_i$ for all $i = 1,2$ because $\overset{*}{A}_\xi$ is $\Gamma(ST(\Lambda_0^3))-$valued.But $\overset{*}{A}_\xi\xi = 0$, so we can give the matrix associated to $\overset{*}{A}_\xi$

relative to the basis $\{E_1, E_2, \xi\}$ *by*

$$\begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

The mean curvature α_p *at point* $p = (x, y, z, t)$ *is given by :* $\alpha_p = -2$ *and let the mean curvature vector field* H *of the lightlike cone* Λ^3_0.
By equation (4.4), *we have*

$$\begin{aligned} \Delta H =& \varphi \sum_i B(E_i, \overset{*}{A}_\xi E_i) H \\ =& -\varphi \sum_i B(E_i, E_i) H \\ =& -\varphi tr(B) H \\ =& 2\varphi H \text{ because } tr(B) = \alpha_p = -2. \end{aligned}$$

Thus the mean curvature vector field H *satisfies the equation* $\Delta H = \lambda H$ *with* $\lambda = 2\varphi$.

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