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NEW GENERALIZATIONS OF THE KIEPERT HYPERBOLA AND THE LESTER CIRCLE

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ABSTRACT. In this paper, we first extend the concept of the Kiepert hyperbola, and based on this generalization, we present a further generalization of the classical Lester circle theorem. In addition, another extension of the Lester circle is also introduced. The tools we employ are both barycentric coordinates, Cartesian coordinates and complex coordinates.

1. INTRODUCTION

The Kiepert hyperbola is a well-known rectangular hyperbola associated with a triangle, passing through several notable points such as the centroid, the orthocenter, the circumcenter, and the Fermat points. Our starting point is to recall the classical definition due to Kiepert.

Theorem 1 (Recall Kiepert hyperbola [6]). *If the three triangles ABC, $A'BC$, $AB'C$, and ABC', constructed on the sides of a triangle ABC as bases, are similar, isosceles, and similarly situated, then the triangles ABC and $A'B'C'$ are in perspective. As the base angle of the isosceles triangles varies between $-\pi/2$ and $\pi/2$, the locus of the center of perspectivity of the triangles ABC and $A'B'C'$ is a hyperbola called the Kiepert hyperbola.*

In what follows, we introduce a parameterized construction of the Kiepert hyperbola based on the first Fermat point. This formulation will serve as a stepping stone towards our generalization.

Theorem 2 (Parameterized Kiepert hyperbola by the first Fermat point). *Let ABC be a triangle and F_1 its first Fermat point. Lines F_1A, F_1B, and F_1C meet the circles (F_1BC), (F_1CA), and (F_1AB) again at A_1, B_1, and C_1, respectively. Let A_2, B_2, and C_2 be the midpoints of sides BC, CA, and AB, respectively. For all real numbers t, define the points A_0, B_0, and C_0 as follows*

$$A_0 = tA_1 + (1-t)A_2, \quad B_0 = tB_1 + (1-t)B_2, \quad C_0 = tC_1 + (1-t)C_2.$$

Then, lines AA_0, BB_0, and CC_0 are concurrent on the Kiepert hyperbola.

We now replace the first Fermat point in Theorem 2 with an arbitrary point P, leading to a generalized hyperbola $\mathcal{K}_P$ that retains many elegant properties of the classical case.

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Theorem 3 (Generalization of Theorem 2)**.** *Let ABC be a triangle and P an arbitrary point. Lines PA, PB, and PC meet the circles (PBC), (PCA), and (PAB) again at A_1, B_1, and C_1, respectively. Let A_2, B_2, and C_2 be the orthogonal projections of A_1, B_1, and C_1 on the sidelines BC, CA, and AB, respectively.*

i) *Lines AA_2, BB_2, and CC_2 are concurrent at a point G_P.*
ii) *Let A_3, B_3, and C_3 be the reflections of A_1, B_1, and C_1 in the sidelines BC, CA, and AB, respectively. Then, lines AA_3, BB_3, and CC_3 are concurrent at a point P^*, the* antigonal conjugate *of P.*
iii) *For all real numbers t, define the points A_0, B_0, and C_0 by*
$$A_0 = tA_1 + (1-t)A_2, \quad B_0 = tB_1 + (1-t)B_2, \quad C_0 = tC_1 + (1-t)C_2.$$
Then, lines AA_0, BB_0, and CC_0 are concurrent on a rectangular hyperbola $\mathcal{K}_P$ passing through G_P and P^*.

Remark. *When $P = F_1$, the first Fermat point of ABC, we recover $P^* = F_2$, the second Fermat point, $G_P = G$, the centroid of ABC, and $\mathcal{K}_P$ becomes the classical Kiepert hyperbola.*

The center of $\mathcal{K}_P$ admits a simple and elegant characterization, linking it to both the midpoint of PP^* and well-known circles associated with ABC.

Theorem 4. *Using the notations of Theorem 3, the midpoint I of PP^* lies on both the circumcircle of $A_2B_2C_2$ and the nine-point circle of ABC, and I is also the center of $\mathcal{K}_P$.*

Remark. *For $P = F_1$, Theorem 4 yields the familiar result that the midpoint of F_1F_2 lies on the nine-point circle and is the center of the Kiepert hyperbola.*

Motivated by Theorem 3, we next explore a family of circles $\mathcal{L}_P$ that generalize the celebrated Lester circle.

Theorem 5 (Generalization of the Lester circle)**.** *Using the notations of Theorem 3, let H be the orthocenter of ABC, and let O_P and N_P divide the segment HG_P in ratios 2 and -2, respectively, i.e.*
$$\frac{\overline{O_PH}}{\overline{O_PG_P}} = 2, \quad \frac{\overline{N_PH}}{\overline{N_PG_P}} = -2.$$
Then, the four points P, P^*, N_P, and O_P lie on a common circle $\mathcal{L}_P$.

Remark. *When $P = F_1$, Theorem 5 reduces to the classical Lester circle.*

The next result is an analogue of a theorem of Gibert, linking chords of $\mathcal{K}_P$ to the pair P, P^*.

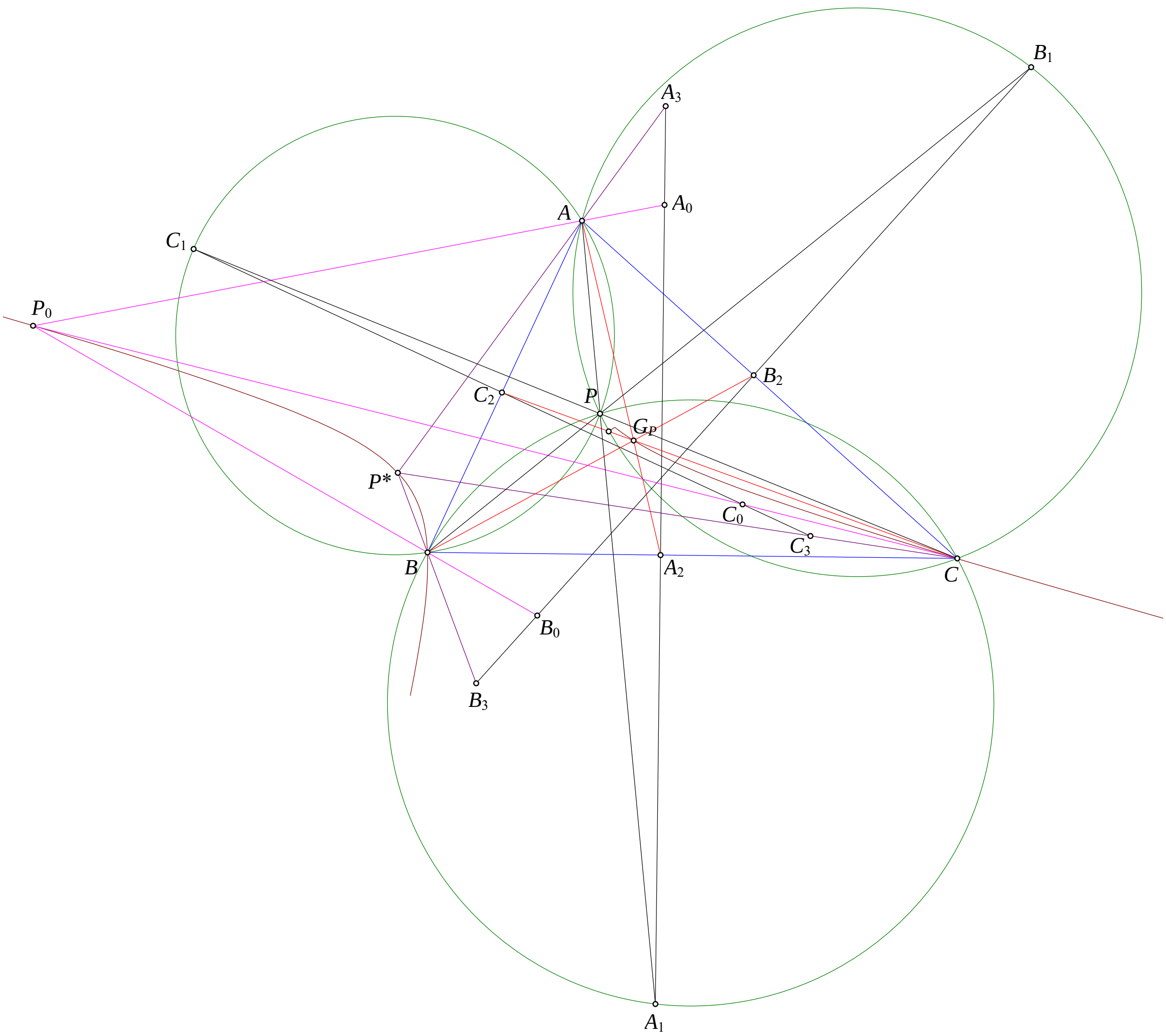
Theorem 6 (Generalization of Gibert's theorem)**.** *Using the notations of Theorem 3, let ℓ be the line passing through G_P and the orthocenter H of ABC. Every circle whose diameter is a chord of $\mathcal{K}_P$ perpendicular to ℓ passes through the points P and P^*.*

Finally, we present a second type of generalization of the Lester circle, built from isogonal conjugacy and harmonic division.

Theorem 7 (Another generalization of the Lester circle)**.** *Let ABC be a triangle with orthocenter H and circumcenter O. Let P be a point on line OH and Q its isogonal conjugate. Points S^+ and S^- lie on OQ such that $(S^+S^-, OQ) = -1$ and S^+, S^- are inverses with respect to (O). Let F^+ and F^- be the isogonal conjugates of S^+ and S^-, respectively, and let R be a point on OH such that $(OR, PH) = -1$. Then, the four points F^+, F^-, O, R lie on a circle.*

Remark. *For $P = F_1$, Theorem 7 recovers Gibert's generalization of the first Lester circle.*

2. Proof of Theorem 3



Proof.

i) Let $A = (1, 0, 0)$, $B = (0, 1, 0)$, $C = (0, 0, 1)$ and $P = (p : q : r)$.
The coordinates of three points A_1, B_1 and C_1 are

$$A_1 = \left(-a^2qr(p+q+r) : q\left(a^2qr + b^2rp + c^2pq\right) : r\left(a^2qr + b^2rp + c^2pq\right)\right)$$
$$B_1 = \left(p\left(a^2qr + b^2rp + c^2pq\right) : -b^2rp(p+q+r) : r\left(a^2qr + b^2rp + c^2pq\right)\right)$$
$$C_1 = \left(p\left(a^2qr + b^2rp + c^2pq\right) : q\left(a^2qr + b^2rp + c^2pq\right) : -c^2pq(p+q+r)\right)$$

Using the formula in [3, p. 21], we have the coordinates of three points A_2, B_2 and C_2 are

$$A_2 = \begin{pmatrix} 0 \\ q\left(\left(r^2+pr-qr\right)a^2+\left(r^2+qr-pr\right)b^2-\left(r^2+pr+qr+2pq\right)c^2\right) \\ r\left(\left(q^2+pq-qr\right)a^2-\left(q^2+pq+qr+2pr\right)b^2+\left(q^2+qr-pq\right)c^2\right) \end{pmatrix}$$
$$B_2 = \begin{pmatrix} p\left(\left(r^2+pr-qr\right)a^2+\left(r^2+qr-pr\right)b^2-\left(r^2+pr+qr+2pq\right)c^2\right) \\ 0 \\ r\left(-\left(p^2+pq+pr+2qr\right)a^2+\left(p^2+pq-pr\right)b^2+\left(p^2+pr-pq\right)c^2\right) \end{pmatrix}$$
$$C_2 = \begin{pmatrix} p\left(\left(q^2+pq-qr\right)a^2-\left(q^2+pq+qr+2pr\right)b^2+\left(q^2+qr-pq\right)c^2\right) \\ q\left(-\left(p^2+pq+pr+2qr\right)a^2+\left(p^2+pq-pr\right)b^2+\left(p^2+pr-pq\right)c^2\right) \\ 0 \end{pmatrix}$$

From here, we have lines AA_2, BB_2 and CC_2 are concurrent at point

$$G_P = \begin{pmatrix} \frac{p}{-(p^2+pq+pr+2qr)a^2+(p^2+pq-pr)b^2+(p^2+pr-pq)c^2} \\ \frac{q}{(q^2+pq-qr)a^2-(q^2+pq+qr+2pr)b^2+(q^2+qr-pq)c^2} \\ \frac{r}{(r^2+pr-qr)a^2+(r^2+qr-pr)b^2-(r^2+pr+qr+2pq)c^2} \end{pmatrix}$$

ii) From this formula, we can calculate the coordinates of three points A_3, B_3, C_3

$$A_3 = \begin{pmatrix} -a^2qr(p+q+r) \\ q\left(\left(a^2+b^2-c^2\right)r^2-c^2pq+\left(a^2-c^2\right)rp+\left(b^2-c^2\right)rq\right) \\ r\left(\left(a^2+c^2-b^2\right)q^2-b^2pr+\left(a^2-b^2\right)qp+\left(c^2-b^2\right)qr\right) \end{pmatrix}$$
$$B_3 = \begin{pmatrix} p\left(\left(a^2+b^2-c^2\right)r^2-c^2pq+\left(a^2-c^2\right)rp+\left(b^2-c^2\right)rq\right) \\ 0 \\ r\left(\left(b^2+c^2-a^2\right)p^2-a^2qr+\left(b^2-a^2\right)pq+\left(c^2-a^2\right)pr\right) \end{pmatrix}$$
$$C_3 = \begin{pmatrix} p\left(\left(a^2+c^2-b^2\right)q^2-b^2pr+\left(a^2-b^2\right)qp+\left(c^2-b^2\right)qr\right) \\ q\left(\left(b^2+c^2-a^2\right)p^2-a^2qr+\left(b^2-a^2\right)pq+\left(c^2-a^2\right)pr\right) \\ 0 \end{pmatrix}$$

We have lines AA_3, BB_3, and CC_3 are concurrent at point

$$P^* = \begin{pmatrix} \frac{p}{((b^2+c^2-a^2)p^2-a^2qr+(b^2-a^2)pq+(c^2-a^2)pr)} \\ \frac{q}{((a^2+c^2-b^2)q^2-b^2pr+(a^2-b^2)qp+(c^2-b^2)qr)} \\ \frac{r}{((a^2+b^2-c^2)r^2-c^2pq+(a^2-c^2)rp+(b^2-c^2)rq)} \end{pmatrix},$$

which is antigonal conjugate of P by this definition.

iii) We have the coordinates of three points A_0, B_0 and C_0 are

$$A_0 = \begin{pmatrix} -2a^2qrt(p+q+r) \\ q\cdot\lambda_r \\ r\cdot\lambda_q \end{pmatrix},$$
$$B_0 = \begin{pmatrix} p\cdot\lambda_r \\ -2b^2pr(p+q+r) \\ r\cdot\lambda_p \end{pmatrix},$$
$$C_0 = \begin{pmatrix} p\cdot\lambda_q \\ q\cdot\lambda_p \\ -2c^2pq(p+q+r) \end{pmatrix}$$

with

$$\begin{aligned}
\lambda_p &= (p^2+pq+pr+2qr-(p^2+pq+pr)t)a^2 + p(pt+qt+rt-p-q+r)b^2 \\
&\quad + p(pt+qt+rt-p+q-r)c^2 \\
\lambda_q &= q(pt+qt+rt-p-q+r)a^2 + (q^2+qp+qr+2pr-(q^2+qp+qr)t)b^2 \\
&\quad + q(pt+qt+rt+p-q-r)c^2 \\
\lambda_r &= r(pt+qt+rt-p+q-r)a^2 + r(pt+qt+rt+p-q-r)b^2 \\
&\quad + (r^2+rp+rq+2pq-(r^2+pr+qr)t)c^2
\end{aligned}$$

Now, we have lines AA_0, BB_0 and CC_0 are concurrent at

$$P_0 = \left(\frac{p}{\lambda_p} : \frac{q}{\lambda_q} : \frac{r}{\lambda_r}\right)$$

The equation of the circumconic $\mathcal{K}_P$ is

$$\lambda_a yz + \lambda_b zx + \lambda_c xy = 0$$

with

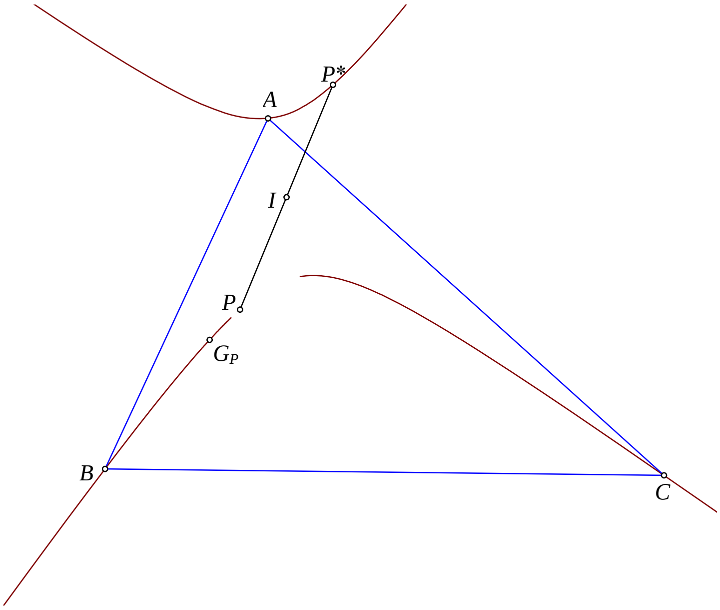
$$\begin{cases} \lambda_a = p\left(a^2r - a^2q + b^2q + b^2r - c^2q - c^2r\right) \\ \lambda_b = q\left(-a^2p - a^2r + b^2p - b^2r + c^2p + c^2r\right) \\ \lambda_c = r\left(a^2p + a^2q - b^2p - b^2q - c^2p + c^2q\right) \end{cases}$$

Since we can check that

$$(b^2+c^2-a^2)\lambda_a + (c^2+a^2-b^2)\lambda_b + (a^2+b^2-c^2)\lambda_c = 0,$$

we have $\mathcal{K}_P$ is a rectangular hyperbola. $\square$

3. Proof of Theorem 4



Proof. We have the barycentric coordinates of point I is

$$I=\frac{P+P^*}{2}=\begin{pmatrix} p\left(-b^2pr-b^2qr+c^2pq+c^2qr\right)\left(a^2q-a^2r-b^2q-b^2r+c^2q+c^2r\right)\\ q\left(a^2p+a^2r-b^2p+b^2r-c^2p-c^2r\right)\left(a^2pr+a^2qr-c^2pq-c^2pr\right)\\ r\left(a^2p+a^2q-b^2p-b^2q-c^2p+c^2q\right)\left(a^2pq+a^2qr-b^2pq-b^2pr\right)\end{pmatrix}$$

The equation of the nine-point circle of ABC is

$$a^2yz+b^2zx+c^2xy-\frac{((b^2+c^2-a^2)x+(a^2+c^2-b^2)y+(a^2+b^2-c^2)z)(x+y+z)}{4}=0$$

The equation of circle $(A_2B_2C_2)$ is

$$a^2yz+b^2zx+c^2xy-(\Lambda_a+\Lambda_b+\Lambda_c)(x+y+z)=0,$$

with

$$\Lambda_a=\frac{\delta_a}{\Delta_a},\ \Lambda_b=\frac{\delta_b}{\Delta_b},\ \Lambda_c=\frac{\delta_c}{\Delta_c},$$

and

$$\begin{aligned}\delta_a=&(a^2p^2+a^2pq+a^2pr+2a^2qr-b^2p^2-b^2pq+b^2pr-c^2p^2+c^2pq-c^2pr)\cdot(a^6p^2q^3r^3+a^6pq^4r^3\\ &+a^6pq^3r^4+a^6q^4r^4+2a^4b^2p^3q^2r^3+a^4b^2p^2q^3r^3+a^4b^2p^2q^2r^4-a^4b^2pq^4r^3+a^4b^2pq^3r^4\\ &+2a^4c^2p^3q^3r^2+a^4c^2p^2q^4r^2+a^4c^2p^2q^3r^3+a^4c^2pq^4r^3-a^4c^2pq^3r^4+a^2b^4p^4qr^3-a^2b^4p^3q^2r^3\\ &-a^2b^4p^3qr^4-2a^2b^4p^2q^3r^3-a^2b^4p^2q^2r^4+a^2b^2c^2p^4q^2r^2-2a^2b^2c^2p^3q^3r^2-2a^2b^2c^2p^3q^2r^3\\ &-a^2b^2c^2p^2q^4r^2-a^2b^2c^2p^2q^2r^4+a^2c^4p^4q^3r-a^2c^4p^3q^4r-a^2c^4p^3q^3r^2-a^2c^4p^2q^4r^2-2a^2c^4p^2q^3r^3\\ &-b^6p^4qr^3-b^6p^4r^4-b^6p^3q^2r^3-b^6p^3qr^4-b^4c^2p^4q^2r^2-c^6p^4q^4-b^4c^2p^4qr^3+b^4c^2p^3q^2r^3\\ &b^4c^2p^3qr^4-b^2c^4p^4q^3r-b^2c^4p^4q^2r^2+b^2c^4p^3q^4r+b^2c^4p^3q^3r^2-c^6p^4q^3r-c^6p^3q^4r-c^6p^3q^3r^2)\end{aligned}$$

$$\begin{aligned}\delta_b =&(a^2pq + a^2q^2 - a^2qr - b^2pq - 2b^2pr - b^2q^2 - b^2qr - c^2pq + c^2q^2 + c^2qr) \cdot (a^6p^2q^3r^3 + a^6pq^4r^3 + \\ & a^6pq^3r^4 + a^6q^4r^4 + 2a^4b^2p^3q^2r^3 + a^4b^2p^2q^3r^3 + a^4b^2p^2q^2r^4 - a^4b^2pq^4r^3 + a^4b^2pq^3r^4 + a^4c^2p^2q^4r^2 \\ & - a^4c^2p^2q^3r^3 + a^4c^2pq^4r^3 - a^4c^2pq^3r^4 + a^2b^4p^4qr^3 - a^2b^4p^3q^2r^3 - a^2b^4p^3qr^4 - 2a^2b^4p^2q^3r^3 \\ & - a^2b^4p^2q^2r^4 + a^2b^2c^2p^4q^2r^2 + 2a^2b^2c^2p^3q^3r^2 - a^2b^2c^2p^2q^4r^2 + 2a^2b^2c^2p^2q^3r^3 + a^2b^2c^2p^2q^2r^4 \\ & - a^2c^4p^4q^3r + a^2c^4p^3q^4r - a^2c^4p^3q^3r^2 + a^2c^4p^2q^4r^2 - b^6p^4qr^3 - b^6p^4r^4 - b^6p^3q^2r^3 - b^6p^3qr^4 \\ & - b^4c^2p^4q^2r^2 - b^4c^2p^4qr^3 - 2b^4c^2p^3q^3r^2 - b^4c^2p^3q^2r^3 + b^4c^2p^3qr^4 + b^2c^4p^4q^3r + b^2c^4p^4q^2r^2 \\ & - b^2c^4p^3q^4r + b^2c^4p^3q^3r^2 + 2b^2c^4p^3q^2r^3 + c^6p^4q^4 + c^6p^4q^3r + c^6p^3q^4r + c^6p^3q^3r^2)\end{aligned}$$

$$\begin{aligned}\delta_c =&(a^2pr - a^2qr + a^2r^2 - b^2pr + b^2qr + b^2r^2 - 2c^2pq - c^2pr - c^2qr - c^2r^2) \cdot (a^6p^2q^3r^3 + a^6pq^4r^3 \\ & + a^6pq^3r^4 + a^6q^4r^4 - a^4b^2p^2q^3r^3 + a^4b^2p^2q^2r^4 - a^4b^2pq^4r^3 + a^4b^2pq^3r^4 + 2a^4c^2p^3q^3r^2 \\ & + a^4c^2p^2q^4r^2 + a^4c^2p^2q^3r^3 + a^4c^2pq^4r^3 - a^4c^2pq^3r^4 - a^2b^4p^4qr^3 - a^2b^4p^3q^2r^3 + a^2b^4p^3qr^4 \\ & + a^2b^4p^2q^2r^4 + a^2b^2c^2p^4q^2r^2 + 2a^2b^2c^2p^3q^2r^3 + a^2b^2c^2p^2q^4r^2 + 2a^2b^2c^2p^2q^3r^3 - a^2b^2c^2p^2q^2r^4 \\ & + a^2c^4p^4q^3r - a^2c^4p^3q^4r - a^2c^4p^3q^3r^2 - a^2c^4p^2q^4r^2 - 2a^2c^4p^2q^3r^3 + b^6p^4qr^3 + b^6p^4r^4 + b^6p^3q^2r^3 \\ & + b^6p^3qr^4 + b^4c^2p^4q^2r^2 + b^4c^2p^4qr^3 + 2b^4c^2p^3q^3r^2 + b^4c^2p^3q^2r^3 - b^4c^2p^3qr^4 - b^2c^4p^4q^3r \\ & - b^2c^4p^4q^2r^2 + b^2c^4p^3q^4r - b^2c^4p^3q^3r^2 - 2b^2c^4p^3q^2r^3 - c^6p^4q^4 - c^6p^4q^3r - c^6p^3q^4r - c^6p^3q^3r^2)\end{aligned}$$

$$\begin{aligned}\Delta_a =&4p^2qr(a^2pq + a^2q^2 + b^2p^2 + b^2pq - c^2pq)(a^2pr + a^2r^2 - b^2pr + c^2p^2 + c^2pr)(a^2qr - b^2qr - b^2r^2 \\ & - c^2q^2 - c^2qr)\end{aligned}$$

$$\begin{aligned}\Delta_b =&4q^2rp(a^2qr - b^2qr - b^2r^2 - c^2q^2 - c^2qr)(a^2pr + a^2r^2 - b^2pr + c^2p^2 + c^2pr)(a^2pq + a^2q^2 + b^2p^2 \\ & + b^2pq - c^2pq)\end{aligned}$$

$$\begin{aligned}\Delta_c =&4pqr^2(a^2qr - b^2qr - b^2r^2 - c^2q^2 - c^2qr)(a^2pr + a^2r^2 - b^2pr + c^2p^2 + c^2pr)(a^2pq + a^2q^2 + b^2p^2 \\ & + b^2pq - c^2pq)\end{aligned}$$

It is not hard to check that I lies on both circle $(A_2B_2C_2)$ and the nine-point circle of triangle ABC, and note that

$$I = (\lambda_a(-\lambda_a + \lambda_b + \lambda_c) : \lambda_b(\lambda_a - \lambda_b + \lambda_c) : \lambda_c(\lambda_a + \lambda_b - \lambda_c)),$$

thus I is the center of $\mathcal{K}_P$. $\square$

4. Proof of Theorem 5

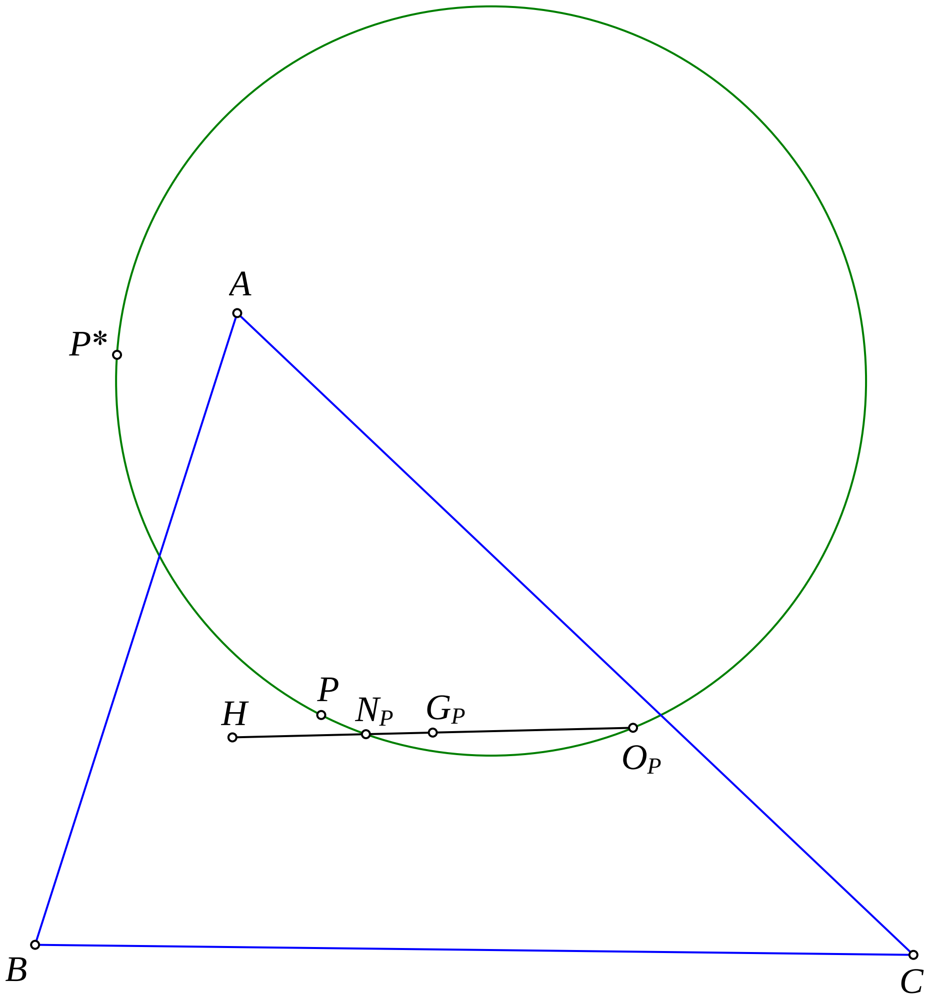


Figure 1

Proof. Let M be the midpoint of segment HG_P.
By the definition of O_P and N_P, we have O_P and N_P divide HG_P harmonically, and combine with Newton's relation, we have

$$\overline{MO_P} \cdot \overline{MN_P} = MH^2$$

The equation of the circle with diameter HG_P is

$$\zeta_1 x^2 + \zeta_2 y^2 + \zeta_3 z^2 + \zeta_4 yz + \zeta_5 zx + \zeta_6 xy = 0$$

with

$$\begin{cases} \zeta_1 = p(a^2-b^2-c^2)(a^2qr-b^2qr-b^2r^2-c^2q^2-c^2qr)(a^2p^2+a^2pq+a^2pr+2a^2qr+b^2p(r-q-p)+c^2p(q-r-p)) \\ \zeta_2 = q(b^2-c^2-a^2)(a^2pr+a^2r^2-b^2pr+c^2p^2+c^2pr)(a^2q(p+q-r)-b^2pq-2b^2pr-b^2q^2-b^2qr+c^2q(q+r-p)) \\ \zeta_3 = r(c^2-a^2-b^2)(a^2pq+a^2q^2+b^2p^2+b^2pq-c^2pq)(a^2r(p+r-q)-b^2pr+b^2qr+b^2r^2-2c^2pq+c^2r(r-q-p)) \end{cases}$$

and

$$\zeta_4 = (a^2p^2 + a^2pq + a^2pr + 2a^2qr - b^2p^2 - b^2pq + b^2pr - c^2p^2 + c^2pq - c^2pr)(a^4pqr - a^2b^2pqr - a^2b^2pr^2 + a^2b^2q^2r - a^2b^2qr^2 - a^2c^2pq^2 - a^2c^2pqr - a^2c^2q^2r + a^2c^2qr^2 - b^4pr^2 - b^4q^2r - b^4qr^2 + b^2c^2pq^2 + b^2c^2pr^2 + 2b^2c^2q^2r + 2b^2c^2qr^2 - c^4pq^2 - c^4q^2r - c^4qr^2)$$

$$\zeta_5 = (a^2pq + a^2q^2 - a^2qr - b^2pq - 2b^2pr - b^2q^2 - b^2qr - c^2pq + c^2q^2 + c^2qr)(a^4p^2r + a^4pr^2 + a^4qr^2 - a^2b^2p^2r + a^2b^2pqr + a^2b^2pr^2 + a^2b^2qr^2 - a^2c^2p^2q - 2a^2c^2p^2r - 2a^2c^2pr^2 - a^2c^2qr^2 - b^4pqr + b^2c^2p^2q + b^2c^2p^2r + b^2c^2pqr - b^2c^2pr^2 + c^4p^2q + c^4p^2r + c^4pr^2)$$

$$\zeta_6 = (a^2pr - a^2qr + a^2r^2 - b^2pr + b^2qr + b^2r^2 - 2c^2pq - c^2pr - c^2qr - c^2r^2)(a^4p^2q + a^4pq^2 + a^4q^2r - 2a^2b^2p^2q - a^2b^2p^2r - 2a^2b^2pq^2 - a^2b^2q^2r - a^2c^2p^2q + a^2c^2pq^2 + a^2c^2pqr + a^2c^2q^2r + b^4p^2q + b^4p^2r + b^4pq^2 + b^2c^2p^2q + b^2c^2p^2r - b^2c^2pq^2 + b^2c^2pqr - c^4pqr)$$

This is represented by the matrix

$$\mathcal{M} = \begin{pmatrix} 2\zeta_1 & \zeta_6 & \zeta_5 \\ \zeta_6 & 2\zeta_2 & \zeta_4 \\ \zeta_5 & \zeta_4 & 2\zeta_3 \end{pmatrix}$$

The coordinates of point P and P^* can be written as

$$P = X + Y, \; P^* = X - Y$$

with

$$X = \frac{(x_X \quad y_X \quad z_X)}{\zeta_X}, \; Y = \frac{(x_Y \quad y_Y \quad z_Y)}{\zeta_Y}$$

and

$$x_X = p(a^2q - a^2r - b^2q - b^2r + c^2q + c^2r)(-b^2pr - b^2qr + c^2pq + c^2qr)$$

$$y_X = q(a^2p + a^2r - b^2p + b^2r - c^2p - c^2r)(a^2pr + a^2qr - c^2pq - c^2pr)$$

$$z_X = r(a^2p + a^2q - b^2p - b^2q - c^2p + c^2q)(a^2pq + a^2qr - b^2pq - b^2pr)$$

$$\zeta_X = 2a^4p^2qr + 2a^4pq^2r + 2a^4pqr^2 + 2a^4q^2r^2 - 4a^2b^2p^2qr - 4a^2b^2pq^2r - 4a^2c^2p^2qr - 4a^2c^2pqr^2 + 2b^4p^2qr + 2b^4p^2r^2 + 2b^4pq^2r + 2b^4pqr^2 - 4b^2c^2pq^2r - 4b^2c^2pqr^2 + 2c^4p^2q^2 + 2c^4p^2qr + 2c^4pq^2r + 2c^4pqr^2$$

$$x_Y = p(2a^4p^2qr + 2a^4pq^2r + 2a^4pqr^2 + 2a^4q^2r^2 - 3a^2b^2p^2qr - a^2b^2p^2r^2 - 2a^2b^2pq^2r - a^2b^2pqr^2 - a^2b^2pr^3 + a^2b^2q^3r - a^2b^2qr^3 - a^2c^2p^2q^2 - 3a^2c^2p^2qr - a^2c^2pq^3 - a^2c^2pq^2r - 2a^2c^2pqr^2 - a^2c^2q^3r + a^2c^2qr^3 + b^4p^2qr + b^4p^2r^2 - b^4pqr^2 - b^4pr^3 - b^4q^3r - 2b^4q^2r^2 - b^4qr^3 + b^2c^2p^2q^2 + 2b^2c^2p^2qr + b^2c^2p^2r^2 + b^2c^2pq^3 + b^2c^2pq^2r + b^2c^2pqr^2 + b^2c^2pr^3 + 2b^2c^2q^3r + 4b^2c^2q^2r^2 + 2b^2c^2qr^3 + c^4p^2q^2 + c^4p^2qr - c^4pq^3 - c^4pq^2r - c^4q^3r - 2c^4q^2r^2 - c^4qr^3)$$

$$y_Y = -q(a^4p^3r + 2a^4p^2r^2 - a^4pq^2r + a^4pqr^2 + a^4pr^3 - a^4q^2r^2 + a^4qr^3 - a^2b^2p^3r + 2a^2b^2p^2qr + 3a^2b^2pq^2r + a^2b^2pqr^2 + a^2b^2pr^3 + a^2b^2q^2r^2 + a^2b^2qr^3 - a^2c^2p^3q - 2a^2c^2p^3r - a^2c^2p^2q^2 - a^2c^2p^2qr - 4a^2c^2p^2r^2 - 2a^2c^2pq^2r - a^2c^2pqr^2 - 2a^2c^2pr^3 - a^2c^2q^2r^2 - a^2c^2qr^3 - 2b^4p^2qr - 2b^4p^2r^2 - 2b^4pq^2r - 2b^4pqr^2 + b^2c^2p^3q + b^2c^2p^3r + b^2c^2p^2q^2 + b^2c^2p^2qr + 3b^2c^2pq^2r + 2b^2c^2pqr^2 - b^2c^2pr^3 + c^4p^3q + c^4p^3r - c^4p^2q^2 + c^4p^2qr + 2c^4p^2r^2 - c^4pq^2r + c^4pr^3)$$

$$z_Y = -r(a^4p^3q + 2a^4p^2q^2 + a^4pq^3 + a^4pq^2r - a^4pqr^2 + a^4q^3r - a^4q^2r^2 - 2a^2b^2p^3q - a^2b^2p^3r - 4a^2b^2p^2q^2 - a^2b^2p^2qr - a^2b^2p^2r^2 - 2a^2b^2pq^3 - a^2b^2pq^2r - 2a^2b^2pqr^2 - a^2b^2q^3r - a^2b^2q^2r^2 - a^2c^2p^3q + 2a^2c^2p^2qr + a^2c^2pq^3 + a^2c^2pq^2r + 3a^2c^2pqr^2 + a^2c^2q^3r + a^2c^2q^2r^2 + b^4p^3q + b^4p^3r + 2b^4p^2q^2 + b^4p^2qr - b^4p^2r^2 + b^4pq^3 - b^4pqr^2 + b^2c^2p^3q + b^2c^2p^3r + b^2c^2p^2qr + b^2c^2p^2r^2 - b^2c^2pq^3 + 2b^2c^2pq^2r + 3b^2c^2pqr^2 - 2c^4p^2q^2 - 2c^4p^2qr - 2c^4pq^2r - 2c^4pqr^2)$$

$$\zeta_Y = 2(p+q+r)(a^4p^2qr + a^4pq^2r + a^4pqr^2 + a^4q^2r^2 - 2a^2b^2p^2qr - 2a^2b^2pq^2r - 2a^2c^2p^2qr - 2a^2c^2pqr^2 + b^4p^2qr + b^4p^2r^2 + b^4pq^2r + b^4pqr^2 - 2b^2c^2pq^2r - 2b^2c^2pqr^2 + c^4p^2q^2 + c^4p^2qr + c^4pq^2r + c^4pqr^2)$$

From here, we have $X\mathcal{M}X^t = Y\mathcal{M}Y^t = \frac{\zeta}{\Xi}$, with

$$\zeta = 8pqr(uv + uw + vw)(p^4q^2u^3 + p^4q^2u^2v + 2p^4qru^3 + p^4r^2u^3 + p^4r^2u^2w - 2p^3q^3u^2v - 2p^3q^3uv^2 - 2p^3q^2ru^2v - 2p^3qr^2u^2w - 2p^3r^3u^2w - 2p^3r^3uw^2 + p^2q^4uv^2 + p^2q^4v^3 - 2p^2q^3ruv^2 + 6p^2q^2r^2uvw - 2p^2qr^3uw^2 + p^2r^4uw^2 + p^2r^4w^3 + 2pq^4rv^3 - 2pq^3r^2v^2w - 2pq^2r^3vw^2 + 2pqr^4w^3 + q^4r^2v^3 + q^4r^2v^2w - 2q^3r^3v^2w - 2q^3r^3vw^2 + q^2r^4vw^2 + q^2r^4w^3)$$

$$\Xi = p^2q^2u^2 + 2p^2q^2uv + p^2q^2v^2 + 2p^2qru^2 - 2p^2qruv - 2p^2qruw - 2p^2qrvw + p^2r^2u^2 + 2p^2r^2uw + p^2r^2w^2 - 2pq^2ruv - 2pq^2ruw + 2pq^2rv^2 - 2pq^2rvw - 2pqr^2uv - 2pqr^2uw - 2pqr^2vw + 2pqr^2w^2 + q^2r^2v^2 + 2q^2r^2vw + q^2r^2w^2$$

and

$$u = \frac{b^2 + c^2 - a^2}{2},\ v = \frac{c^2 + a^2 - b^2}{2},\ w = \frac{a^2 + b^2 - c^2}{2}$$

With this, we have

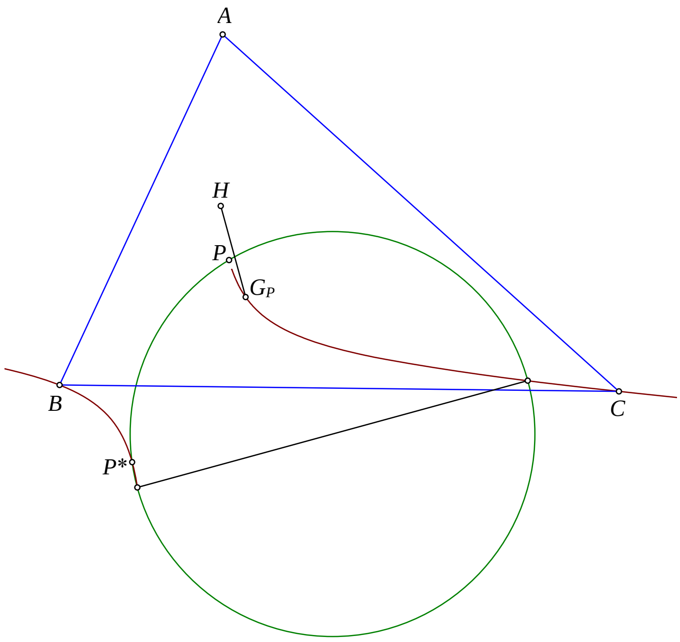
$$P\mathcal{M}P^{*^t} = (X+Y)\mathcal{M}(X-Y)^t = X\mathcal{M}X^t - Y\mathcal{M}Y^t = 0$$

This show that the points P and P^* are inverse in the circle have diameter HG_P.
Since

$$\overline{MP} \cdot \overline{MP^*} = MH^2 = \overline{MO_P} \cdot \overline{MN_P},$$

we have four points P, P^*, N_P, and O_P lie on the same circle $\mathcal{L}_P$. □

5. Proof of Theorem 6



Proof. The equation of the line HG_P (also line ℓ) is

$$\eta_a x + \eta_b y + \eta_c z = 0$$

with

$$\begin{cases} \eta_a = (a^2-b^2-c^2)\left(a^2q-a^2r-b^2q-b^2r+c^2q+c^2r\right)\left(a^2p^2+a^2pq+a^2pr+2a^2qr-b^2p^2-b^2pq+b^2pr+c^2p(q-r-p)\right) \\ \eta_b = (a^2-b^2+c^2)\left(a^2p+a^2r-b^2p+b^2r-c^2p-c^2r\right)\left(a^2pq+a^2q^2-a^2qr-b^2pq-2b^2pr-b^2q^2-b^2qr+c^2q(q+r-p)\right) \\ \eta_c = (c^2-a^2-b^2)\left(a^2p+a^2q-b^2p-b^2q-c^2p+c^2q\right)\left(a^2pr-a^2qr+a^2r^2-b^2pr+b^2qr+b^2r^2-2c^2pq-c^2r(p+q+r)\right) \end{cases}$$

The equation of the line ℓ_1 tangent to $\mathcal{K}_P$ at P is

$$\theta_a x + \theta_b y + \theta_c z = 0$$

with

$$\begin{cases} \theta_a = qr\left(-a^2q+a^2r+b^2q+b^2r-c^2q-c^2r\right) \\ \theta_b = pr\left(-a^2p-a^2r+b^2p-b^2r+c^2p+c^2r\right) \\ \theta_c = pq\left(a^2p+a^2q-b^2p-b^2q-c^2p+c^2q\right) \end{cases}$$

The equation of the line ℓ_2 tangent to $\mathcal{K}_P$ at P^* is

$$\iota_a x + \iota_b y + \iota_c z = 0$$

with

$$\begin{cases} \iota_a = qr\left(-a^2q+a^2r+b^2q+b^2r-c^2q-c^2r\right)\left(a^2p^2+a^2pq+a^2pr+a^2qr-b^2p^2-b^2pq-c^2p^2-c^2pr\right)^2 \\ \iota_b = rp\left(-a^2p-a^2r+b^2p-b^2r+c^2p+c^2r\right)\left(a^2pq+a^2q^2-b^2pq-b^2pr-b^2q^2-b^2qr+c^2q^2+c^2qr\right)^2 \\ \iota_c = pq\left(a^2p+a^2q-b^2p-b^2q-c^2p+c^2q\right)\left(a^2pr+a^2r^2+b^2qr+b^2r^2-c^2pq-c^2pr-c^2qr-c^2r^2\right)^2 \end{cases}$$

It is not hard to check that

$$\begin{vmatrix} \eta_a & \eta_b & \eta_c \\ \theta_a & \theta_b & \theta_c \\ 1 & 1 & 1 \end{vmatrix} = \begin{vmatrix} \eta_a & \eta_b & \eta_c \\ \iota_a & \iota_b & \iota_c \\ 1 & 1 & 1 \end{vmatrix} = 0$$

Therefore, we have $\ell \parallel \ell_1 \parallel \ell_2$.
Now, we use Cartesian coordinates as follows.
In a Cartesian coordinate system, let the rectangular hyperbola $\mathcal{K}_P$ be represented by $xy = a$, and $P\left(x_0, \frac{a}{x_0}\right)$ and $P^*\left(-x_0, \frac{-a}{x_0}\right)$ is two antipodal points.
Let P_+P_- be a chord of $\mathcal{K}_P$, and $P_+P_- \perp \ell$.
The slope of the tangents at P and P^* is $\frac{-a}{x_0^2}$.
Since $\ell \parallel \ell_1 \parallel \ell_2$, the slope of line ℓ is $\frac{-a}{x_0^2}$, thus the slope of line P_+P_- is $\frac{x_0^2}{a}$.
The coordinates of the points P_+ and P_- are

$$P_+ = \left(x_1, \frac{a}{x_1}\right),\ P_- = \left(x_2, \frac{a}{x_2}\right)$$

We have

$$\frac{\frac{a}{x_1} - \frac{a}{x_2}}{x_1 - x_2} = \frac{x_0^2}{a} \implies \frac{-a}{x_1x_2} = \frac{x_0^2}{a} \implies x_0^2x_1x_2 = -a^2$$

Note that

$$\begin{aligned} \overrightarrow{PP_+} \cdot \overrightarrow{PP_-} &= (x_0 - x_1)(x_0 - x_2) + \left(\frac{a}{x_0} - \frac{a}{x_1}\right)\left(\frac{a}{x_0} - \frac{a}{x_2}\right) \\ &= (x_0 - x_1)(x_0 - x_2)\left(1 + \frac{a^2}{x_0^2x_1x_2}\right) = 0 \end{aligned}$$

$$\begin{aligned} \overrightarrow{P^*P_+} \cdot \overrightarrow{P^*P_-} &= (-x_0 - x_1)(-x_0 - x_2) + \left(-\frac{a}{x_0} - \frac{a}{x_1}\right)\left(-\frac{a}{x_0} - \frac{a}{x_2}\right) \\ &= (x_0 + x_1)(x_0 + x_2)\left(1 + \frac{a^2}{x_0^2x_1x_2}\right) = 0 \end{aligned}$$

therefore we have P, P^* lie on the circle have diameter P_+P_-.

□

6. Proof of Theorem 7

Proof. Consider this configuration in the complex plane, and let (O) be the unit circle. We also let

$$\begin{cases} u = a + b + c, \\ v = ab + bc + ca, \\ w = abc \end{cases}$$

Note that

$$\bar{u} = \frac{v}{w},\ \bar{v} = \frac{u}{w},\ \bar{w} = \frac{1}{w}$$

Since P is a point on line OH, we have

$$p = kh = k(a + b + c) = ku,\ k \in \mathbb{R}$$

Since Q is isogonal conjugate of P with respect to triangle ABC, we have

$$q = \frac{abc\bar{p}^2 - (ab+bc+ca)\bar{p} - p + a + b + c}{1 - p\bar{p}} = \frac{(1-k)(kv^2 - uw)}{k^2uv - w}, \ (1)$$

Now, we let

$$s^+ = k_1 q, \ s^- = k_2 q, \ (k_1, k_2 \in \mathbb{R})$$

Since $(S^+S^-, OQ) = -1$ and S^+, S^- are inversion with (O), we have

$$\frac{s^+}{s^-} \div \frac{q - s^+}{q - s^-} = -1, \ s^- = \frac{1}{\overline{s^+}}$$

thus

$$\begin{cases} k_1 k_2 = \frac{1}{q\bar{q}} \\ k_1 + k_2 = 2k_1k_2 = \frac{2}{q\bar{q}} \end{cases}$$

We have k_1, k_2 are the roots of the equation

$$X^2 - \frac{2}{q\bar{q}}X + \frac{1}{q\bar{q}} = 0, \ (*)$$

Since F^+, F^- are isogonal conjugate of S^+, S^-, we have

$$f^+ = \frac{abc\overline{s^+}^2 - (ab+bc+ca)\overline{s^+} - s^+ + a + b + c}{1 - s^+\overline{s^+}} = \frac{w\overline{s^+}^2 - v\overline{s^+} - s^+ + u}{1 - s^+\overline{s^+}}$$

$$f^- = \frac{abc\overline{s^-}^2 - (ab+bc+ca)\overline{s^-} - s^- + a + b + c}{1 - s^-\overline{s^-}} = \frac{w\overline{s^-}^2 - v\overline{s^-} - s^- + u}{1 - s^-\overline{s^-}}$$

Since R is on OH such that $(OR, PH) = -1$, we have

$$\frac{p}{p-r} \div \frac{h}{h-r} = -1 \implies r = \frac{2ku}{k+1}$$

From $(*)$ and (1), we have

$$\begin{aligned} &k^4k_1^2u^2v^2 - 2k^4k_1u^2v^2 - k^3k_1^2u^3w - 2k^3k_1^2u^2v^2 + k^4u^2v^2 \\ &- k^3k_1^2v^3 + 2k^2k_1^2u^3w + k^2k_1^2u^2v^2 + k^2k_1^2uvw + 2k^2k_1^2v^3 \\ &- kk_1^2u^3w + 4k^2k_1uvw - 2kk_1^2uvw - kk_1^2v^3 - 2k^2uvw \\ &+ k_1^2uvw - 2k_1w^2 + w^2 = 0 \end{aligned}$$

This leads us to the fact that

$$\begin{aligned} &\frac{f^+}{f^+ - r} \div \overline{\left(\frac{f^+}{f^+ - r}\right)} - \frac{k^2u^3w + 2k^2u^2v^2 + k^2v^3 + 2ku^3w - 2ku^2v^2 - 2kuvw - 2kv^3 + u^3w - 2uvw + v^3}{k^2u^3w + 2k^2u^2v^2 + k^2v^3 - 2ku^3w - 2ku^2v^2 - 2kuvw + 2kv^3 + u^3w - 2uvw + v^3} \\ &= \frac{2k}{\epsilon}(u^3w - v^3)(k^4k_1^2u^2v^2 - 2k^4k_1u^2v^2 - k^3k_1^2u^3w - 2k^3k_1^2u^2v^2 + k^4u^2v^2 - k^3k_1^2v^3 + 2k^2k_1^2u^3w + k^2k_1^2u^2v^2 \\ &+ k^2k_1^2uvw + 2k^2k_1^2v^3 - kk_1^2u^3w + 4k^2k_1uvw - 2kk_1^2uvw - kk_1^2v^3 - 2k^2uvw + k_1^2uvw - 2k_1w^2 + w^2) = 0 \end{aligned}$$

with

$$\begin{aligned}\epsilon = {} & k^5k_1^2u^4vw + 2k^5k_1^2u^3v^3 + k^5k_1^2uv^4 + k^5k_1u^4vw + 2k^5k_1u^3v^3 - 3k^4k_1^2u^4vw - 4k^4k_1^2u^3v^3 + k^5k_1uv^4 \\ & - 4k^4k_1^2u^2v^2w - 3k^4k_1^2uv^4 - k^4k_1u^4vw + 3k^3k_1^2u^4vw + 2k^3k_1^2u^3v^3 - 2k^4k_1u^2v^2w - k^4k_1uv^4 + 2k^4u^3v^3 \\ & + k^3k_1^2u^3w^2 + 6k^3k_1^2u^2v^2w + 3k^3k_1^2uv^4 - k^3k_1u^4vw - 2k^3k_1u^3v^3 - k^2k_1^2u^4vw + k^3k_1^2v^3w - k^3k_1u^3w^2 \\ & - 2k^3k_1u^2v^2w - k^3k_1uv^4 - k^2k_1^2u^3w^2 - k^2k_1^2uv^4 + k^2k_1u^4vw - k^3k_1v^3w - k^2k_1^2v^3w + k^2k_1u^3w^2 \\ & + 2k^2k_1u^2v^2w + k^2k_1uv^4 - kk_1^2u^3w^2 - 2kk_1^2u^2v^2w + 2k^2k_1uvw^2 + k^2k_1v^3w - 4k^2u^2v^2w - kk_1^2v^3w \\ & + kk_1u^3w^2 + 2kk_1u^2v^2w + k_1^2u^3w^2 + kk_1v^3w + k_1^2v^3w - k_1u^3w^2 - 2k_1uvw^2 - k_1v^3w + 2uvw^2\end{aligned}$$

Similarly, we have

$$\begin{aligned}&\frac{f^-}{f^- - r} \div \overline{\left(\frac{f^-}{f^- - r}\right)} \\ &= \frac{k^2u^3w + 2k^2u^2v^2 + k^2v^3 + 2ku^3w - 2ku^2v^2 - 2kuvw - 2kv^3 + u^3w - 2uvw + v^3}{k^2u^3w + 2k^2u^2v^2 + k^2v^3 - 2ku^3w - 2ku^2v^2 - 2kuvw + 2kv^3 + u^3w - 2uvw + v^3}\end{aligned}$$

thus

$$\frac{f^+}{f^+ - r} \div \overline{\left(\frac{f^+}{f^+ - r}\right)} = \frac{f^-}{f^- - r} \div \overline{\left(\frac{f^-}{f^- - r}\right)}$$

From here, we can conclude that four points F^+, F^-, O, R lie on a circle. □

7. Conclusion

In this paper we introduced a new framework that extends the classical Kiepert hyperbola and the Lester circle. Starting from a parameterized construction involving an arbitrary point P, we defined a rectangular hyperbola $\mathcal{K}_P$ associated with triangle ABC. This curve naturally generalizes the classical Kiepert hyperbola obtained when P is the first Fermat point. A key feature of this construction is the appearance of the point P^*, the antigonal conjugate of P, and the point G_P, which plays a role analogous to the centroid in the classical configuration.

We showed that the midpoint of PP^* serves simultaneously as the center of the generalized hyperbola $\mathcal{K}_P$, while also lying on both the nine-point circle of ABC and the circumcircle of $A_2B_2C_2$. This provides a natural geometric link between the generalized Kiepert configuration and classical triangle centers.

Based on this hyperbolic structure, we established a generalized version of the Lester circle, proving that the four points P, P^*, N_P, and O_P are concyclic. In addition, we obtained a generalized form of Gibert's theorem, showing that certain circles determined by chords of $\mathcal{K}_P$ pass through the pair P and P^*. Finally, we presented another extension of the Lester circle arising from a configuration involving isogonal conjugacy and harmonic division along the Euler line.

These results demonstrate that the classical relationships among the Fermat points, the Kiepert hyperbola, and the Lester circle belong to a broader geometric structure that persists for an arbitrary point P. It would be interesting to investigate further properties of the hyperbola $\mathcal{K}_P$, its interaction with other well-known triangle centers, and possible extensions in the settings of projective or inversive geometry.

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