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SOLVING THE $\partial\overline{\partial}$ FOR EXTENDABLE CURRENTS WITHOUT VANISHING THE BOUNDARY COHOMOLOGY GROUP

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ABSTRACT. In this paper, we consider the problem of solving the $\partial\overline{\partial}$ equation with prescribed support for a forms in a relatively compact domain Ω of a complex manifold X. As a consequence, we have the solution of the $\partial\overline{\partial}$ equation for extendable currents without the vanishing assumption of the De Rham cohomology group of the boundary.

1. INTRODUCTION AND MOTIVATIONS

In this paper, we are interested in the $\partial\overline{\partial}$ problem with prescribed support for differential forms and currents. The problem is as follows, let X be an complex manifold and $\Omega \subset X$ a relatively compact domain such that $X \setminus \Omega$ is connected. Then, under certain assumptions on the vanishing of the De Rham and the Dolbeault cohomology groups with prescribed support, we have

$$H^{1,1}_{BC,\overline{\Omega}}(X) = 0.$$

Our first result is the generalization of [5, Theorem 1.2] to the case of a Stein manifold, under the vanishing assumption of the second De Rham cohomology group with compact support (see Theorem 3.1). Our second improvement (see Theorem 3.2) is a more general result in the case of any complex manifold X with the vanishing assumptions of the following De Rham and Dolbeault cohomology groups

$$H^2_c(X) = H^{0,1}_c(X) = 0.$$

The corollary 3.1 provides a solution to the $\partial\overline{\partial}$-problem for a $(1,1)$-forms with prescribed support in a completely pseudoconvex smooth-boundary domain. And, by duality, we have the solution of $\partial\overline{\partial}$ for a $(n-1, n-1)$-extendable currents without the vanishing hypothesis on the De Rham cohomology group of the boundary see corollary 3.2.
In the last section, Theorem 4.1 gives an extension of the previous results to the (p,q)-forms, and by duality, we get the solution of the $\partial\overline{\partial}$-problem for the $(n-p, n-q)$-extendable currents (see Corollary 4.1.).

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In general, the vanishing assumption of the De Rham cohomology group on the boundary is a necessary condition to get a solution to the $\partial\overline{\partial}$-problem for extendable currents with exact support, as mentioned in [3, 4]. In this paper, we remove such ambiguity.

2. Notations

We fix the following notations: Let X be a complex manifold

- $\mathcal{E}^k(X) :=$ the space of smooth k-forms defined on X;
- $\mathcal{D}^k(X) :=$ the space of smooth k-forms with compact support on X;
- $\mathcal{D}'^k(X) :=$ the space of k-current on X.

For every $k \in \{0, 1, \dots, 2n\}$, we have

$$\mathcal{E}^k(X) = \oplus_{p+q=k}\mathcal{E}^{p,q}(X)$$

and the complex structure of X induices a splitting

$$d = \partial + \overline{\partial}.$$

For every $p, q \in \{1, \dots, n\}$, we define the Bott-Chern cohomology group of smooth (p, q)-forms on X as

$$H^{p,q}_{BC}(X) = \frac{\left(\ker\partial : \mathcal{E}^{p,q}(X) \to \mathcal{E}^{p+1,q}(X)\right) \cap \left(\ker\overline{\partial} : \mathcal{E}^{p,q}(X) \to \mathcal{E}^{p,q+1}(X)\right)}{Im\partial\overline{\partial} : \mathcal{E}^{p-1,q-1}(X) \to \mathcal{E}^{p,q}(X).}$$

$H^{p,q}_{BC,\overline{\Omega}}(X)$ is the Bott-Chern cohomology group of smooth (p, q)-forms with support in $\overline{\Omega}$.

We define the Aeppli cohomology group of smooth (p, q)-forms on X as

$$H^{p,q}_{A}(X) = \frac{\ker(\partial\overline{\partial}) : \mathcal{E}^{p,q}(X) \to \mathcal{E}^{p+1,q+1}(X)}{Im\partial : \mathcal{E}^{p-1,q}(X) \to \mathcal{E}^{p,q}(X) + Im\overline{\partial} : \mathcal{E}^{p,q-1}(X) \to \mathcal{E}^{p,q}(X).}$$

Case of either $p = 0$ or $q = 0$. For example, if $q = 0$, then the $(p, 0)$ Bott-Chern cohomology group is given, from definition, by

$$H^{p,0}_{BC}(X) = \{f \in \Gamma(X, \Omega^p_X), \mid \partial f := 0\},$$

where Ω^p_X is the sheaf of holomorphic p-forms on X. Thanks to the symmetric property of Bott-Chern cohomology group, we have

$$H^{0,p}_{BC}(X) = \overline{H^{p,0}_{BC}(X)}.$$

For $p = q = 0$, we have

$$H^{0,0}_{BC}(X) = \{f \in \Gamma(X, \Omega^0_X) \mid df = 0\}.$$

So $H^{0,0}_{BC}(X) = \mathbb{C}^d$, where d is the number of connected component of X.

$H^{0,0}_{A}(X)$ is the set of pluriharmonic functions on X.

The Bott-Chern cohomology group of (p, q)-forms with compact support is defined as

$$H^{p,q}_{BC,c}(X) = \frac{\left(\ker\partial : \mathcal{D}^{p,q}(X) \to \mathcal{D}^{p+1,q}(X)\right) \cap \left(\ker\overline{\partial} : \mathcal{D}^{p,q}(X) \to \mathcal{D}^{p,q+1}(X)\right)}{Im\partial\overline{\partial} : \mathcal{D}^{p-1,q-1}(X) \to \mathcal{D}^{p,q}(X).}$$

3. Solving the $\partial\overline{\partial}$ for extendable currents

The following result generalizes [5, Theorem 1.2],

Theorem 3.1.

Let X be a Stein manifold of complex dimension $n \geq 2$ such that

$$H_c^2(X) = 0.$$

Let $\Omega \subset X$ be a relatively compact domain such that $X \setminus \Omega$ is connected. Then

$$H^{1,1}_{BC,\overline{\Omega}}(X) = 0.$$

Proof.

Let f be a $(1,1)$-form of class C^∞, d-closed with support on $\overline{\Omega}$. Since $H_c^2(X) = 0$, there exists a 1-form g of class C^∞ with compact support such that $dg = f$. Without loss of generality, we can decompose g into

$$g = g_1 + g_2$$

where g_1 and g_2 are respectively a $(0,1)$-form of class C^∞ with compact support and a $(1,0)$-form of class C^∞ with compact support. We have

$$dg = dg_1 + dg_2 = f.$$

Since $d = \partial + \overline{\partial}$ and for bidegree reasons we have $\partial g_2 = 0$ and $\overline{\partial} g_1 = 0$. Since X is Stein, there exist functions w_1 and w_2 of class C^∞ with compact support such that

$$g_1 = \overline{\partial} w_1 \text{ and } g_2 = \partial w_2.$$

So

$$f = \partial g_1 + \overline{\partial} g_2 = \partial\overline{\partial} w_1 + \overline{\partial}\partial w_2 = \partial\overline{\partial}(w_1 - w_2).$$

Let $v = w_1 - w_2$, then v is compactly supported and $\partial\overline{\partial} v = 0$ on $X \setminus \overline{\Omega}$. According to [5, theorem 1.1], there exists a pluriharmonic function $\tilde{v}$ defined on X such that $\tilde{v}_{X\setminus\overline{\Omega}} = v$. We set

$$\check{v} = v - \tilde{v}.$$

Then we have $supp(\check{v}) \subset \overline{\Omega}$ and $\partial\overline{\partial}\check{v} = f$. So

$$H^{1,1}_{BC,\overline{\Omega}}(X) = 0.$$

□

More generally, we have the following result

Theorem 3.2.

Let X be a complex manifold of complex dimension $n \geq 2$ such that

$$H_c^2(X) = H_c^{0,1}(X) = 0,$$

and $\Omega \subset X$ be a relatively compact domain such that $X \setminus \Omega$ is connected.
Then,

$$H^{1,1}_{BC,\overline{\Omega}}(X) = 0.$$

Proof.

Let f be a $(1,1)$-form of class C^∞, d-closed with support on $\overline{\Omega}$. Since $H^2_c(X) = 0$, there exists a 1-form g of class C^∞ with compact support such that $dg = f$. Without loss of generality, we can decompose g into

$$g = g_1 + g_2$$

where g_1 and g_2 are respectively a $(0,1)$-form of class C^∞ with compact support and a $(1,0)$-form of class C^∞ with compact support. We have

$$dg = dg_1 + dg_2 = f.$$

Since $d = \partial + \overline{\partial}$ and for bidegree reasons we have $\partial g_2 = 0$ and $\overline{\partial} g_1 = 0$. Since

$$H^{0,1}_c(X) = 0,$$

there exist functions w_1 and w_2 of class C^∞ with compact support such that $g_1 = \overline{\partial} w_1$ and $g_2 = \partial w_2$.

$$f = \partial g_1 + \overline{\partial} g_2 = \partial\overline{\partial} w_1 + \overline{\partial}\partial w_2 = \partial\overline{\partial}(w_1 - w_2).$$

Let $v = w_1 - w_2$, v is compactly supported and $\partial\overline{\partial} v = 0$ on $X \setminus \overline{\Omega}$. Then v is a pluriharmonic function on $X \setminus \overline{\Omega}$ and which is vanishing on $X \setminus suppv$. Suppose that $\overline{\Omega}$ is strictly included in $suppv$. So $v = 0$ on $X \setminus \overline{\Omega}$ which is absurd. So $suppv \subset \overline{\Omega}$. This implies that

$$H^{1,1}_{BC,\overline{\Omega}}(X) = 0.$$

□

Corollary 3.1.

Let X be a complex manifold of complex dimension $n \geq 2$ and $\Omega \subset X$ a completely strictly pseudoconvex domain with smooth boundary. Then

$$H^{1,1}_{BC,\overline{\Omega}}(X) = 0.$$

Proof.

Since Ω has a smooth boundary, according to [2, Theorem 2.3], there exists a completely strictly pseudoconvex domain Ω' such that $\Omega \subset \Omega' \subset\subset X$ and that $H^r(\Omega') = 0$ for $1 \leq r \leq 2$. By Poincaré duality

$$H^{n-r}_c(\Omega') = 0 \Longleftrightarrow H^j_c(\Omega') = 0$$

for $0 \leq j \leq n-2$. In particular $H^2_c(\Omega') = 0$ and Ω' is also a Stein domain. According to theorem 3.1,

$$H^{1,1}_{BC,\overline{\Omega}}(\Omega') = H^{1,1}_{BC,\overline{\Omega}}(X) = 0.$$

□

Remark 3.1. *Corollary 3.1 shows that if f is a (1.1)-form of class C^∞ with support in $\overline{\Omega}$ and d-closed, then there exists a function u of class C^∞ with support in $\overline{\Omega}$ such that $\partial\overline{\partial} u = f$.*

By duality, we have the following result.

Corollary 3.2.
Under the assumptions of the corollary 3.1, if T is a $(n-1, n-1)$-extendable current, d-closed and defined on Ω, then there exists a $(n-2, n-2)$-extendable current S such that $\partial\overline{\partial}S = T$.

Proof.
We have $\check{D}^{n-1,n-1}(\Omega) = (D^{1,1}(\overline{\Omega}))'$. Let us show that

$$\check{D}^{n-1,n-1}(\Omega) \cap \ker d = \partial\overline{\partial}\check{D}^{n-2,n-2}(\Omega).$$

Let $T \in \check{D}^{n-1,n-1}(\Omega) \cap \ker d$. We define

$$\begin{aligned} L_T : \partial\overline{\partial}D^{1,1}(\overline{\Omega}) &\longrightarrow \mathbb{C} \\ \partial\overline{\partial}\varphi &\longmapsto \langle T, \varphi\rangle, \end{aligned}$$

L_T is well defined because $\partial\overline{\partial}D^{n-2,n-2}(\overline{\Omega}) = D^{n-1,n-1}(\overline{\Omega}) \cap \ker d$ (cf corollaire 3.1).
If φ and φ' are two $(1,1)$-forms such that

$$\partial\overline{\partial}\varphi = \partial\overline{\partial}\varphi',$$

then

$$\varphi - \varphi' = \partial\overline{\partial}\theta$$

et

$$\langle T, \partial\overline{\partial}\theta\rangle = \lim_{j\to+\infty}\langle T, \partial\overline{\partial}\theta_j\rangle = 0 \Rightarrow \langle T, \varphi\rangle = \langle T, \varphi'\rangle$$

where $(\theta_j)_{j\in\mathbb{N}}$ is a sequence of elements of $D^{2,2}(\Omega)$ which converge uniformly towards θ.
L_T is linear because

$$\partial\overline{\partial} : D^{2,2}(\overline{\Omega}) \longrightarrow \partial\overline{\partial}D^{2,2}(\overline{\Omega})$$

is a linear and continuous map because the inverse image of an open set U of $\mathbb{C}$ by L_T is an open set. So, we have $L_T \circ \partial\overline{\partial} = T$ since

$$L_T^{-1}(U) = \partial\overline{\partial} \circ T^{-1}(U).$$

According to the Hahn-Banach theorem, we can extend L_T into a linear and continuous operator

$$\tilde{L}_T : D^{1,1}(\overline{\Omega}) \longrightarrow \mathbb{C},$$

it is an extendable current and

$$\partial\overline{\partial}\tilde{L}_T\varphi = T,$$

then

$$\langle\partial\overline{\partial}\tilde{L}_T, \varphi\rangle = \langle T, \varphi\rangle.$$

Set $S = \tilde{L}_T$, S is an extendable current and verifies the relationship $\partial\overline{\partial}S = T$.

□

Remark 3.2. *The corollary 3.2 is an improvement of* [1, Theorem 1.1] *without the assumption $H^j(b\Omega) = 0$ for all $1 \le j < 2n-1$.*

4. The case of differential (p,q)-forms and extendable currents

In this section we generalize these results to the (p,q)-forms.Then we have:

Theorem 4.1.
Let X be a complex manifold of complex dimension n and $\Omega \subset X$ a smooth boundary domain and $1 \leq p,q \leq n$ such that

$$H^{p+q}_{c,d}(X) = H^{p-1,q}_{c,\partial}(X) = H^{p,q-1}_{c,\overline{\partial}}(X) = 0.$$

If the natural mapp

$$H^{p-1,q-1}_{A}(X) \quad\longrightarrow\quad H^{p-1,q-1}_{A}(X\backslash\Omega)$$

is an isomorphism then

$$H^{p,q}_{BC,\overline{\Omega}}(X) = 0.$$

Proof.
Let $[f] \in H^{p,q}_{BC,\overline{\Omega}}(X) \Longrightarrow f$ is a $(p+q)$-form d-closed with compact support. Since

$$H^{p+q}_{c,d}(X) = 0,$$

then there exists a $(p+q-1)$-form g with compact support such that $dg = f$. Without loss of generality, we can decompose g into

$$g = g_1 + g_2$$

where g_1 is a $(p-1,q)$-form with compact support and g_2 is a $(p,q-1)$-form with compact support. The saturation of the bidegree holds $\overline{\partial}g_1 = 0$ and $\partial g_2 = 0$. Since

$$H^{p,q-1}_{c,\overline{\partial}}(X) = 0,$$

so there are two forms h_1 and h_2 with compact support such that

$$g_1 = \overline{\partial}h_1 \text{ and } g_2 = \partial h_2.$$

Hence

$$\begin{aligned} f &= \partial g_1 + \overline{\partial} g_2 \\ &= \partial\overline{\partial}h_1 + \overline{\partial}\partial h_2 \\ &= \partial\overline{\partial}(h_1 - h_2). \end{aligned}$$

The solution $h_1 - h_2$ we get has compact support. We end the proof by correcting this solution to new one with exact support.

- If $p = q = 1$, then $h_1 - h_2$ is a pluriharmonic function defined in $X \setminus \overline{\Omega}$. Under the theorem's assumptions, we have a natural isomorphism map

$$H^{0,0}_{A}(X) \quad\longrightarrow\quad H^{0,0}_{A}(X\backslash\Omega).$$

Then, there is u a pluriharmonic function on X such that

$$[u] = [(h_1 - h_2)_{|X\backslash\overline{\Omega}}] \Longrightarrow u_{|X\backslash\overline{\Omega}} = (h_1 - h_2)_{|X\backslash\overline{\Omega}}.$$

Set

$$\widetilde{h_1 - h_2} = h_1 - h_2 - u$$

which is compactly supported on Ω and

$$\partial\bar{\partial}(\widetilde{h_1 - h_2}) = f \text{ on } \Omega.$$

- If $p, q \geq 2$, $h_1 - h_2$ is a $(p-1, q-1)$ form with compact support.

We have

$$\partial\bar{\partial}(h_1 - h_2) = 0 \text{ on } X \backslash \overline{\Omega}.$$

Under the theorem's assumptions, we have a natural isomorphism map

$$H_A^{p-1,q-1}(X) \quad \longrightarrow \quad H_A^{p-1,q-1}(X \backslash \Omega).$$

Then, there is u a $(p-1, q-1)$ pluriharmonic form on X such that

$$[u] = [(h_1 - h_2)_{|X\backslash\overline{\Omega}}] \Longrightarrow u_{|X\backslash\overline{\Omega}} = (h_1 - h_2)_{|X\backslash\overline{\Omega}} + \partial\bar{\partial}v,$$

where v is a $(p-2, q-2)$ form defined in $X \setminus \overline{\Omega}$. Hence

$$(h_1 - h_2)_{|X\backslash\overline{\Omega}} = u - \partial\bar{\partial}\tilde{v}$$

where $\tilde{v}$ is an extension of v. Set

$$\widetilde{h_1 - h_2} = h_1 - h_2 - u + \partial\tilde{\bar{\partial}}\tilde{v}$$

which is compactly supported on Ω and

$$\partial\bar{\partial}(\widetilde{h_1 - h_2}) = f \text{ on } \Omega.$$

□

As a consequence, we have:

Corollary 4.1. *Let X be a complex manifold of complex dimension $n \geq 2$ and $\Omega \subset X$ a completely pseudoconvex domain with smooth boundary. If T is a $(n-p, n-q)$-extendable current, d-closed and defined on Ω, then there exists a $(n-p-1, n-q-1)$-extendable current S such that $\partial\bar{\partial}S = T$.*

Proof. The proof is identical to that of the corollary 3.2. □

5. Conclusion

In this paper, we solve the $\partial\bar{\partial}$-problem by a direct method based on Bott-Chern cohomology. What is original in this case is that, unlike in papers [1],[2],[3] and [4], we did not use the null hypothesis of the boundary cohomology group of the domain, which was a necessary condition for the solution with prescribed support.

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