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A MARSDEN-WEINSTEIN TYPE REDUCTION SCHEME FOR CONTACT PAIRS

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ABSTRACT. This paper generalizes the classical Marsden-Weinstein reduction theory for symmetric symplectic manifolds to the context of Contact Pairs. We demonstrate that, for a Contact Pair structure, it is possible to establish a reduction scheme that is analogous to the well-known procedure for symplectic manifolds.

1. INTRODUCTION AND MOTIVATIONS

The Marsden-Weinstein reduction theorem is a cornerstone of symplectic geometry, providing a powerful method for simplifying the study of Hamiltonian systems that possess symmetries. In essence, it allows a complex dynamical system on a high-dimensional phase space to be reduced to a simpler, lower-dimensional system.
In physics and mathematics, a Hamiltonian system describes the evolution of a physical system over time using a function called the Hamiltonian, which often represents the total energy. A symplectic manifold serves as the phase space for such a system. When a system exhibits a symmetry, it means that its dynamics remain unchanged under the action of a group of transformations (a Lie group).
The Marsden-Weinstein theorem formalizes this idea of reducing the system by its symmetries. It takes a symplectic manifold (M, ω) and a Lie group G that acts on it in a way that preserves both the symplectic form ω and the Hamiltonian function H. The theorem then constructs a new, smaller manifold, called the symplectic quotient or reduced phase space. This reduced space inherits a natural symplectic structure, and the original dynamics of the system descend to it, allowing for a much simpler analysis. Albert has generalized in [17] the Marsden-Weinstein reduction theory for symplectic manifolds with symmetry to cosymplectic manifolds and to general contact manifolds. In doing so he has conceived a unified approach to cosymplectic and contact structures by identifying them as limiting cases of a transitive almost contact structure. In a recent paper (see[14]), Y. Mbodji and H. Dathe extended the Marsden-Weinstein reduction theory originally developed for symmetric symplectic manifolds to contact-symplectic pairs after first establishing the required notions of Hamiltonian and gradient vector fields for this framework. Contact pairs have been identified and studied by G.BANDE in [5]. The

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concept of a Contact Pair allows for the study of dynamical systems that are simultaneously governed by two types of non-integrability or two distinct time directions, each encoded by α and η. The study of Contact Pairs paves the way for potential generalizations of the reduction theory and the concepts of Hamiltonian and gradient vector fields to geometric contexts where the dynamics are constrained by two independent Pfaffian structures, thereby considerably enriching the landscape of geometric mechanics. In this paper, we generalized the Marsden-Weinstein reduction theory for symmetric symplectic manifolds to the context of contact pairs. This result provides a significant extension of geometric reduction methods to a class of even-dimensional manifolds governed by two independent Pfaffian structures.

2. MAIN RESULTS

2.1. **Contact pairs.** In this subsection, we recall the notion of contact pair structures. For more information see [5].

Definition 2.1. *A contact pair of type (k,h) on a manifold $M^{2k+2h+2}$ of dimension $(2k+2h+2)$ is a pair of Pfaffian forms (η,α) such that:*

$\eta \wedge (d\eta)^k \wedge \alpha \wedge d\alpha^h$ *is a volume form on* $M^{2k+2h+2}$, *and* $d\eta^{k+1}=0$, $d\alpha^{h+1}=0$.

The forms η and α are so necessarily constant class $2k+1$ and $2h+1$. To a such structure are naturally associated two distributions: the distribution of vector fields which annul η and $d\eta$, and the distribution of vector fields which annul α and $d\alpha$. This distributions are completely integrable because the forms η and ω are constant class $2k+1$ and $2h+1$. They determine the characteristic foliations of η and α, which we will note $\mathfrak{H}$ and $\mathfrak{G}$.
The characteristic foliation of η is of codimension $2h+1$ and her leaves are contact manifolds of contact structure associated α. The characteristic foliation of η is of codimension $2k+1$ and her leaves are contact manifold of contact form associated η.
The foliations $\mathfrak{H}$ and $\mathfrak{G}$ are hollowing out transversal and additional. A vectors fields Z_η, Z_α known as the Reeb vectors fields, is carried by manifolds with contact-symplectic pairs and is uniquely identified by:

$$\eta(Z_\eta)=1 \text{ and } i_{Z_\eta}(\alpha\wedge d\alpha^h\wedge d\eta^k)=0.$$

and

$$\alpha(Z_\alpha)=1 \text{ and } i_{Z_\alpha}(\eta\wedge d\eta^k\wedge d\alpha^h)=0.$$

The Reeb vector field Z_η, Z_α of contact pair (η,α) commute and satisfy the following condition:

$$\eta(Z_\alpha)=0,\, i_{Z_\alpha}d\eta=i_{Z_\alpha}d\eta=0,\, \alpha(Z_\eta)=0,\, i_{Z_\eta}d\alpha=i_{Z_\eta}d\alpha=0,$$

So, Z_η (resp. Z_α) is tangent to characteristic foliation of η (resp. α) and it is the Reeb vector field (in the classical sense) of contact form induced by η (resp. α) on each leaf of their foliation. Contrary to riemannian manifolds, the contact pairs have not local invariants. That is to do to theorem which follows, called the Darboux theorem. It establishes an unique local model of contact pairs.

Theorem 2.1. [5] *Let (η,α) be a contact pair on $M^{2k+2h+2}$ and p a point of $M^{2k+2h+2}$. Then, it exists a local coordinates system*
$(U,x_1,\cdots,x_k,y_1,\cdots,y_k,z_1,p_1,\cdots,p_h,q_1,\cdots,q_h,z_2)$ *around of p, such that*

$$\eta_{|U}=\sum_{i=1}^k x_i dy_i + dz_1,\, \alpha_{|U}=\sum_{i=1}^k p_i dq_i + dz_2,\, Z_{\eta|U}=\partial_{z_1},\, Z_{\alpha|U}=\partial_{z_2}$$

The coordinates system $(U, x_1, \cdots, x_k, y_1, \cdots, y_k, z_1, p_1, \cdots, p_h, q_1, \cdots, q_h, z_2)$ is called coordinates system of Darboux. All contact pair is locally the produce of contact manifolds.

2.2. **Gradient vector fields and Hamiltonian vector fields on contact-symplectic pairs.** Let $(M^{2k+2h+2}, \eta, \alpha)$ be a contact pair. Consider the bundle homomorphism $\flat_{(\eta,\omega)}$: $\chi(M) \longrightarrow \Omega^1(M)$ defined by:

$$\flat_{(\eta,\alpha)}(X) = \eta(X).\eta + \alpha(X).\alpha + i_X(d\eta + d\alpha), \forall X \in \chi(M).$$

We have the following proposition.

Proposition 2.1. *The map $\flat_{(\eta,\alpha)}$ is a smooth vector bundle isomorphism.*

Proof. Let us show that $\flat_{(\eta,\alpha)}$ is an isomorphism. Observe that it suffices to show that the map $\flat_{(\eta,\alpha)}$ is injective. Let $(U, x_1, \cdots, x_k, y_1, \cdots, y_k, z_1, p_1, \cdots, p_h, q_1, \cdots, q_h, z_2)$ be a Darboux coordinates system and
$X = \Sigma_{i=1}^k a_i \partial x_i + \Sigma_{i=1}^k b_i \partial y_i + \Sigma_{i=1}^h c_i \partial p_i + \Sigma_{i=1}^h d_i \partial q_i + e\partial z_1 + f\partial z_2$ be a vectors field on U. We assume that $\flat_{(\eta,\alpha)}(X) = 0$. So we have

$$\begin{aligned} \eta(X).\eta(X) + \alpha(X).\alpha + i_X(d\eta + d\alpha)(X) &= 0 \\ \eta^2(X) + \alpha^2(X) &= 0 \\ \eta(X) = \alpha(X) &= 0. \end{aligned}$$

Thus, we obtain $i_X(d\eta + d\alpha) = 0$. Consequently,

$$\begin{aligned} i_X(d\eta + d\alpha)(\partial x_i) &= 0 \\ b_i &= 0, \end{aligned} \tag{2.1}$$

$$\begin{aligned} i_X(d\alpha + d\eta)(\partial y_i) &= 0 \\ a_i &= 0, \end{aligned} \tag{2.2}$$

$$\begin{aligned} i_X(d\alpha + d\eta)(\partial p_i) &= 0 \\ d_i &= 0, \end{aligned} \tag{2.3}$$

$$\begin{aligned} i_X(d\alpha + d\eta)(\partial q_i) &= 0 \\ c_i &= 0, \end{aligned} \tag{2.4}$$

$$\begin{aligned} \eta(X) &= 0 \\ e &= 0. \end{aligned} \tag{2.5}$$

$$\begin{aligned} \alpha(X) &= 0 \\ f &= 0. \end{aligned} \tag{2.6}$$

According to the relations (2.1),(2.2),(2.3),(2.4),(2.5) and (2.6) we deduce that $X = 0$ and later on the map $\flat_{(\eta,\alpha)}$ is injective. □

Thanks to this isomorphism, we can associate at every function
$\varphi \in C^\infty(M^{2k+2h+2}, \mathbb{R})$ an unique vectors field $grad\varphi \in \chi(M)$, called gradient of φ, which is defined by

Definition 2.2. *Let $(M^{2k+2h+2}, \eta, \alpha)$ be a contact pair and $\varphi \in \mathcal{C}^\infty(M^{2k+2h+2}, \mathbb{R})$ a function. Is called gradient vectors field of φ the single vectors field $grad\varphi$ defined by*

$$grad\varphi = \flat^{-1}_{(\eta,\alpha)}(d\varphi)$$

. Likewise

$$(2.1)\begin{cases} i_{grad\varphi}(\omega + d\eta) = d\varphi - Z_\eta(\varphi)\eta - Z_\alpha(\varphi)\alpha \\ \eta(grad\varphi) = Z_\eta(\varphi), \alpha(grad\varphi) = Z_\alpha(\varphi). \end{cases}$$

We define also the Hamiltonian vectors field associated to $\varphi \in C^\infty(M^{2k+2h+2}, \mathbb{R})$.

Definition 2.3. *Let $(M^{2k+2h+2}, \eta, \alpha)$ be a contact pair and $\varphi \in \mathcal{C}^\infty(M^{2k+2h+2}, \mathbb{R})$ a function. Is called Hamiltonian vectors field of φ the single vectors field X_φ defined by*

$$X_\varphi = \flat^{-1}_{(\eta,\alpha)}(d\varphi - Z_\eta(\varphi)\eta - Z_\alpha(\varphi)\alpha).$$

Likewise,

$$(2.2)\begin{cases} i_{X_\varphi}(d\eta + d\alpha) = d\varphi - Z_\eta(\varphi)\eta - Z_\alpha(\varphi)\alpha \\ \eta(X_\varphi) = \alpha(X_\varphi) = 0 \end{cases}$$

The gradient vectors field $grad\varphi$ and the Hamiltonian vector field X_φ have the same horizontal component and

$$X_\varphi = grad\varphi \text{ if only if } Z_\eta(\varphi) = Z_\alpha(\varphi) = 0. \ (*)$$

Definition 2.4. *The Hamiltonian vectors fields which verify the relation $(*)$ are called the restricted Hamiltonian vectors fields.*

By elementary calculus we obtain the following proposition

Proposition 2.2. *Let $(U, x_1, \cdots, x_k, y_1, \cdots, y_k, z_1, p_1, \cdots, p_h, q_1, \cdots, q_h, z_2)$ be a Darboux coordinates system on $(M^{2k+2h+2}, \eta, \alpha)$ and $\varphi \in C^\infty(M^{2k+2h+2}, \mathbb{R})$. So we have*

$$grad\varphi_{|U} = (\frac{\partial\varphi}{\partial y_i} - x_i\frac{\partial\varphi}{\partial_{z_1}})\partial_{x_i} - \frac{\partial\varphi}{\partial x_i}\partial_{y_i} + (\frac{\partial\varphi}{\partial z_1} + x_i\frac{\partial\varphi}{\partial x_i})\partial z_1 + (\frac{\partial\varphi}{\partial q_i} - p_i\frac{\partial\varphi}{\partial z_2})\partial p_i - \frac{\partial\varphi}{\partial p_i}\partial q_i + (\frac{\partial\varphi}{\partial z_2} + p_i\frac{\partial\varphi}{\partial p_i})\partial z_2$$

$$X_{\varphi|U} = (\frac{\partial\varphi}{\partial y_i} - x_i\frac{\partial\varphi}{\partial_{z_1}})\partial_{x_i} - \frac{\partial\varphi}{\partial x_i}\partial_{y_i} + (x_i\frac{\partial\varphi}{\partial x_i})\partial z_1 + (\frac{\partial\varphi}{\partial q_i} - p_i\frac{\partial\varphi}{\partial z_2})\partial p_i - \frac{\partial\varphi}{\partial p_i}\partial q_i + (p_i\frac{\partial\varphi}{\partial p_i})\partial z_2$$

In particular

$$gradx_j = -\partial_{y_j} + x_j\partial z_1; \, gradp_j = -\partial q_j + p_j\partial_{z_2}$$
$$grady_j = \partial x_j; \, gradq_j = \partial p_j$$
$$gradz_1 = -x_i\partial x_i + \partial z_1; \, gradz_2 = -p_i\partial p_i + \partial z_2; \, X_{x_j} = -\partial y_j + x_j\partial_{z_1}$$
$$X_{y_j} = \partial x_j; \, X_{p_j} = -\partial q_i + p_j\partial z_2$$
$$X_{q_j} = \partial p_j; \, X_{z_1} = -x_i\partial x_i; \, X_{z_2} = -p_i\partial p_i$$

2.3. **Restricted Hamiltonian action on contact pairs.** In all that follow, the Lie algebra of Lie group G will be denoted by $\mathcal{G}$ and its dual by $\mathcal{G}^*$.

Definition 2.5. *An action of Lie group G on contact pair $(M^{2k+2h+2}, \eta, \alpha)$ is a $\mathcal{C}^\infty$ map $\Phi : G \times M^{2k+2h+2} \longrightarrow M^{2k+2h+2}$ such that for all g and $g' \in G$, $x \in M^{2k+2h+2}$,*

$$\Phi(g, \Phi(g', x)) = \Phi(gg', x) \text{ et } \Phi(e, x) = x$$

Observe that Φ_g is a diffeomorphism, of reciprocal $\Phi_{g^{-1}}$ and for all g and $g' \in G$,

$$\Phi_g \circ \Phi_{g'} = \Phi_{gg'}.$$

Thus the map $g \longmapsto \Phi_g$ is a homomorphism. For all $x \in M^{2k+2h+2}$, we denote respectively by $\mathcal{O}_x$ and G_x the orbit and isotropy sub-group of x. $\mathcal{O}_x$ is a sub-manifold of $M^{2k+2h+2}$ not necessary embedding and G_x is a Lie sub-group of G.

Definition 2.6. *Let Φ be an action of Lie group G on contact pair $(M^{2k+2h+2}, \eta, \alpha)$. We call fundamental vectors field associated to an element ξ of Lie algebra $\mathcal{G}$, the vectors field ξ_M on $M^{2k+2h+2}$ defined by*

$$\xi_M(x) = \frac{d}{dt}_{|t=0} \Phi_{expt\xi}(x).$$

Consequence 2.1. [8]

(1) *Fundamental vector fields ξ_M are complete.*
(2) *For all $\xi \in \mathcal{G}$ and $x \in M^{2k+2h+2}$ the fundamental vector field ξ_M is tangent to $\mathcal{O}_x$.*
(3) *The map $\xi \longmapsto \xi_M$ is a homomorphism of Lie algebra.*

Definition 2.7. *An action Φ of Lie group G on contact pair $(M^{2k+2h+2}, \eta, \alpha)$ is called an automorphism action, or G is said to be an automorphism group of $(M^{2k+2h+2}, \eta, \alpha)$, if for each $g \in G$, the corresponding Φ_g is an automorphism of the contact pair structure, i.e.*

$$\Phi_g^*\eta = \eta,\ \Phi_g^*\alpha = \alpha$$

It then follows that the fundamental vectors fields are infinitesimal automorphisms; i.e.

$$\mathcal{L}_{\xi_M}\eta = 0,\ \mathcal{L}_{\xi_M}\alpha = 0$$

for all $\xi \in \mathcal{G}$.

Definition 2.8. *An automorphism action Φ will be called a restricted Hamiltonian action if furthermore, for each $\xi \in \mathcal{G}$, the associated fundamental vectors field ξ_M is a restricted Hamiltonian vectors field.*

Proposition 2.3. *Let Φ be a Hamiltonian action on contact-symplectic manifold $(M^{2k+2h+2}, \eta, \alpha)$. For all $\xi \in \mathcal{G}$, we note by J_ξ the Hamiltonian associated to fundamental vector field ξ_M*

(1) *It exist a linear map $J : \xi \in \mathcal{G} \longrightarrow J_\xi \in \mathcal{C}^\infty(M^{2k+2h+2}, \mathbb{R})$ such that for all $\xi \in \mathcal{G}$, J_ξ is a Hamiltonian function for the fundamental vectors field ξ_M. The map J is called generalized Hamiltonian of action Φ*
(2) *To generalized Hamiltonian $\xi \longmapsto J_\xi$, we can associate a $\mathcal{C}^\infty$ map $J : M^{2k+2h+2} \longrightarrow \mathcal{G}^*$, called the momentum map of action Φ defined by*

$$J_x(\xi) =< J(x), \xi >= J_\xi(x),\ \xi \in \mathcal{G},\ x \in M^{2k+2h+2}$$

Proof. Suppose that G is of dimension n and let $\xi_1, \cdots, \xi_n$ be a basis of $\mathcal{G}$. We denote by J_{ξ_i} the Hamiltonian function associated to the fundamental vector field ξ_{iM}. Then for any $\xi \in \mathcal{G}$, there exists a unique n-tuple $(\beta_1, \cdots, \beta_n) \in \mathbb{R}^n$ such that $\xi = \sum_{i=1}^n \beta_i \xi_i$ and, moreover, $\xi_M = \sum_{i=1}^n \beta_i \xi_{iM}$. Now for any $\xi = \sum_{i=1}^n \beta_i \xi_i$ we set

$$J_\xi = \sum_{i=1}^n \beta_i J_{\xi_i}.$$

Let us show that J_ξ is well-defined. We have

$$\begin{aligned}
\eta(\xi_M)\eta + \alpha(\xi_M)\alpha + i_{\xi_M}(d\eta + d\alpha) &= \eta(\sum_{i=1}^{n}\beta_i\xi_{iM})\eta + \alpha(\sum_{i=1}^{n}\beta_i\xi_{iM})\alpha + \sum_{i=1}^{n}\beta_i i_{\xi_{iM}}(d\eta + d\alpha)\\
&= \sum_{i=1}^{n}\beta_i\eta(\xi_{iM})\eta + \sum_{i=1}^{n}\beta_i\alpha(\xi_{iM})\alpha + \sum_{i=1}^{n}\beta_i i_{\xi_{iM}}(d\eta + d\alpha)\\
&= \sum_{i=1}^{n}\beta_i dJ_{\xi_i}\\
&= d\sum_{i=1}^{n}\beta_i J_{\xi_i} \qquad (2.7)
\end{aligned}$$

According to the relation (2.7), the map J is well-defined. Let us show that J is linear. Let $\xi = \sum_{i=1}^{n}\beta_i\xi_i$, $\xi' = \sum_{i=1}^{n}\delta_i\xi_i$ belonging to $\mathcal{G}$ and $\lambda \in \mathbb{R}$, we have $\xi + \lambda\xi' = \sum_{i=1}^{n}(\beta_i + \lambda\delta_i)\xi_i$.
We put $\xi'' = \xi + \lambda\xi'$, thus

$$\begin{aligned}
J_{\xi''} &= J_{\sum_{i=1}^{n}(\beta_i+\lambda\delta_i)\xi_i}\\
&= \sum_{i=1}^{n}(\beta_i + \lambda\delta_i)J_{\xi_i}\\
&= \sum_{i=1}^{n}\beta_i J_{\xi_i} + \sum_{i=1}^{n}\lambda\delta_i J_{\xi_i}\\
&= \sum_{i=1}^{n}\beta_i J_{\xi_i} + \lambda\sum_{i=1}^{n}\delta_i J_{\xi_i}\\
&= J_\xi + \lambda J_{\xi'}. \qquad (2.8)
\end{aligned}$$

Consequently, according to the relation (2.8) the map J is linear. □

2.4. **Elementary properties of momentum map.**

Proposition 2.4. *Let Φ be a restricted Hamiltonian action on contact-pair manifold $(M^{2k+2h+2}, \eta, \alpha)$. Then for all $\xi \in \mathcal{G}$ and $g \in G$,*

$$(\Phi_g)_*\xi_M = (Ad_g\xi)_M.$$

Moreover the function Hamiltonian associated to $(\Phi_g)_*\xi_M$ is $J_\xi \circ \Phi_{g^{-1}}$.

Proof. Let $\xi \in \mathcal{G}$ and $g \in G$, we have

$$\begin{aligned}
(\Phi_g)_*\xi_M(x) &= T_{\Phi_{g^{-1}(x)}}\Phi_g(\xi_M(\Phi_{g^{-1}}(x)))\\
&= T_{\Phi_{g^{-1}(x)}}\Phi_g(\frac{d}{dt}_{|t=0}\Phi_{exp-t\xi}\circ\Phi_{g^{-1}}(x))\\
&= \frac{d}{dt}_{|t=0}\Phi_g\circ\Phi_{exp-t\xi}\circ\Phi_{g^{-1}}(x)\\
&= \frac{d}{dt}_{|t=0}\Phi_{g(exp-t\xi)g^{-1}}(x)\\
&= (Ad_g\xi)_M. \qquad (2.9)
\end{aligned}$$

Moreover we have,

$$\begin{aligned} i_{(\Phi_g)_*\xi_M}(d\eta + d\alpha) &= i_{(\Phi_g)_*\xi_M}(\Phi_{g^{-1}})^*(d\eta + d\alpha) \\ &= \Phi^*_{g^{-1}}(i_{\xi_M}(d\eta + d\alpha)) \\ &= \Phi^*_{g^{-1}} dJ_\xi \\ &= d(J_\xi \circ \Phi_{g^{-1}}) \end{aligned} \tag{2.10}$$

□

The following proposition shows that, even in the generalized framework of contact-pair manifolds, the variation of the momentum (measured by T_xJ) is entirely controlled by the fundamental geometric form $(d\eta + d\alpha)$ that defines the dynamics on the manifold M

Proposition 2.5. *Let Φ be a restricted Hamiltonian action on contact-symplectic manifold $(M^{2k+2h+2}, \eta, \alpha)$, admitting a momentum map J. For all $x \in M^{2k+2h+2}$ the transposed to linear map $T_xJ : T_xM^{2k+2h+2} \longrightarrow \mathcal{G}^*$ is the map $\xi \longmapsto i_{\xi_M}(\omega + d\eta)(x)$*

Proof. Let $\xi \in \mathcal{G}$, for all $x \in M^{2k+2h+1}$ and $v \in T_xM^{2k+2h+1}$. we have

$$\begin{aligned} T_xJ^t(\xi)(v) &= \xi \circ T_xJ(v) \\ &= T_xJ_\xi(v) \\ &= i_{\xi_{M(x)}}(d\eta + d\alpha)_x(v) + Z_\eta(J_\xi)_x\eta_x(v) + Z_\alpha(J_\xi)_x\alpha_x(v). \end{aligned} \tag{2.11}$$

According to definition of restricted Hamiltonian action we have $Z_\eta(J_\xi) = Z_\alpha(J_\xi) = 0$. Consequently, we obtain

$$i_{\xi_{M(x)}}(\omega + d\eta)(x)(v) = T_xJ^t(\xi)(v).$$

□

According to proposition we obtain the following consequence.

Consequence 2.2. (1) *For all $x \in M^{2k+2h+2}$, the set $T_xJ(T_xM)$ is an annulator $\mathcal{G}_x^0$ of isotropy algebra $\mathcal{G}_x$ of point x, which is defined by*

$$\mathcal{G}_x = \{X \in \mathcal{G} | X_M(x) = 0\}.$$

The image of T_xJ is indeed the annihilator of the kernel of the transposed map.

(2) *For all $x \in M^{2k+2h+2}$, the rank of linear map T_xJ is the dimension of orbit $\mathcal{O}(x)$. In fact, let $orth(\ker T_xJ)$ denote the orthogonal complement of $\ker T_xJ$. The set $orth(\ker T_xJ)$ is given by*

$$orth(\ker T_xJ) = \{v \in T_xM : \forall w \in \ker T_xJ, (d\eta + d\alpha)_x(v, w) = 0\}.$$

Note that $dimM^{2k+2h+2} = dim(\ker(T_xJ)) + dim(orth(\ker T_xJ))$. However, since $T_x\mathcal{O}_x$ is generated by the fundamental vector fields at x, we then have $dim(T_x\mathcal{O}_x) = dim(orth(\ker T_xJ))$. Therefore, we obtain the relation

$$rank(T_xJ) = dimM^{2k+2h+2} - dim(\ker(T_xJ)) = dim(T_x\mathcal{O}_x).$$

In particular J is a submersion at x if only if the orbit have the same dimension that G, i.e the isotropy group is a dimension nil.

The theorem that follow is the generalized version of Noether's Theorem for systems defined on a contact pairs manifold.

Theorem 2.2. *Let Φ be a restricted Hamiltonian action on contact-symplectic manifold $(M^{2k+2h+2}, \eta, \alpha)$, admitting a momentum map J. Let H be a Hamiltonian function on $(M^{2k+2h+2}, \eta, \alpha)$, invariant under action Φ. Then the momentum map J is constant on each integral curve of Hamiltonian vector field X_H.*

Proof. Let $\xi \in \mathcal{G}$, and $\gamma(t)$ an integral curve of X_H. We have

$$\begin{aligned}
\frac{d}{dt}J_\xi(\gamma(t)) &= d_{\gamma(t)}J_\xi(X_{H\gamma(t)}) \\
&= i_{X_{H\gamma(t)}}d_{\gamma(t)}J_\xi \\
&= (d\eta + d\alpha)_{\gamma(t)}(X_{M\gamma(t)}, X_{H\gamma(t)}) - Z_\eta(J_X)_{\gamma(t)}\eta_{\gamma(t)}(X_{H\gamma(t)}) - Z_\alpha(J_X)_{\gamma(t)}\alpha_{\gamma(t)}(X_{H\gamma(t)}) \\
&= (d\eta + d\alpha)_{\gamma(t)}(X_{M\gamma(t)}, X_{H\gamma(t)}) \\
&= -i_{X_{M\gamma(t)}}d_{\gamma(t)}H \\
&= -\frac{d}{ds}H(\Phi_{expsX}(\gamma(t)) \\
&= -\frac{d}{ds}H(\gamma(t)) \\
. &= 0
\end{aligned}$$

□

2.5. **Equivariant of momentum map.** Recall the following lemma

Lemma 2.1. [8] *Let A be a linear representation of Lie group in a finite dimension vector space V, and $\theta : G \longrightarrow V$ a differential map. We put for all $g \in G$ and $v \in V$,*

$$a(g, v) = A_g(v) + \theta(g).$$

The map $a : G \times V \longrightarrow V$ is an action of G on V if only if the map θ verified, for all $g_1, g_2 \in G$,

$$\theta(g_1 g_2) = A_{g_1}(\theta(g_2)) + \theta(g_1). \tag{2.12}$$

θ is a 1-cocycle associated to linear representation A, and a is a G-affine action having for linear part A. This lemma allows us to obtain the following proposition, which plays an essential role in the reduction of contact pairs.

Proposition 2.6. *Let Φ be a restricted Hamiltonian action of Lie group G on a connected contact-symplectic manifold $(M^{2k+2h+2}, \eta, \alpha)$, and let $J : M^{2k+2h+2} \longrightarrow \mathcal{G}^*$ be a momentum map for this action. Let $x \in M^{2k+2h+2}$, so the map $\sigma : G \longrightarrow \mathcal{G}^*$ defined by*

$$(\forall g \in G), \sigma(g) = J_{\Phi_g(x)} - Ad^*_g J_x$$

only depends on the choice of the momentum map J and not on the particular choice of $x \in M^{2k+2h+2}$. σ is a 1-cocycle of G with values in $\mathcal{G}^*$. The cohomology class of σ only depends on the given action Φ and not on the particular choice of the momentum map. Moreover the map $\Psi : G \times \mathcal{G}^* \longrightarrow \mathcal{G}^*$ defined by

$$(\forall g \in G, \mu \in \mathcal{G}^*), \Psi_g(\mu) = Ad^*_g\mu + \sigma(g)$$

is an affine action of G on $\mathcal{G}^*$ such that J is equivariant with respect to Φ and Ψ, i.e.

$$J \circ \Phi_g = \Psi_g \circ J$$

for each $g \in G$.

Proof. Let $g \in G$ and $\xi \in \mathcal{G}$, according to proposition2.4, the vector field $(\Phi_{g^{-1}})_*\xi_M$ is a restricted Hamiltonian vector field with Hamiltonian $J_\xi \circ \Phi_g =< J_{\Phi_g}, \xi >$. It is also a fundamental vector field associated to $Ad_{g^{-1}}\xi$. So it admits like Hamiltonian the function $< J, Ad_{g^{-1}}\xi >=< Ad_g^* \circ J, \xi >$. Since the Hamiltonians $J_\xi \circ \Phi_g$ and $< Ad_g^* \circ J, \xi >$ are associated to the same restricted Hamiltonian vector field $(\Phi_{g^{-1}})_*\xi_M$, then we have

$$\begin{aligned} d(J_\xi \circ \Phi_g - < Ad_g^* \circ J, \xi >) &= dJ_\xi \circ \Phi_g - d < Ad_g^* \circ J, \xi > \\ &= i_{(\Phi_{g^{-1}})_*\xi_M}(d\eta + d\alpha) - i_{(\Phi_{g^{-1}})_*\xi_M}(d\eta + d\alpha) \\ &= 0. \end{aligned} \tag{2.13}$$

Thus, according to the relation (2.13), the map $\sigma(g) = J(\Phi_g(x)) - Ad_g^*J(x)$ is constant locally on $M^{2k+2h+2}$. And as the manifold $M^{2k+2h+2}$ is connected, therefore the map $\sigma(g)$ is constant on $M^{2k+2h+2}$. Hence, $\sigma(g)$ only depends on the choice of the momentum map J and not on the particular choice of $x \in M^{2k+2h+2}$.
Let g and $g' \in G$. We have

$$\begin{aligned} \sigma(gg') &= J(\Phi_{gg'}(x)) - Ad_{gg'}^*J(x) \\ &= J(\Phi_g(\Phi_{g'}(x))) - Ad_g^* \circ Ad_{g'}^*J(x) \\ &= \sigma(g) + Ad_g^*(J(\Phi_{g'}(x)) - Ad_{g'}^*J(x)) \\ &= \sigma(g) + Ad_g^*\sigma(g') \end{aligned}$$

So the map σ is a 1-cocycle of G with values in $\mathcal{G}^*$, for the coadjoint representation. Consequently, according to lemma2.1, the map $\Psi : G \times \mathcal{G}^* \longrightarrow \mathcal{G}^*$ defined by

$$(\forall g \in G, \mu \in \mathcal{G}^*), \Psi_g(\mu) = Ad_g^*\mu + \sigma(g).$$

is an affine action of G on $\mathcal{G}^*$ and by construction J is equivariant with respect to Φ and Ψ. In fact, we have

$$\begin{aligned} \Psi_{gg'} &= Ad_{gg'}^* + \sigma(gg') \\ &= Ad_g^* \circ Ad_{g'}^* + Ad_g^*(\sigma(g')) + \sigma(g') \\ &= Ad_g^*(Ad_{g'}^* + \sigma(g')) + \sigma(g') \\ &= \Psi_g \circ \Psi_{g'}. \end{aligned}$$

□

3. The reduction theorem for contact pairs

Let $\Phi : G \times M^{2k+2h+2} \longrightarrow M^{2k+2h+2}$ be a restricted Hamiltonian action of a Lie group G on a contact pair manifold $(M^{2k+2h+2}, \eta, \alpha)$. Let J be a momentum map and Ψ the affine action of G on $\mathcal{G}^*$. For all $\mu \in \mathcal{G}^*$ we denote by $M_\mu = J^{-1}(\{\mu\})$,

$$G_\mu = \{g \in G \mid \Psi_g(\mu) = \mu\}$$

the isotropy group of μ with respect to the affine action Ψ and by $\mathcal{P}_\mu$, the orbit space M_μ/G_μ. We recall that G_μ is a closed sub-group of G, so according to Cartan theorem it is a Lie sub-group having for Lie algebra

$$\mathcal{G}_\mu = \{\xi \in \mathcal{G} \mid (\forall \xi' \in \mathcal{G})(d\eta + d\alpha)(\xi_M, \xi'_M) = 0\}.$$

It will always be tacitly assumed that both G and $M^{2k+2h+2}$ are connected. The Lie algebra of G will be denoted by $\mathcal{G}$ and its dual by $\mathcal{G}^*$. It will always be assumed that G_μ is connected such that the fibres of π_μ are also connected. To prove the contact pair reduction theorem, we need the following proposition.

Proposition 3.1. *Let (M, η, α) be a connected manifold, where η and α are non-closed 1-forms such that $d\eta$ and $d\alpha$ have constant rank. We assume that there exist two vector fields R and Z on M such that $\eta(R) = \alpha(Z) = 1$, $\eta(Z) = \alpha(R) = 0$, $i_R d\eta = i_R d\alpha = 0$ and $i_Z d\eta = i_Z d\alpha = 0$. if the map $\flat_{(\eta,\omega)} : \chi(M) \longrightarrow \Omega^1(M)$ defined by:*

$$\flat_{(\eta,\alpha)}(X) = \eta(X).\eta + \alpha(X).\alpha + i_X(d\eta + d\alpha), \forall X \in \chi(M).$$

is a smooth vector bundle isomorphism, then (M, η, α) is a contact pair.

Proof. Let x be an element of M, we put

$$\mathfrak{F}_x = \ker \eta_x \cap \ker(d\eta)_x \text{ and } \xi_x = \ker \alpha_x \cap \ker \eta_x \cap \ker(d\eta)_x.$$

Let us consider the map $\delta : \xi_x \longrightarrow \mathfrak{F}_x^*$ defined by

$$\delta(v) = i_v(d\alpha)_x,$$

the map δ is injective. Indeed, let v be an element of ξ_x. Assume that $\delta(v) = 0$, then we have $\flat_{(\eta,\alpha)}(v) = 0$, and since $\flat_{(\eta,\alpha)}$ is bijective, therefore $v = 0$. Hence, the map δ is injective. Consequently, $(d\alpha)_x$ is of maximum rank, and furthermore, we have

$$rank(\delta) = \dim(\xi_x) = \dim(Im(\delta))$$

. However, since $d\alpha$ has constant rank, $rank(\delta) = rank((d\alpha)_x) = 2h$, which means that

$$\dim(Im(\delta)) = \dim(\xi_x) = 2h.$$

Now let H be the vector subspace of $\mathfrak{F}_x$ spanned by Z_x. We denote its annihilator by H^0. Let us show that H^0 is equal to $Im(\delta)$. Let β be an element of H^0. Since the map $\flat_{(\eta,\alpha)} : \mathfrak{F}_x \longrightarrow \mathfrak{F}_x^*$ is bijective, there exists $v \in \mathfrak{F}_x$ such that:

$$\begin{aligned} \beta &= \flat_{(\eta,\alpha)}(v) \\ &= \eta_x(v)\eta_x + \alpha_x(v)\alpha_x + i_v(d\eta + d\alpha)_x \\ &= \alpha_x(v)\alpha_x + i_v(d\alpha)_x. \end{aligned} \tag{3.1}$$

However, since $\beta \in H^0$, then $\beta(Z_x) = 0$,Which means, according to relation (14), that $\alpha_x(v) = 0$. Consequently, $\beta = i_v(d\alpha)_x$, hence β belongs to $Im(\delta)$, thus

$$H^0 \subseteq Im(\delta). \tag{3.2}$$

Conversely, let β be an element of $Im(\delta)$. Then there exists v such that $\beta = \delta(v)$. We have

$$\begin{aligned} \beta(Z_x) &= \delta(v) \\ &= i_v(d\alpha)_x(Z_x) \\ &= 0, \end{aligned} \tag{3.3}$$

which means, according to relation (3.3), that β is an element of H^0, thus

$$Im(\delta) \subseteq H^0. \tag{3.4}$$

From relations (3.2) and (3.4), we can conclude that $H^0 = Im(\delta)$. According to Annihilator Theorem we have

$$\dim H + \dim H^0 = \dim \mathfrak{F}_x,$$

and since $\dim(H) = 1$ and $\dim H^0 = \dim(Im(\delta)) = 2h$ we obtain

$$\dim \mathfrak{F}_x = 2h + 1. \tag{3.5}$$

Now let us consider the spaces $\mathfrak{G}$ and ξ' defined by

$$\mathfrak{G}_x = \ker \alpha_x \cap \ker(d\alpha)_x \text{ and } \xi_x = \ker \eta_x \cap \ker \alpha_x \cap \ker(d\alpha)_x.$$

Let us consider the map $\delta' : \xi'_x \longrightarrow \mathfrak{G}^*_x$ defined by

$$\delta'(v) = i_v(d\eta)_x,$$

the map δ' is injective. Indeed, let v be an element of ξ'_x. Assume that $\delta'(v) = 0$, then we have $\flat_{(\eta,\alpha)}(v) = 0$, and since $\flat_{(\eta,\alpha)}$ is bijective, therefore $v = 0$. Hence, the map δ' is injective. Consequently, $(d\eta)_x$ is of maximum rank, and furthermore, we have

$$rank(\delta') = \dim(\xi'_x) = \dim(Im(\delta'))$$

. However, since $d\eta$ has constant rank, $rank(\delta) = rank((d\eta)_x) = 2k$, which means that

$$\dim(Im(\delta')) = \dim(\xi'_x) = 2k.$$

Now let K be the vector subspace of $\mathfrak{G}_x$ spanned by R_x. We denote its annihilator by K^0. Let us show that K^0 is equal to $Im(\delta')$. Let β be an element of K^0. Since the map $\flat_{(\eta,\alpha)} : \mathfrak{G}_x \longrightarrow \mathfrak{G}^*_x$ is bijective, there exists $v \in \mathfrak{G}_x$ such that:

$$\begin{aligned} \beta &= \flat_{(\eta,\alpha)}(v) \\ &= \eta_x(v)\eta_x + \alpha_x(v)\alpha_x + i_v(d\eta + d\alpha)_x \\ &= \eta_x(v)\eta_x + i_v(d\eta)_x. \end{aligned} \tag{3.6}$$

However, since $\beta \in K^0$, then $\beta(R_x) = 0$,Which means, according to relation (3.6), that $\eta_x(v) = 0$. Consequently, $\beta = i_v(d\eta)_x$, hence β belongs to $Im(\delta')$, thus

$$K^0 \subseteq Im(\delta'). \tag{3.7}$$

Conversely, let β be an element of $Im(\delta')$. Then there exists v such that $\beta = \delta'(v)$. We have

$$\begin{aligned} \beta(R_x) &= \delta'(v) \\ &= i_v(d\eta)_x(R_x) \\ &= 0, \end{aligned} \tag{3.8}$$

which means, according to relation (3.8), that β is an element of K^0, thus

$$Im(\delta') \subseteq K^0. \tag{3.9}$$

From relations (3.7) and (22), we can conclude that $K^0 = Im(\delta')$. According to Annihilator Theorem we have

$$\dim K + \dim K^0 = \dim \mathfrak{G}_x,$$

and since $\dim(K) = 1$ and $\dim K^0 = \dim(Im(\delta')) = 2k$ we obtain

$$\dim \mathfrak{G}_x = 2k + 1. \tag{3.10}$$

Relations (3.5) and (3.10) imply that M is $(2k + 2h + 2)$-dimensional, and moreover $\eta \wedge (d\eta)^k \wedge \alpha \wedge (d\alpha)^h$ is a volume forme on M. Hence, (η, α) is a contact pair on M. □

Now, we will define the notion of a weakly regular value of the momentum map J.

Definition 3.1. *$\mu \in \mathcal{G}^*$ is a weakly regular value if $M_\mu = J^{-1}\{\mu\}$ is a closed submanifold and for all $x \in M_\mu$, $T_x M_\mu = \ker d_x J$.*

From this definition, the following consequences follow

Consequence 3.1. *if μ is a weakly regular value (and thus, in particular, a regular value) of J, then for all $x \in M_\mu$,*

$$T_x M_\mu = \{v \in T_x M \mid (\forall \xi \in \mathcal{G})(\omega + d\eta)(v, \xi_M(x)) = 0\} \tag{3.11}$$

Proposition 3.2. *If μ is a weakly regular value (and thus, in particular, a regular value) of J, then the set M_μ is a submanifold of $M^{2k+2h+2}$ and Φ induces a smooth action of G_μ on M_μ.*

Proof. By definition, it is clear that M_μ is a submanifold of $M^{2k+2h+2}$. Let x be an element of M_μ. For every g belonging to G_μ, we have from Proposition2.6,

$$\begin{aligned} J_{\Phi_g(x)} &= \Psi_g(J_x) \\ &= \Psi_g(\mu) \\ &= \mu. \end{aligned}$$

Thus $\Phi_g(x)$ is an element of M_μ. Hence the set M_μ is invariant for the restriction of Φ to G_μ. Therefore, Φ induces a smooth action of G_μ on M_μ. □

Theorem 3.1. *Given a restricted Hamiltonian action of a Lie group G on a contact pair manifold $(M^{2k+2h+2}, \eta, \alpha)$ with momentum map J, let $\mu \in \mathcal{G}^*$ be a weakly regular value of J and assumed the induced action of G_μ on $M_\mu = J^{-1}\{\mu\}$ is simple. Then the quotient space $\mathcal{P}_\mu = M_\mu / G_\mu$ admits a contact pair structure (η_μ, α_μ) such that*

$$j_\mu^*(\eta) = \pi_\mu^*(\eta_\mu) \text{ and } j_\mu^*(\alpha) = \pi_\mu^*(\alpha_\mu)$$

with $j_\mu : M_\mu \longrightarrow M^{2k+2h+2}$ the inclusion map and $\pi_\mu : M_\mu \longrightarrow \mathcal{P}_\mu$ the canonical projection. Furthermore if $\varphi \in \mathcal{C}^\infty(M, \mathbb{R})$ is invariant under Φ such that $Z_\eta(\varphi) = Z_\alpha(\varphi) = 0$, then X_φ is tangent to $J^{-1}\{\mu\}$ and its restriction to $J^{-1}\{\mu\}$ projects onto the reduced contact- pair $(\mathcal{P}_\mu, \eta_\mu, \alpha_\mu)$. Moreover, its projection $(X_\varphi)_\mu$, is again a restricted Hamiltonian vector field, i.e.

$$(X_\varphi)_\mu = X_{\varphi_\mu},$$

for some φ_μ satisfying $\varphi_{|J^{-1}\{\mu\}} = \pi_\mu^* \varphi_\mu$.

Proof. Since the induced action of G_μ on M_μ is simple, then the orbit space $\mathcal{P}_\mu$ admits a smooth manifold structure such that the canonical projection

$$\pi_\mu : M_\mu \longrightarrow \mathcal{P}_\mu$$

is a surjective submersion. The fundamental vector fields of this action are the ξ_{M_μ} where $\xi \in \mathcal{G}_\mu$, which show that the pair (η, α) is basic. So, it exist an unique forms η_μ and α_μ on $\mathcal{P}_\mu$ such that

$$\eta_{|M_\mu} = \pi_\mu^* \eta_\mu \text{ and } \alpha_{|M_\mu} = \pi_\mu^* \alpha_\mu.$$

Now, we show that (η_μ, α_μ) is a contact pair on $\mathcal{P}_\mu$.
Observe that since

$$\pi_\mu^* d\eta_\mu = d\eta_{|M_\mu} \text{ and } \pi_\mu^* d\alpha_\mu = d\alpha_{|M_\mu},$$

and since the forms $\eta_{|M_\mu}$ and $\alpha_{|M_\mu}$ are non-closed, then the forms η_μ and α_μ are non-closed. Furthermore, let x be an element of M_μ, we have for all $\xi \in \mathcal{G}$,

$$\begin{aligned} T_x J(Z_{\eta x})(\xi) &= d_x J_\xi(Z_{\eta x}) \\ &= 0 \end{aligned}$$

and

$$\begin{aligned} T_x J(Z_{\alpha x})(\xi) &= d_x J_\xi(Z_{\alpha x}) \\ &= 0. \end{aligned}$$

This means that $Z_{\eta x}$ and $Z_{\alpha x}$ belong to the kernel of $T_x J$, and since μ is weakly regular, $Z_{\eta x}$ and $Z_{\alpha x}$ belong to $T_x M_\mu$. Hence, the Reeb vector fields Z_η and Z_α are adapted to M_μ. Also, for every ξ in $\mathcal{G}$, we have

$$\begin{aligned} \eta([\xi_M, Z_\eta]) &= L_{Z_\eta}(\eta(\xi_M)) - L_{\xi_M}(\eta(Z_\eta)) - d\eta(\xi_M, Z_\eta) \\ &= L_{Z_\eta}(0) - L_{\xi_M}(1) + i_{Z_\eta} d\eta(\xi_M) \\ &= 0, \end{aligned} \tag{3.12}$$

$$\begin{aligned} \alpha([\xi_M, Z_\eta]) &= L_{Z_\eta}(\alpha(\xi_M)) - L_{\xi_M}(\alpha(Z_\eta)) - d\alpha(\xi_M, Z_\eta) \\ &= L_{Z_\eta}(0) - L_{\xi_M}(0) + i_{Z_\eta} d\alpha(\xi_M) \\ &= 0, \end{aligned} \tag{3.13}$$

$$\begin{aligned} i_{[\xi_M, Z_\eta]} d\eta &= L_{\xi_M}(i_{Z_\eta} d\eta) - i_{Z_\eta}(L_{\xi_M} d\eta) \\ &= L_{\xi_M}(0) - i_{Z_\eta}(0) \\ &= 0 \end{aligned} \tag{3.14}$$

and

$$\begin{aligned} i_{[\xi_M, Z_\eta]} d\alpha &= L_{\xi_M}(i_{Z_\eta} d\alpha) - i_{Z_\eta}(L_{\xi_M} d\alpha) \\ &= L_{\xi_M}(0) - i_{Z_\eta}(0) \\ &= 0. \end{aligned} \tag{3.15}$$

From relations (3.12), (3.13), (3.14), and (3.15), we deduce that $\flat_{(\eta,\alpha)}([\xi_M, Z_\eta])$ is zero, and since $\flat_{(\eta,\alpha)}$ is an isomorphism, then $[\xi_M, Z_\eta] = 0$. Consequently, Z_η is invariant under G_μ. Likewise, we have

$$\begin{aligned} \eta([\xi_M, Z_\alpha]) &= L_{Z_\alpha}(\eta(\xi_M)) - L_{\xi_M}(\eta(Z_\alpha)) - d\eta(\xi_M, Z_\alpha) \\ &= L_{Z_\alpha}(0) - L_{\xi_M}(1) + i_{Z_\alpha} d\eta(\xi_M) \\ &= 0, \end{aligned} \tag{3.16}$$

$$\begin{aligned}\alpha([\xi_M, Z_\alpha]) &= L_{Z_\alpha}(\alpha(\xi_M)) - L_{\xi_M}(\alpha(Z_\alpha)) - d\alpha(\xi_M, Z_\alpha)\\ &= L_{Z_\alpha}(0) - L_{\xi_M}(0) + i_{Z_\alpha}d\alpha(\xi_M)\\ &= 0, \end{aligned} \tag{3.17}$$

$$\begin{aligned}i_{[\xi_M, Z_\alpha]}d\eta &= L_{\xi_M}(i_{Z_\alpha}d\eta) - i_{Z_\alpha}(L_{\xi_M}d\eta)\\ &= L_{\xi_M}(0) - i_{Z_\alpha}(0)\\ &= 0 \end{aligned} \tag{3.18}$$

and

$$\begin{aligned}i_{[\xi_M, Z_\alpha]}d\alpha &= L_{\xi_M}(i_{Z_\alpha}d\alpha) - i_{Z_\alpha}(L_{\xi_M}d\alpha)\\ &= L_{\xi_M}(0) - i_{Z_\alpha}(0)\\ &= 0. \end{aligned} \tag{3.19}$$

From relations (3.16), (3.17), (3.18), and (3.19),, we deduce that $\flat_{(\eta,\alpha)}([\xi_M, Z_\alpha])$ is zero, and since $\flat_{(\eta,\alpha)}$ is an isomorphism, then $[\xi_M, Z_\alpha] = 0$. Consequently, Z_α is invariant under G_μ.

Since the Reeb vector fields Z_η and Z_α are adapted to M_μ and invariant under G_μ, they are projectable to Z_{η_μ} and Z_{α_μ} on $\mathcal{P}_\mu$. The vector fields Z_{η_μ} and Z_{α_μ} satisfy the following relations:

$$\begin{aligned}\pi_\mu^*\eta_\mu(Z_\eta) &= \eta(Z_\eta)\\ &= 1. \end{aligned}$$

Now we know that,

$$\begin{aligned}\pi_\mu^*\eta_\mu(Z_\eta) &= \eta_\mu(\pi_{\mu*}Z_\eta)\\ &= \eta_\mu(Z_{\eta\mu}) \end{aligned}$$

Thus, we obtain

$$\eta_\mu(Z_{\eta\mu}) = 1. \tag{3.20}$$

$$\begin{aligned}\pi_\mu^*\eta_\mu(Z_\alpha) &= \eta(Z_\alpha)\\ &= 0. \end{aligned}$$

Now we know that,

$$\begin{aligned}\pi_\mu^*\eta_\mu(Z_\eta) &= \eta_\mu(\pi_{\mu*}Z_\alpha)\\ &= \eta_\mu(Z_{\alpha\mu}) \end{aligned}$$

Thus, we obtain

$$\eta_\mu(Z_{\alpha\mu}) = 0. \tag{3.21}$$

$$\begin{aligned}\pi_\mu^*\alpha_\mu(Z_\alpha) &= \alpha(Z_\alpha)\\ &= 1. \end{aligned}$$

Now we know that,

$$\begin{aligned}\pi_\mu^*\alpha_\mu(Z_\alpha) &= \alpha_\mu(\pi_{\mu*}Z_\alpha)\\ &= \alpha_\mu(Z_{\alpha\mu}) \end{aligned}$$

Thus, we obtain
$$\alpha_\mu(Z_{\alpha\mu}) = 1. \tag{3.22}$$
$$\begin{aligned} \pi_\mu^*\alpha_\mu(Z_\eta) &= \alpha(Z_\eta) \\ &= 0. \end{aligned}$$
Now we know that,
$$\begin{aligned} \pi_\mu^*\alpha_\mu(Z_\eta) &= \alpha_\mu(\pi_{\mu*}Z_\eta) \\ &= \alpha_\mu(Z_{\eta\mu}) \end{aligned}$$
Thus, we obtain
$$\alpha_\mu(Z_{\eta\mu}) = 0. \tag{3.23}$$
$$\begin{aligned} i_{Z_\eta}\pi_\mu^* d\eta_\mu &= i_{Z_\eta} d\eta \\ &= 0. \end{aligned}$$
Now we know that,
$$\begin{aligned} i_{Z_\eta}\pi_\mu^* d\eta_\mu &= d\eta_\mu(\pi_{\mu*}Z_\eta) \\ &= d\eta_\mu(Z_{\eta\mu}) \end{aligned}$$
Thus, we obtain
$$i_{Z_{\eta\mu}} d\eta_\mu = 0. \tag{3.24}$$
$$\begin{aligned} i_{Z_\alpha}\pi_\mu^* d\eta_\mu &= i_{Z_\alpha} d\eta \\ &= 0. \end{aligned}$$
Now we know that,
$$\begin{aligned} i_{Z_\alpha}\pi_\mu^* d\eta_\mu &= d\eta_\mu(\pi_{\mu*}Z_\alpha) \\ &= d\eta_\mu(Z_{\alpha\mu}) \end{aligned}$$
Thus, we obtain
$$i_{Z_{\alpha\mu}} d\eta_\mu = 0. \tag{3.25}$$
$$\begin{aligned} i_{Z_\eta}\pi_\mu^* d\alpha_\mu &= i_{Z_\eta} d\alpha \\ &= 0. \end{aligned}$$
Now we know that,
$$\begin{aligned} i_{Z_\eta}\pi_\mu^* d\alpha_\mu &= d\alpha_\mu(\pi_{\mu*}Z_\eta) \\ &= d\alpha_\mu(Z_{\eta\mu}) \end{aligned}$$
Thus, we obtain
$$i_{Z_{\eta\mu}} d\alpha_\mu = 0. \tag{3.26}$$
$$\begin{aligned} i_{Z_\alpha}\pi_\mu^* d\alpha_\mu &= i_{Z_\alpha} d\alpha \\ &= 0. \end{aligned}$$
Now we know that,
$$\begin{aligned} i_{Z_\alpha}\pi_\mu^* d\alpha_\mu &= d\alpha_\mu(\pi_{\mu*}Z_\alpha) \\ &= d\alpha_\mu(Z_{\alpha\mu}) \end{aligned}$$

Thus, we obtain
$$i_{Z_{\alpha\mu}}d\alpha_\mu = 0. \tag{3.27}$$
Let us consider the map $\flat_{(\eta_\mu,\alpha_\mu)} : T\mathcal{P}_\mu \longrightarrow T^*\mathcal{P}_\mu$ defined by
$$\eta_\mu(X_\mu)\eta_\mu + \alpha_\mu(X_\mu)\alpha_\mu + i_{X_\mu}(d\eta_\mu + d\alpha_\mu)$$
. Let v be an element of $T\mathcal{P}_\mu$. Suppose that $\flat_{(\eta_\mu,\alpha_\mu)}(v) = 0$. Then, according to the relations (3.20),(3.21),(3.22),(3.23),(3.24),(3.25),(3.26),(3.27) we obtain
$$\eta_\mu(v) = 0, \alpha_\mu(v) = 0 \text{ and } i_v(d\eta_\mu + d\alpha_\mu) = 0.$$
Since $v \in T\mathcal{P}_\mu$, then there exists $u \in TM_\mu$ such that $\pi_{\mu*}u = v$. Hense, we have
$$\begin{aligned} \alpha(u) &= \pi_\mu^*\alpha_\mu(u) \\ &= \alpha_\mu(\pi_{\mu*}u) \\ &= \alpha_\mu(v) \\ &= 0 \end{aligned} \tag{3.28}$$
$$\begin{aligned} \eta(u) &= \pi_\mu^*\eta_\mu(u) \\ &= \eta_\mu(\pi_{\mu*}u) \\ &= \eta_\mu(v) \\ &= 0 \end{aligned} \tag{3.29}$$
$$\begin{aligned} i_u(d\eta + d\alpha) &= i_u(d\pi_\mu^*\eta_\mu + d\pi_\mu^*\alpha_\mu) \\ &= i_u\pi_\mu^*(d\eta_\mu + d\alpha_\mu) \\ &= i_{\pi_{\mu*}u}(d\eta_\mu + d\alpha_\mu) \\ &= i_v(d\eta_\mu + d\alpha_\mu) \\ &= 0. \end{aligned} \tag{3.30}$$
Relations (3.28), (3.29) and (3.30) give $\flat_{(\eta,\alpha)}(u) = 0$, and since $\flat_{(\eta,\alpha)}$ is an isomorphism, then $u = 0$. This implies that $v = 0$. Consequently, $\flat_{(\eta_\mu,\alpha_\mu)}$ is an isomorphism.
To complete the proof, we now demonstrate that the two forms $d\eta_\mu$ and $d\alpha_\mu$ have constant rank. let $x \in \mathcal{P}_\mu$. We have
$$rank(d\eta_{\mu x}) = dim\mathcal{P}_\mu - dim\ker(d\eta_{\mu x}),$$
where
$$\ker(d\eta_{\mu x}) = \{v \in T_x\mathcal{P}_\mu : i_v d\eta_{\mu x} = 0\}.$$
Since π_μ is surjective there exists $p \in M_\mu$ such that $x = \pi_\mu(p)$. Let v be an element of $d_p\pi_\mu(\ker d\eta_{|M_{\mu p}})$. Then there exists $u \in T_pM_\mu$ such that $v = d_p\pi(u)(u)$ and $i_u d\eta_{|M_{\mu p}} = 0$, we have
$$\begin{aligned} i_v d\eta_{\mu x} &= i_{d_p\pi_\mu(u)}d\eta_{\mu x} \\ &= (\pi_\mu^* d\eta_\mu)_p(u,) \\ &= i_u d\eta_{|M_{\mu p}} \\ &= 0. \end{aligned} \tag{3.31}$$

The relation (3.31) implies that

$$d_p\pi_\mu(\ker d\eta_{|M_{\mu p}}) \subset \ker d\eta_{\mu x}. \tag{3.32}$$

Let v be an element of $\ker d\eta_{\mu x}$, then we have $i_v d\eta_{\mu x} = 0$. Since π_μ is a submersion then there exist $u \in T_pM_\mu$ such that $v = d_p\pi_\mu(u)$. So we have

$$\begin{aligned} i_{d_p\pi_\mu(u)}d\eta_{\mu x} &= 0 \\ (\pi_\mu^* d\eta_\mu)_p(u,) &= 0 \\ i_u d\eta_{|M_{\mu p}} &= 0. \end{aligned} \tag{3.33}$$

The relation (3.33) implies that

$$\ker d\eta_{\mu x} \subset d_p\pi_\mu(\ker d\eta_{|M_{\mu p}}) \tag{3.34}$$

According to the relation (3.32) and (3.34), we obtain

$$\ker d\eta_{\mu x} = d_p\pi_\mu(\ker d\eta_{|M_{\mu p}}). \tag{3.35}$$

We denote by $\mathfrak{F}_1 = \ker\alpha \cap \ker\eta \cap \ker d\eta$ and $\mathfrak{F}_2 = \ker\eta \cap \ker\alpha \cap \ker d\alpha$. It is clear that the restriction of $d\alpha$ to $\mathfrak{F}_1$ is symplectic, as is the restriction of $d\eta$ to $\mathfrak{F}_2$. We observe that T_pM decomposes uniquely as follows $T_pM = \mathfrak{F}_{1p} \oplus \mathfrak{F}_{2p} \oplus vect\{Z_{\eta p}, Z_{\alpha p}\}$. This decomposition induces the following decomposition

$$T_pM_\mu = \mathfrak{F}_{1p\mu} \oplus \mathfrak{F}_{2p\mu} \oplus \mathfrak{G}_0 \tag{3.36}$$

where

$$\begin{aligned} \mathfrak{F}_{1p\mu} &= \{v \in \mathfrak{F}_{1p} / \forall\xi \in \mathcal{G}, (d\eta + d\alpha)(v, \xi_{Mp}) = 0\} \\ &= \{v \in \mathfrak{F}_{1p} / \forall\xi \in \mathcal{G}, d\eta(v, \xi_{Mp}) = 0\} \\ &= (T_p\mathcal{O} \cap \mathfrak{F}_{1p})^\perp, \end{aligned} \tag{3.37}$$

where $(T_p\mathcal{O} \cap \mathfrak{F}_{1p})^\perp$ design orthogonal symplectic of $(T_p\mathcal{O} \cap \mathfrak{F}_{1p})^\perp$ compared with a restriction of 2-form $d\alpha_p$ on $\mathfrak{F}_{1p}$,

$$\begin{aligned} \mathfrak{F}_{2p\mu} &= \{v \in \mathfrak{F}_{2p} / \forall\xi \in \mathcal{G}, (d\eta + d\alpha)(v, \xi_{Mp}) = 0\} \\ &= \{v \in \mathfrak{F}_{2p} / \forall\xi \in \mathcal{G}, d\alpha(v, \xi_{Mp}) = 0\} \\ &= (T_p\mathcal{O} \cap \mathfrak{F}_{2p})^\perp, \end{aligned} \tag{3.38}$$

where $(T_p\mathcal{O} \cap \mathfrak{F}_{2p})^\perp$ design orthogonal symplectic of $(T_p\mathcal{O} \cap \mathfrak{F}_{2p})^\perp$ compared with a restriction of 2-form $d\eta_p$ on $\mathfrak{F}_{2p}$. And $\mathfrak{G}_0 = vect\{Z_{\eta p}, Z_{\alpha p}\}$. Now let v be an element of T_pM_μ, then there exists $v_1 \in \mathfrak{F}_{1p\mu}$, $v_2 \in \mathfrak{F}_{2p\mu}$ and $v_3 \in \mathfrak{G}_0$ such that $v = v_1 + v_2 + v_3$. We assume that v be an element of $\ker d\eta_{|M_{\mu p}}$, we obtain

$$\begin{aligned} i_v d\eta_{|M_{\mu p}} &= 0 \\ i_{v_1}d\eta_{|M_{\mu p}} + i_{v_2}d\eta_{|M_{\mu p}} + i_{v_3}d\eta_{|M_{\mu p}} &= 0 \\ i_{v_2}d\eta_{|M_{\mu p}} &= 0. \end{aligned} \tag{3.39}$$

Relation (3.37) implies that v be an element of $\mathfrak{F}_{1p\mu} \oplus \ker(d\eta_{|M_{\mu p}})_{\mathfrak{F}_{1p\mu}} \oplus \mathfrak{G}_0$, hence

$$\ker d\eta_{|M_{\mu p}} = \mathfrak{F}_{1p\mu} \oplus \ker(d\eta_{|M_{\mu p}})_{\mathfrak{F}_{1p\mu}} \oplus \mathfrak{G}_0.$$

Thus we obtain

$$\dim\ker d\eta_{|M_{\mu p}} = \dim \mathfrak{F}_{1p\mu} + \dim\ker(d\eta_{|M_{\mu p}})_{|\mathfrak{F}_{1p\mu}} + \dim \mathfrak{G}_0$$

Now the 2-form $(d\eta_{|M_{\mu p}})_{|\mathfrak{F}_{1p\mu}}$ is symplectic on $\mathfrak{F}_{1p\mu}$, so $\dim\ker(d\eta_{|M_{\mu p}})_{|\mathfrak{F}_{1p\mu}} = 0$. which implies, according to relation (3.38) that

$$\begin{aligned}
\dim\ker d\eta_{|M_{\mu p}} &= \dim \mathfrak{F}_{1p\mu} + 2 \\
&= \dim \mathfrak{F}_{1p\mu} + 2 \\
&= \dim \mathfrak{F}_{1p} - \dim T_p\mathcal{O} \cap \mathfrak{F}_{1p} + 2 \\
&= 2k + 2 - \dim T_p\mathcal{O} \cap \mathfrak{F}_{1p}.
\end{aligned}$$

And since the dimension of the intersection $\dim T_p\mathcal{O} \cap \mathfrak{F}_{1p}$ is independent of the point p, then $\dim\ker d\eta_{|M_{\mu p}}$ is constant. Hence, according to relation (3.35), the form $d\eta_\mu$ has constant rank. Likewise we have

$$\ker d\alpha_{\mu x} = d_p\pi_\mu(\ker d\alpha_{|M_{\mu p}}) \tag{3.40}$$

and

$$\dim\ker d\alpha_{|M_{\mu p}} = \dim \mathfrak{F}_{2p\mu} + \dim\ker(d\alpha_{|M_{\mu p}})_{|\mathfrak{F}_{2p\mu}} + \dim \mathfrak{G}_0$$

Now the 2-form $(d\alpha_{|M_{\mu p}})_{|\mathfrak{F}_{2p\mu}}$ is symplectic on $\mathfrak{F}_{2p\mu}$, so $\dim\ker(d\alpha_{|M_{\mu p}})_{|\mathfrak{F}_{2p\mu}} = 0$. which implies, according to relation (3.38) that

$$\begin{aligned}
\dim\ker d\alpha_{|M_{\mu p}} &= \dim \mathfrak{F}_{2p\mu} + 2 \\
&= \dim \mathfrak{F}_{2p\mu} + 2 \\
&= \dim \mathfrak{F}_{2p} - \dim T_p\mathcal{O} \cap \mathfrak{F}_{2p} + 2 \\
&= 2h + 2 - \dim T_p\mathcal{O} \cap \mathfrak{F}_{2p}.
\end{aligned}$$

And since the dimension of the intersection $\dim T_p\mathcal{O} \cap \mathfrak{F}_{2p}$ is independent of the point p, then $\dim\ker d\alpha_{|M_{\mu p}}$ is constant. Hence, according to relation (3.35), the form $d\alpha_\mu$ has constant rank. Therefore, according to the proposition3.1, the pair (η_μ, α_μ) is a contact pair on $\mathcal{P}_\mu$. Now, let φ be an Hamiltonian function invariant under Φ. Since φ is invariant under Φ, then according to theorem2.2 J is constant along the orbits of X_φ, which means that for all $x \in J^{-1}\{\mu\}$,

$$(X_\varphi)_x \in \ker T_x J.$$

Consequently X_φ is tangent to M_μ. Moreover, for every ξ in $\mathcal{G}$, we have

$$\begin{aligned}
\eta([\xi_M, X_\varphi]) &= L_{X_\varphi}(\eta(\xi_M)) - L_{\xi_M}(\eta(X_\varphi)) - d\eta(\xi_M, X_\varphi) \\
&= L_{X_\varphi}(0) - L_{\xi_M}(0) - i_{\xi_M} d\eta(X_\varphi) \\
&= -i_{\xi_M} d\eta(X_\varphi),
\end{aligned} \tag{3.41}$$

$$\begin{aligned}
\alpha([\xi_M, X_\varphi]) &= L_{X_\varphi}(\alpha(\xi_M)) - L_{\xi_M}(\alpha(X_\varphi)) - d\alpha(\xi_M, X_\varphi) \\
&= L_{X_\varphi}(0) - L_{\xi_M}(0) - i_{\xi_M} d\alpha(X_\varphi) \\
&= -i_{\xi_M} d\alpha(X_\varphi),
\end{aligned} \tag{3.42}$$

$$\begin{aligned} i_{[\xi_M, X_\varphi]} d\eta &= L_{\xi_M}(i_{X_\varphi} d\eta) - i_{X_\varphi}(L_{\xi_M} d\eta) \\ &= L_{\xi_M}(i_{X_\varphi} d\eta) - i_{X_\varphi}(0) \\ &= L_{\xi_M}(i_{X_\varphi} d\eta) \end{aligned} \tag{3.43}$$

and

$$\begin{aligned} i_{[\xi_M, X_\varphi]} d\alpha &= L_{\xi_M}(i_{X_\varphi} d\alpha) - i_{X_\varphi}(L_{\xi_M} d\alpha) \\ &= L_{\xi_M}(i_{X_\varphi} d\alpha) - i_{X_\varphi}(0) \\ &= L_{\xi_M}(i_{X_\varphi} d\alpha). \end{aligned} \tag{3.44}$$

From relations (3.41), (3.42), (3.43), and (3.44), we deduce that

$$\begin{aligned} \flat_{(\eta,\alpha)}([\xi_M, Z_\eta]) &= L_{\xi_M}(i_{X_\varphi}(d\eta + d\alpha)) - i_{\xi_M}(d\eta(X_\varphi) + d\alpha(X_\varphi)) \\ &= ddJ_\xi(X_\varphi) - i_{\xi_M} d \circ d\varphi + dJ_\xi(X_\varphi) \\ &= 0 \end{aligned} \tag{3.45}$$

, and since $\flat_{(\eta,\alpha)}$ is an isomorphism, then $[\xi_M, X_\varphi] = 0$. Consequently, X_φ is invariant under G_μ. This implies that the induced action of G_μ on M_μ commutes with the flow of $X_{\varphi|M_\mu}$. Therefore Hence, the latter induces canonically a flow on $\mathcal{P}_\mu$ and we denote its infinitesimal generator by $(X_\varphi)_\mu$. Let $\varphi_\mu \in \mathcal{C}^\infty(\mathcal{P}_\mu)$ be the unique function determined by $\varphi_{|M_\mu} = \pi_\mu^* \varphi_\mu$. We have

$$\begin{aligned} i_{X_{\varphi|M_\mu}}(j_\mu^* d\alpha + j_\mu^* d\eta) + j_\mu^* \eta(X_{\varphi|M_\mu}) j_\mu^* \eta + j_\mu^* \alpha(X_{\varphi|M_\mu}) j_\mu^* \alpha &= d\varphi_{|M_\mu} \\ i_{X_{\varphi|M_\mu}}(\pi_\mu^* d\alpha_\mu + \pi_\mu^* d\eta_\mu) + \pi_\mu^* \eta_\mu(X_{\varphi|M_\mu}) \pi_\mu^* \eta_\mu + \pi_\mu^* \alpha_\mu(X_{\varphi|M_\mu}) \pi_\mu^* \alpha_\mu &= d\varphi_{|M_\mu}. \end{aligned}$$

Since $\pi_\mu^*(X_\varphi)_\mu = X_{\varphi_{|M_\mu}}$ and $\varphi_{|M_\mu} = \pi_\mu^* \varphi_\mu$, then we obtain

$$i_{\pi_\mu^*(X_\varphi)_\mu}(\pi_\mu^* d\alpha_\mu + \pi_\mu^* d\eta_\mu) + \pi_\mu^* \eta_\mu(\pi_\mu^*(X_\varphi)_\mu) \pi_\mu^* \eta_\mu + \pi_\mu^* \alpha_\mu(\pi_\mu^*(X_\varphi)_\mu) \pi_\mu^* \alpha_\mu = \pi_\mu^* d\varphi_\mu.$$

Consequently we have

$$i_{(X_\varphi)_\mu}(d\alpha_\mu + d\eta_\mu) + \eta_\mu((X_\varphi)_\mu)\eta_\mu + \alpha_\mu((X_\varphi)_\mu)\alpha_\mu = d\varphi_\mu,$$

i.e

$$\begin{aligned} \flat_{(\eta_\mu,\alpha_\mu)}((X_\varphi)_\mu) &= d\varphi_\mu \\ \flat_{(\eta_\mu,\alpha_\mu)}((X_\varphi)_\mu) &= \flat_{(\eta_\mu,\alpha_\mu)}(X_{\varphi_\mu}). \end{aligned}$$

Now since $\flat_{(\eta_\mu,\alpha_\mu)}$ is an isomorphism then

$$(X_\varphi)_\mu = X_{\varphi_\mu}.$$

□

4. Some examples

Example 4.1. *Let $(M^{2k+2h+2}, \eta, \alpha)$ be a contact pair manifold and φ_t a complete flow of restricted Hamiltonian vector field X_H. φ_t is a restricted Hamiltonian $\mathbb{R}$-action of momentum map $H : M^{2k+2h+2} \longrightarrow \mathbb{R}$. The affine action associated to φ_t is defined by $\psi_t(a) = a$. Let a be a regular value of H. We know that $M_a = H^{-1}(a)$ is a sub-manifold of $M^{2k+2h+2}$, of codimension one and the isotropy group G_a is equal to $\mathbb{R}$. Since H is constant a long of integral curves of X_H, then the* *orbits of induced action of $\mathbb{R}$ on M_a are the integral curves of X_H. If the induced action of $\mathbb{R}$ on M_a is simple then the set $\mathcal{P}_a$ of curve integral of X_H on M_a admits a contact pair (η_a, α_a) such that*

$$j_a^*(\eta) = \pi_a^*(\eta_a) \text{ and } j_a^*(\alpha) = \pi_a^*(\alpha_a)$$

with $j_a : M_a \longrightarrow M^{2k+2h+2}$ the inclusion map and $\pi_a : M_a \longrightarrow \mathcal{P}_a$ the canonical projection. The reduced Hamiltonian $\widehat{H} : \mathcal{P}_a \longrightarrow \mathbb{R}$ is constant equal to a and the Hamiltonian vector field $X_{\widehat{H}}$ is nil identically.

Example 4.2. *We consider $\mathbb{R}^{2k+2h+2}$ equipped to standard contact pair structure $\eta = \Sigma_{i=1}^k x_i dy_i + dz_1$ $\alpha = \Sigma_{i=1}^h p_i dq_i + dz_2$. We take the Hamiltonian*

$$H = \frac{1}{2}(\Sigma_{i=1}^k (x_i^2 + y_i^2) + \Sigma_{i=1}^h (p_i^2 + q_i^2)).$$

The corresponding Hamiltonian vector field is

$$X_H = \Sigma_{i=1}^k (y_i \partial x^i - x_i \partial y_i + x_i^2 \partial z_1) + \Sigma_{i=1}^h (q_i \partial p_i - p_i \partial q_i + p_i^2 \partial z_2).$$

The Hamilton's equation are given by

$$\begin{cases} \dot{x}_i = y_i \\ \dot{y}_i = -x_i \\ \dot{p}_i = q_i \\ \dot{q}_i = -p_i \dot{z_1} = x_i^2 \dot{z_2} = p_i^2 \end{cases}$$

Their solution taking the value $x = (x_i^0, y_i^0, z_1, p_i^0, q_i^0, z_2)$ at $t = 0$ is

$$\varphi_t(x) = (ty_i + x_i^0, -tx_i + y_i^0, tx_i^2, tq_i + p_i^0, -tp_i + q_i^0, tp_i^2).$$

Observe that 0 is an unique critical value of H. Let μ be a number reel.

- *If $\mu < 0$, $H^{-1}(\mu) = \emptyset$.*
- *If $\mu > 0$, $H^{-1}(\mu)$ is the fibre in sphere $\sqrt{2\mu}$ radius $\mathbb{S}_{2\mu}^{2(k+h)-1} \times \mathbb{R}^2$ over $\mathbb{R}$. The integral curves of X_H situated on the fibre $\mathbb{S}_{2\mu}^{2(k+h)-1} \times \mathbb{R}^2$ are a super circle, orbit of $\mathbb{S}^1$-action on $\mathbb{S}_{2\mu}^{2(k+h)-1}$.*

  The reduced contact pair manifold is the fibre $P^{2k+2h}\mathbb{R} \times \mathbb{R}^2$. The restriction $\pi : \mathbb{S}^{2(k+h)-1} \longrightarrow P^{2k+2h}\mathbb{R}^2$ of the canonical projection π_μ is called the Hopf's fibration. In homogeneous coordinates the induced contact-symplectic structure on the reduced manifold $P^{2k+2h}\mathbb{R} \times \mathbb{R}^2$ is

  $$\eta_\mu = \mu \frac{\Sigma_{i=1}^h x_i dy_i}{H} + dz_1 \text{ and } \alpha_\mu = \mu \frac{\Sigma_{i=1}^h p_i dq_i}{H} + dz_2$$

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