Global Journal of Advanced Research
on Classical and Modern Geometries
ISSN: 2284-5569, Vol. 15, (2026), Issue 1, pp.9-25

A NEW APPROACH OF THE FERMAT-TORRICELLI CONFIGURATION (II)

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ABSTRACT. In this paper we continue the study on the *Fermat-Torricelli configuration* started in [2]. Given a triangle ABC, instead of the six equilateral triangles constructed (inside or outside) on its sides, we consider three equilateral triangles AA_1A_2, BB_1B_2, CC_1C_2 with the same orientation as the given triangle and such that A_1, $A_2 \in BC$, B_1, $B_2 \in CA$, C_1, $C_2 \in AB$ (Fig. 1). Two equilateral triangles $V_+^aV_+^bV_+^c$ and $V_-^aV_-^bV_-^c$ are introduced - *1st and 2nd vertex triangles* -, and their properties are studied (Fig. 3 and 4). Their close connection with Napoleon triangles is highlighted.

1. NOTATIONS AND PRELIMINARIES

Consider an arbitrary triangle ABC. Let A_+, A_- be the apexes of the equilateral triangles built on the side BC outside and inside, respectively; similarly, points B_+, B_-, C_+, C_- are defined. The *classic Fermat-Torricelli configuration* starts with points A, B, C, A_+, A_-, B_+, B_-, C_+, C_-. The circumcircles of equilateral triangles A_+BC and A_-BC are denoted by $\mathcal{T}_+^a$ and $\mathcal{T}_-^a$ –*Torricelli circles* relative to side BC –, and their centers by N_+^a, N_-^a, respectively; similarly, we define the Torricelli circles $\mathcal{T}_+^b, \mathcal{T}_-^b, \mathcal{T}_+^c, \mathcal{T}_-^c$ and the points N_+^b, N_-^b, N_+^c, N_-^c. Triangles $N_+^aN_+^bN_+^c$ and $N_-^aN_-^bN_-^c$ are called the *first and second Napoleon triangles* of the given triangle, respectively; their circumcircles are denoted by $\mathcal{N}_+$ and $\mathcal{N}_-$. On this subject there is a vast specialized literature - treatises, textbooks and papers: [1], [6], [5], [9], [7] etc. In [2], new properties of this configuration are highlighted starting from another six-point system: $A, B, C, A_1, A_2, B_1, B_2, C_1, C_2$, where $A_1, A_2 \in BC$ so that triangle AA_1A_2 is equilateral and with the same orientation as the given triangle, and $B_1, B_2; C_1, C_2$ are defined similarly (Fig. 1).
Hereafter, the circumcircle of a triangle XYZ will be denoted (XYZ). Also, we denote $\mathcal{C} = (ABC)$.
In the new approach to the Fermat-Torricelli configuration, in addition to the starting triangles AA_1A_2, BB_1B_2, CC_1C_2, an important role is played by the triangles AB_2C_1, BC_2A_1, CA_2B_1; AB_1C_2, BC_1A_2, CA_1B_2. Their circumcenters are denoted by V_+^a, V_+^b, V_+^c; V_-^a, V_-^b, V_-^c, respectively. Triangles $V_+^aV_+^bV_+^c$ and $V_-^aV_-^bV_-^c$ will be called the *1st vertex triangle* and respectively *2nd vertex triangle* associated with the given one (Fig 2). We denote $\mathcal{V}_+ = \left(V_+^aV_+^bV_+^c\right)$, $\mathcal{V}_- = \left(V_-^aV_-^bV_-^c\right)$. The purpose of this note is to highlight the properties of these triangles and their close connection with Napoleon triangles.

2010 *Mathematics Subject Classification.* 51M05.
Key words and phrases. Fermat-Torricelli configuration; Torricelli circles; Napoleon triangles; Euler line; nine-point center; 1st and 2nd vertex triangles; similarity; perspectivity; orthology.

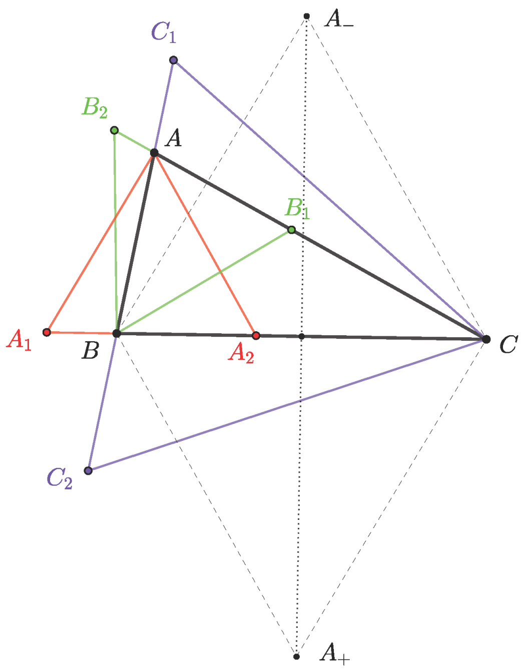

Figure 1. Triangles $AA_1A_2, BB_1B_2, CC_1C_2$

We recall some simple properties of the Fermat-Torricelli configuration established in [2] that will be useful in what follows. They can be easily proven by the reader.
(1) each of the triangles AB_2C_1, BC_2A_1, CA_2B_1; AB_1C_2, BC_1A_2, CA_1B_2 is *inversely similar* to the triangle ABC, and two of them are *directly similar* ([2], Proposition 4.1);
(2) the points A, H, V_+^a, V_-^a are collinear ([2], Corollary 4.1);
(3) the lines B_2C_1 and B_1C_2 are *parallel*, and each of them is *antiparallel* to BC.
Properties similar to (2) and (3) are also true for the sides CA and AB.
The position of points A_1, A_2, B_1, B_2, C_1, C_2, on the sides depends on the shape of triangle ABC. Thus, to prove a statement it is often necessary to consider several cases regarding the given triangle. Since the arguments used for one case can be adapted to the other cases, we will limit ourselves to proving only one, the one that appears in the associated figure.

2. Triangles $V_+^aV_+^bV_+^c$ and $V_-^aV_-^bV_-^c$ vs. Napoleon triangles

We begin with some elementary properties.

Proposition 2.1. *We have:*
(i) the lines V_+^aH, N_+^aO, $V_+^bN_+^c$, $V_+^cN_+^b$ are perpendicular to BC;
(ii) $V_+^aH = N_+^aO = V_+^bN_+^c = V_+^cN_+^b$.
Relative to the sides CA and AB similar properties occur.

Proof. (i) According to property (2) above, point V_+^a is on the altitude corresponding to side BC, therefore $V_+^aH \perp BC$. Point N_+^a is on the perpendicular bisector of BC, so $N_+^aO \perp BC$. The segments $V_+^bN_+^c$ and A_1B are the center line and the common chord of the circles $\mathcal{T}_+^c$ and the

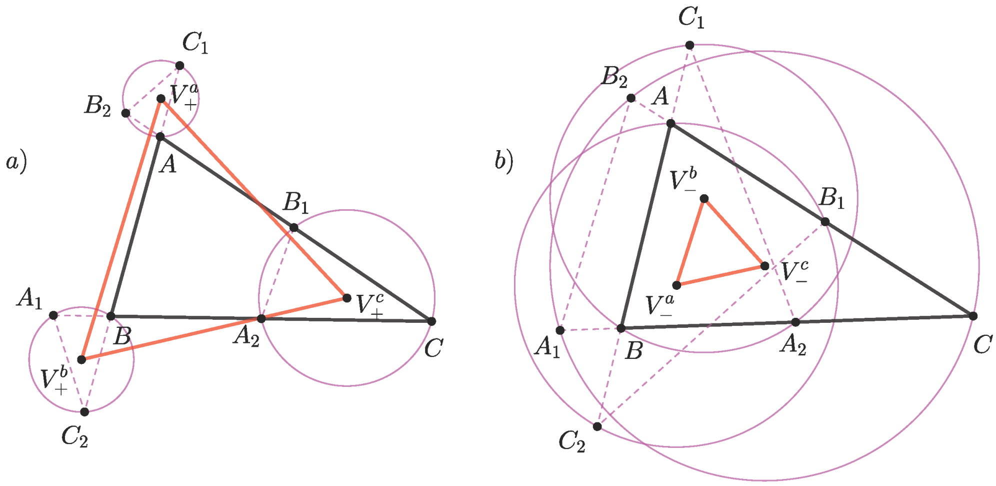

Figure 2. Triangles $V_+^a V_+^b V_+^c$ and $V_-^a V_-^b V_-^c$

circumcircle of triangle BC_2A_1, respectively; therefore, $V_+^b N_+^c \perp BC$. Finally, using $\mathcal{T}_+^b$ and the circumcircle of triangle CA_2B_1, with the same argument we conclude that $V_+^c N_+^b \perp BC$.

(ii) We consider only the case of acute-angled triangles that verify the condition $A > B > \frac{\pi}{3} > C$ (Fig. 3). We will show that the common length of the segments in (ii) is equal to $\frac{2\sqrt{3}R}{3}\sin\left(A+\frac{\pi}{3}\right)$, and from this it follows that statement (ii) is true.

Now, we have:

$$V_+^a H = V_+^a A + AH = V_+^a A + 2R\cos A,$$

and $V_+^a A$ is the circumradius of triangle AB_2C_1. From $AB_2C_1 \sim ABC$ (property (1) above),

$$V_+^a A = R\frac{AB_2}{AB} = R\frac{\sin\left(A-\frac{\pi}{3}\right)}{\sin\frac{\pi}{3}} = \frac{2\sqrt{3}R}{3}\sin\left(A-\frac{\pi}{3}\right)$$

(the sine formula was used in triangle ABB_2). Then,

$$V_+^a H = \frac{2\sqrt{3}R}{3}\sin\left(A-\frac{\pi}{3}\right) + 2R\cos A = \frac{2\sqrt{3}R}{3}\sin\left(A+\frac{\pi}{3}\right).$$

In triangle $BV_+^b N_+^c$, we have:

$$\begin{aligned}
V_+^b N_+^c &= V_+^b B\cos\widehat{BV_+^b N_+^c} + N_+^c B\cos\widehat{BN_+^c V_+^b} \\
&= V_+^b B\cos\frac{\widehat{BV_+^b A_1}}{2} + N_+^c B\cos\frac{\widehat{BN_+^c A_1}}{2} \\
&= V_+^b B\cos\widehat{BC_2A_1} + N_+^c B\cos\widehat{BAA_1} = V_+^b B\cos C + N_+^c B\cos\left(B-\frac{\pi}{3}\right) \\
&= R\frac{A_1B}{c}\cos C + \frac{c}{\sqrt{3}}\cos\left(B-\frac{\pi}{3}\right) \\
&= R\frac{2}{\sqrt{3}}\sin\left(B-\frac{\pi}{3}\right)\cos C + \frac{c}{\sqrt{3}}\cos\left(B-\frac{\pi}{3}\right) = \ldots
\end{aligned}$$

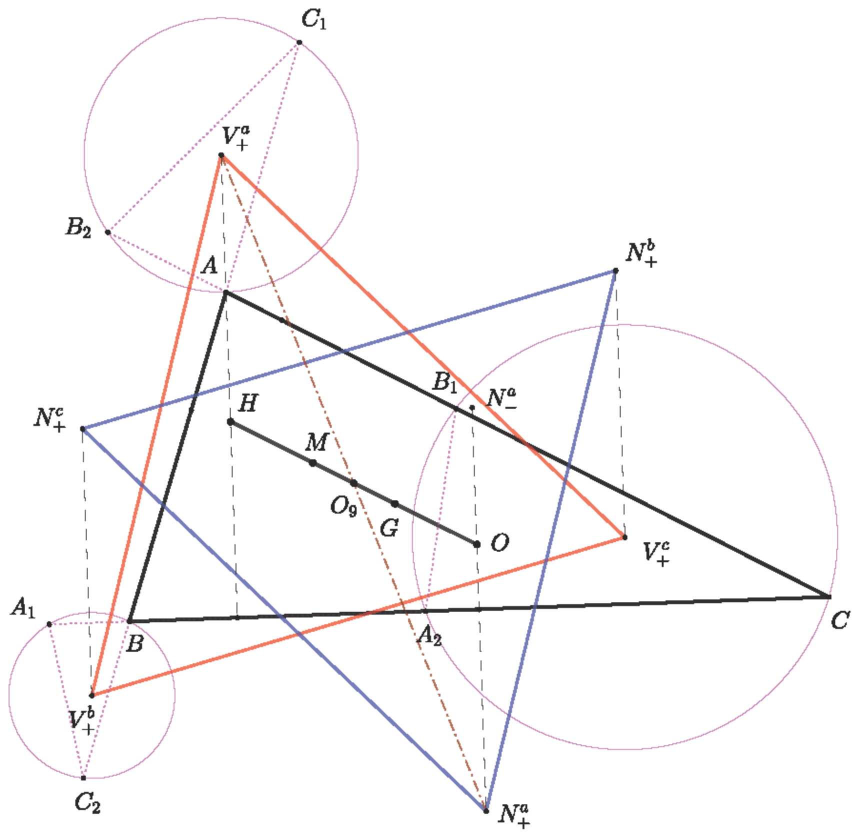

Figure 3. 1st vertex triangle and 1st Napoleon triangle

$$= \frac{2\sqrt{3}R}{3}\sin\left(A+\frac{\pi}{3}\right).$$

To show that

$$V_+^c N_+^b = \frac{2\sqrt{3}R}{3}\sin\left(A+\frac{\pi}{3}\right),$$

we do similar calculations in the triangle $CV_+^c N_+^b$.
Denote by $\overline{N}$ the point at which O and N_+^a project onto BC. Then,

$$\begin{aligned} N_+^a O &= N_+^a\overline{N} + \overline{N}O = \frac{a}{2\sqrt{3}} + 2R\cos A \\ &= \frac{2\sqrt{3}R}{3}\sin\left(A+\frac{\pi}{3}\right). \end{aligned}$$

Statement (ii) is completely proven. □

Remark 2.1. Above, using Proposition 4.7 in [2], the direct calculation of the circumradii $V_+^a A$ and $V_+^b B$ of the triangles AB_2C_1 and BC_2A_1 could be avoide.
A similar result is valid for the triangles $V_-^a V_-^b V_-^c$ and $N_-^a N_-^b N_-^c$ (shown with the same arguments) (Fig. 4). Formally, the sign "+" is changed everywhere to "–".

Proposition 2.2. *We have:*
(i) the lines $V_-^a H$, $N_-^a O$, $V_-^b N_-^c$, $V_-^c N_-^b$ *are perpendicular to* BC*;*
(ii) $V_-^a H = N_-^a O = V_-^b N_-^c = V_-^c N_-^b$.
Relative to the sides CA *and* AB *similar properties occur.*

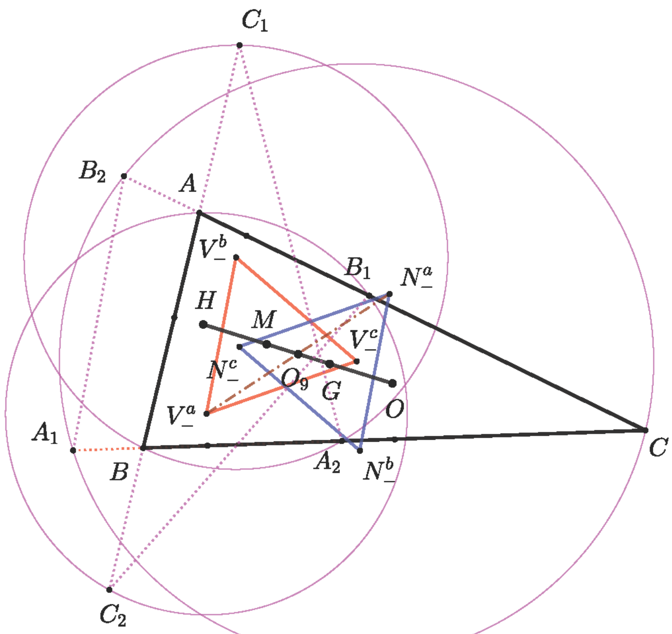

Figure 4. 2nd vertex triangle and 2nd Napoleon triangle

Denote by O_9 the center of the nine-point circle. Also, by $\mathscr{S}_X$ we denote the symmetry with respect to the point (center) X.

Proposition 2.3. *(i)* $V_+^a V_+^b V_+^c = \mathscr{S}_{O_9}(N_+^a N_+^b N_+^c)$ *(Fig. 3).*
(ii) $V_-^a V_-^b V_-^c = \mathscr{S}_{O_9}(N_-^a N_-^b N_-^c)$ *(Fig. 4).*

Proof. (i) From Proposition 2.1 we have that $V_+^a H \parallel N_+^a O$ and $V_+^a H = N_+^a O$, so the quadrilateral $V_+^a H N_+^a O$ is a parallelogram. Its diagonals intersect in the middle of the segment HO, that is, at point O_9. It follows that $V_+^a O_9 = N_+^a O_9$. The statement (i) is true.
(ii) From Proposition 2.2 and with the same arguments. □

Corollary 2.1. *(i) Triangles* $V_+^a V_+^b V_+^c$ *and* $N_+^a N_+^b N_+^c$ *are equal, have the same orientation and the corresponding sides are parallel (Fig. 3).*
(ii) Triangles $V_-^a V_-^b V_-^c$ *and* $N_-^a N_-^b N_-^c$ *have the same properties (Fig. 4).*

We denote by M the midpoint of the segment HG. It is the center of the *orthocentroidal circle.*

Corollary 2.2. *Triangles* $V_+^a V_+^b V_+^c$ *and* $V_-^a V_-^b V_-^c$ *are equilateral with the same center, the point* M *(Fig. 3 and 4).*

Proof. It is known that G is the center of both Napoleon triangles ([8]; [5], p. 65). Obviously, the symmetric of G with respect to O_9 is the point M. So, by Proposition 2.3, M is the center of the two vertex triangles. □

Corollary 2.3. *The circles $\mathcal{V}_+$ and $\mathcal{N}_+$, as well as $\mathcal{V}_-$ and $\mathcal{N}_-$, intersect on the perpendicular bisector of segment HO (Fig. 5).*

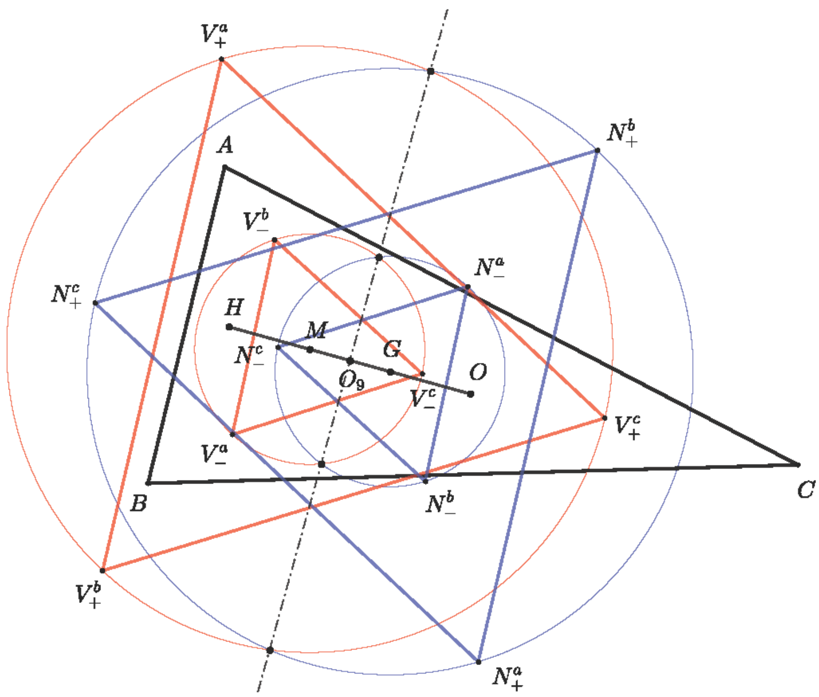

Figure 5. Vertex and Napoleon triangles and their circumcircles

Corollary 2.4. *The sidelengths of triangles $V_+^aV_+^bV_+^c$ and $V_-^aV_-^bV_-^c$ are given by the formulas*

$$(V_+^aV_+^b)^2 = \frac{1}{6}(a^2+b^2+c^2+4\sqrt{3}\Delta),$$

$$(V_-^aV_-^b)^2 = \frac{1}{6}(a^2+b^2+c^2-4\sqrt{3}\Delta),$$

where Δ is the area of triangle ABC.

Proof. It follows from the fact that $V_+^aV_+^b = N_+^aN_+^b$ and the formula $(N_+^aN_+^b)^2 = \dfrac{1}{6}(a^2+b^2+c^2+4\sqrt{3}\Delta)$ ([5], p.64). □

Using notations

$$l_+^2 = \frac{1}{2}\left(a^2+b^2+c^2+4\sqrt{3}\Delta\right), \quad l_-^2 = \frac{1}{2}\left(a^2+b^2+c^2-4\sqrt{3}\Delta\right),$$

the previous formulas are written:

$$V_+^aV_+^b = \frac{\sqrt{3}}{3}l_+, \quad V_-^aV_-^b = \frac{\sqrt{3}}{3}l_-,$$

and the circumradii of triangles $V_+^aV_+^bV_+^c$ and $V_-^aV_-^bV_-^c$ are egal to $\dfrac{l_+}{3}$ and $\dfrac{l_-}{3}$ respectively. Recall that l_+ (resp. l_-) is the common length of the segments AA_+, BB_+, CC_+ (resp. AA_-, BB_-, CC_-)

and that these quantities are very useful in metric questions of the Fermat-Torricelli configuration ([3], [4]).

Corollary 2.5. *The difference in the areas of triangles* $V_+^a V_+^b V_+^c$ *and* $V_-^a V_-^b V_-^c$ *is equal to* Δ.

Proof. Because Napoleon triangles have this property ([5], p. 64). □

Remark 2.2. The following table indicates a certain correspondence between geometric objects associated with the classical Fermat-Torricelli configuration and our proposed one:

classical approach	proposed approach
$A_+, A_-;\ B_+, B_-;\ C_+, C_-$	$A_1, A_2;\ B_1, B_2;\ C_1, C_2$
$A_+BC,\ B_+CA,\ C_+AB$	$AB_2C_1,\ BC_2A_1,\ CA_2B_1$
$A_-BC,\ B_-CA,\ C_-AB$	$AB_1C_2,\ BC_1A_2,\ CA_1B_2$
$N_+^a,\ N_+^b,\ N_+^c$	$V_+^a,\ V_+^b,\ V_+^c$
$N_-^a,\ N_-^b,\ N_-^c$	$V_-^a,\ V_-^b,\ V_-^c$
1st Napoleon triangle	1st vertex triangle
2nd Napoleon triangle	2nd vertex triangle
G	M

Proposition 2.4. *Points* N_+^a, N_+^b, N_+^c *and* V_+^a, V_+^b, V_+^c *are conconic. Points* N_-^a, N_-^b, N_-^c *and* V_-^a, V_-^b, V_-^c *are also conconic.*

Proof. We consider the hexagon $N_+^a V_+^c N_+^b V_+^a N_+^c V_+^b$ (Fig. 6).

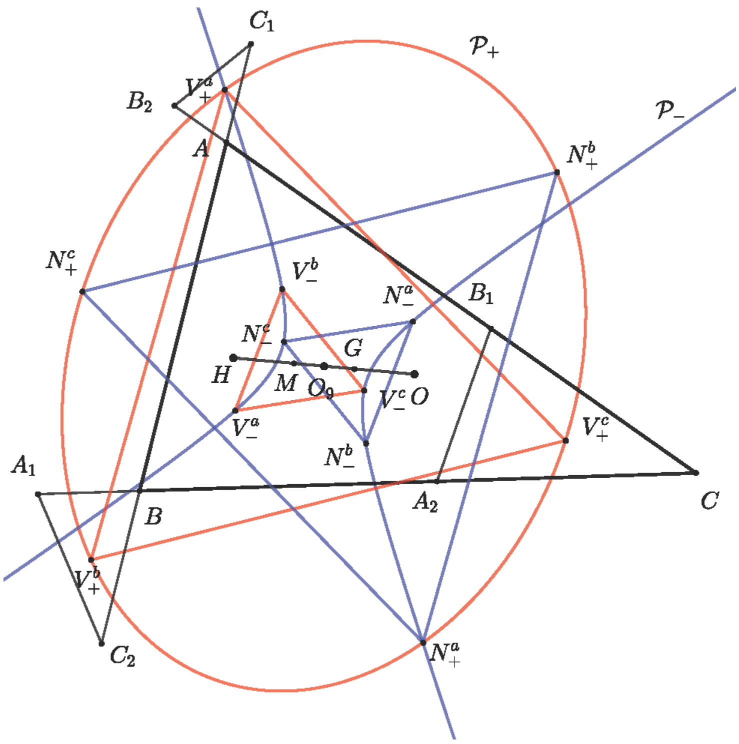

Figure 6. Conics $\mathscr{P}_+$ and $\mathscr{P}_-$

According to Proposition 2.1, we have $N_+^a V_+^c \parallel V_+^a N_+^c$, $V_+^c N_+^b \parallel N_+^c V_+^b$, $N_+^b V_+^a \parallel V_+^b N_+^a$. So, the opposite sides of the hexagon are parallel. Then, by the converse of Pascal's theorem ([5], p. 76), there is a conic passing through the six vertices of the hexagon. The second statement is shown in the same way. $\square$

We denote by $\mathscr{P}_+$ the conic in which the 1st Napoleon triangle and the 1st vertex triangle are inscribed. $\mathscr{P}_-$ has obvious significance.

Proposition 2.5. *The conics $\mathscr{P}_+$ and $\mathscr{P}_-$ have the same center, namely, the nine-point center O_9.*

Proof. It follows immediately from Proposition 2.3. $\square$

3. Similarity of triangles $V_+^a V_+^b V_+^c$ and $V_-^a V_-^b V_-^c$

Obviously, these triangles are similar and have opposite orientations. According to the general theory of similarity, each of them is the image of the other through a homothety followed by a reflection in a line. In the particular case we are in, we rely on the following result.

Lemma 3.1. *Let ABC and $A'B'C'$ be two equilateral triangles inscribed in the same circle and inversely oriented. Let $A"$, $B"$, $C"$ be the antipodal points of A', B', C', respectively (Fig. 7). Then*
(i) triangle $A'B'C'$ is the reflexion of triangle ABC in the common perpendicular bisector of segments AA', BB', CC' (or common angle-bisector of angles $\widehat{AMA'}$, $\widehat{BMB'}$, $\widehat{CMC'}$) and vice versa;
(ii) triangle $A"B"C"$ is the reflexion of triangle ABC in the perpendicular bisector of $AA"$, $BB"$, $CC"$;
(iii) the axes of symmetry of the pair of triangles $(ABC, A'B'C')$ and $(ABC, A"B"C")$ are perpendicular.

Proof. (i) Let's show that the perpendicular bisector of the chords AA', BB', CC' coincide. To this end, we first show that these chords are parallel. Indeed, $BB' \parallel CC'$ because the arcs $\widehat{BC'}$ and $\widehat{B'C}$ are equal (we have: $\widehat{BC'} = \widehat{B'C'} - \widehat{B'B} = 120^\circ - \widehat{B'B} = \widehat{BC} - \widehat{B'B} = \widehat{B'C}$). It also shows that $AA' \parallel CC'$. Hence, $AA' \parallel BB' \parallel CC'$.
Taking into account the fact that the perpendicular bisector of a chord passes through the center of the circle, it follows that the chords AA', BB', CC' have a common perpendicular bisector. Obviously, this line will be the axis of symmetry of the triangles ABC and $A'B'C'$.
(ii) Obviously, triangles ABC and $A"B"C"$ have different orientations. Use (ii) for the pair $(ABC, A"B"C")$.
(iii) Since $A'A"$ is a diameter, we have $AA' \perp AA"$. This leads to (iii). $\square$

Recall that triangles $V_+^a V_+^b V_+^c$ and $V_-^a V_-^b V_-^c$ are equilateral, concyclic, and have different orientations. Their common center is M. The line MV_-^a intersects the circumcircle of triangle $V_+^a V_+^b V_+^c$ at the points α_1 and α_2. Points β_1, β_2 and γ_1, γ_2 are defined similarly (Fig. 8). We also denote by d_1 and d_2 the perpendicular bisector of the chords $V_+^a \alpha_1$ and $V_+^a \alpha_2$, respectively. Now, we are able to indicate the geometric transformations by which the 1st vertex triangle passes into the 2nd vertex triangle.
Further, we denote by $\mathscr{S}_d$ the reflection in the line d and by $\mathscr{H}(X, k)$ the homothety of center X and ratio k.

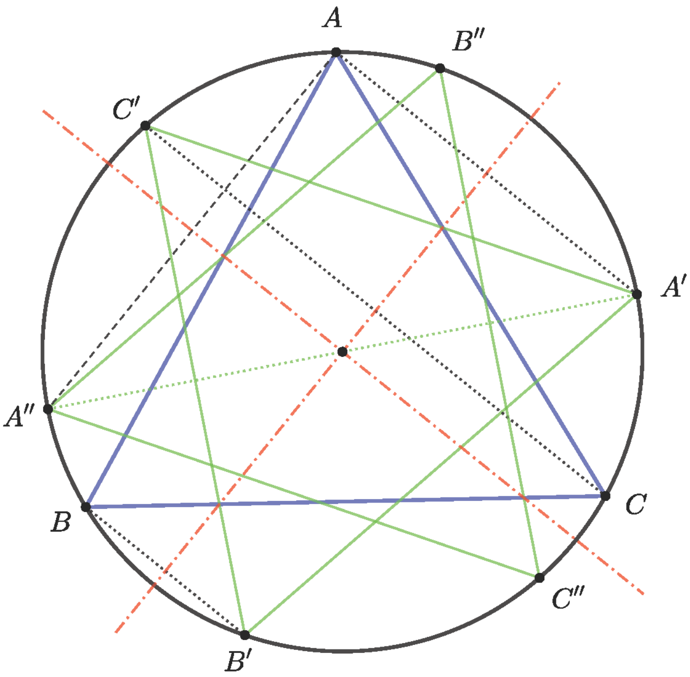

Figure 7. The axes of symmetry of pairs $(ABC, A'B'C')$ and $(ABC, A"B"C")$

Proposition 3.1. *With the previous notations, we have:*

$$V_+^a V_+^b V_+^c = \mathscr{S}_{d_1} \circ \mathscr{H}(M, -\frac{l_+}{l_-})(V_-^a V_-^b V_-^c) \;\; or \;\; V_+^a V_+^b V_+^c = \mathscr{S}_{d_2} \circ \mathscr{H}(M, \frac{l_+}{l_-})(V_-^a V_-^b V_-^c).$$

Proof. Taking into account Corollary 2.4, the ratio of the sides of triangles $V_+^a V_+^b V_+^c$ and $V_-^a V_-^b V_-^c$ is given by

$$\frac{V_+^a V_+^b}{V_-^a V_-^b} = \frac{l_+}{l_-}.$$

Therefore,

$$\alpha_1 \alpha_2 \alpha_3 = \mathscr{H}(M, -\frac{l_+}{l_-}) \left(V_-^a V_-^b V_-^c \right).$$

On the other hand, according to Lemma 3.1, we have:

$$V_+^a V_+^b V_+^c = \mathscr{S}_{d_1}(\alpha_1 \alpha_2 \alpha_3).$$

Combining, we obtain the first required formula. The second formula is proven in the same way. □

Remark 3.1. Above, we noted the analogy that exists between vertex triangles and Napoleon triangles. The counterpart of Proposition 3.1 for Napoleon triangles is obvious: the role of point M is played by G and the parallels through G at d_1 and d_2 are the axes of similitude of triangles $N_+^a N_+^b N_+^c$ and $N_-^a N_-^b N_-^c$. Proposition 2.3 is a "bridge" from Napoleon triangles to vertex triangles and vice versa.

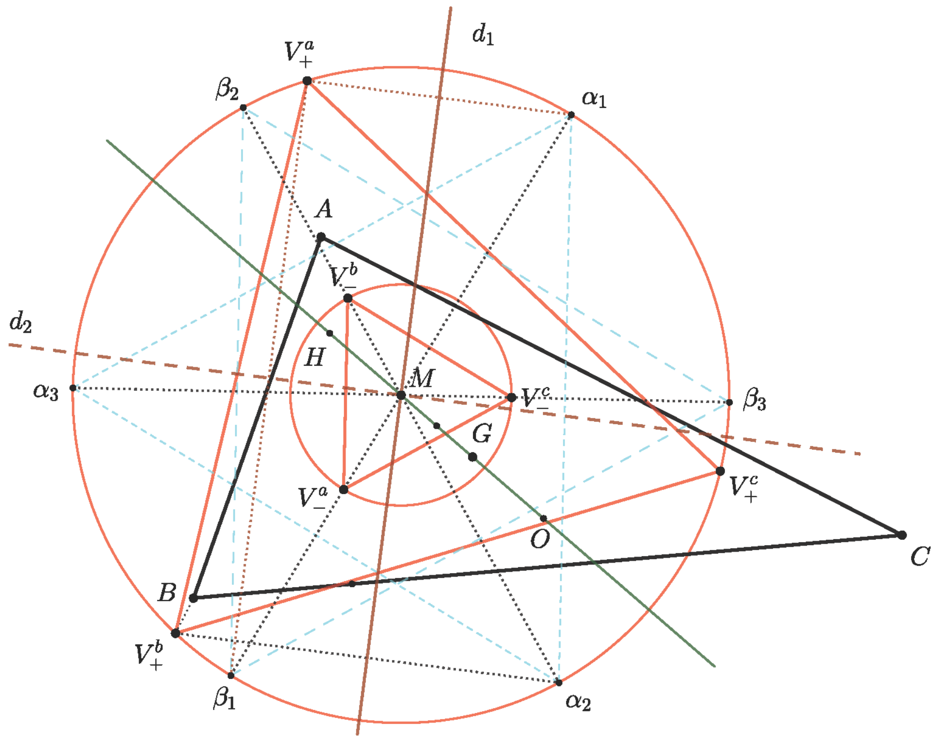

Figure 8. The axes of similitude of triangles $V_+^a V_+^b V_+^c$ and $V_-^a V_-^b V_-^c$

4. Radical axes and centers regarding circles $\mathcal{V}_+$, $\mathcal{V}_-$ $\mathcal{N}_+$, $\mathcal{N}_-$

First, we consider only the circles $\mathscr{C}$, $\mathcal{V}_+$, $\mathcal{V}_-$.

Proposition 4.1. *(i) The radical axes of the pairs of circles $(\mathscr{C}, \mathcal{V}_+)$ and $(\mathscr{C}, \mathcal{V}_-)$ are perpendicular to Euler line of the given triangle (Fig. 9);*
(ii) The radical axis of $\mathcal{V}_+$ and $\mathcal{V}_-$ is the line at infinity.

Proof. By Corollary 2.2, the circles $\mathcal{V}_+$ and $\mathcal{V}_-$ are concyclic and their center is M.
(i) The radical axis of $\mathscr{C}$ and $\mathcal{V}_+$ (or $\mathscr{C}$ and $\mathcal{V}_-$) is perpendicular to the center line, i.e. to the line OM.
(ii) Because $\mathcal{V}_+$ and $\mathcal{V}_-$ are concyclic. □

Now, let's introduce the circles $\mathcal{N}_+$ and $\mathcal{N}_-$ into the discussion. They are concyclic and their center is G (Fig. 9). According to Proposition 2.3, for these circles statements similar to (i) and (ii) of the previous sentence hold. The next sentence is just as easy to prove.

Proposition 4.2. *We have:*
(i) the pairs of circles $(\mathcal{V}_+, \mathcal{N}_+)$ and $(\mathcal{V}_-, \mathcal{N}_-)$ have the same radical axis, namely the perpendicular to the Euler line at point O_9;
(ii) the radical axes of the pairs of circles $(\mathcal{V}_+, \mathcal{N}_-)$ and $(\mathcal{V}_-, \mathcal{N}_+)$ are perpendicular to the Euler line.

Proof. (i) It follows immediately from Corollary 2.3.

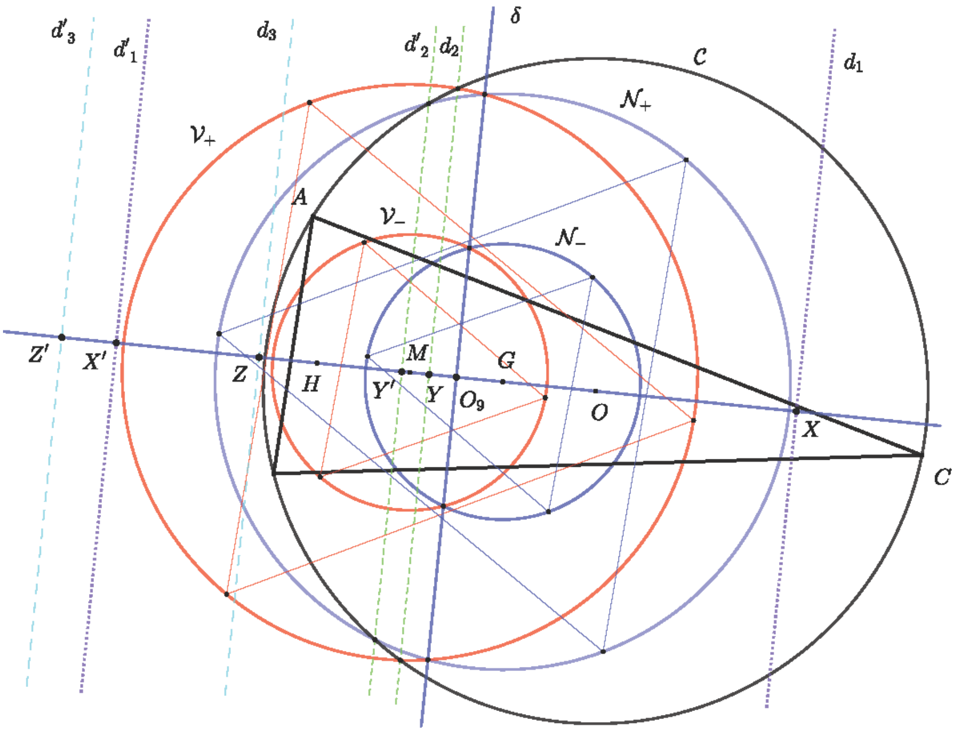

Figure 9. Eight radical axes and their relative position

(ii) The circles $\mathcal{V}_+$, $\mathcal{V}_-$, $\mathcal{N}_+$, $\mathcal{N}_-$ have their centers on the Euler line. So, the radical axes of the pairs $(\mathcal{V}_+, \mathcal{N}_-)$ and $(\mathcal{V}_-, \mathcal{N}_+)$ are perpendicular to the Euler line (the purple lines in Fig. 9). □

With the circles $\mathscr{C}$, $\mathcal{V}_+$, $\mathcal{V}_-$, $\mathcal{N}_+$, $\mathcal{N}_-$ we can form ten distinct pairs. So, we will consider ten radical axes. But, the line at infinity is the radical axis of the pairs $(\mathcal{V}_+, \mathcal{V}_-)$ and $(\mathcal{N}_+, \mathcal{N}_-)$. Also, the perpendicular to the Euler line at point O_9 is the radical axis of the pairs $(\mathcal{V}_+, \mathcal{N}_+)$ and $(\mathcal{V}_-, \mathcal{N}_-)$. So, we only have to examine six radical axes.

Proposition 4.3. *Let δ be the perpendicular to Euler line in O_9. We have:*
(i) the radical axes of the pairs $(\mathcal{V}_+, \mathcal{N}_-)$ and $(\mathcal{N}_+, \mathcal{V}_-)$ are symmetric with respect to δ;
(ii) the distance between the radical axis of the pair $(\mathscr{C}, \mathcal{V}_+)$ and δ is half the distance between the radical axis of the pair $(\mathscr{C}, \mathcal{N}_+)$ and δ;
(iii) the distance between the radical axis of the pair $(\mathscr{C}, \mathcal{V}_-)$ and δ is half the distance between the radical axis of the pair $(\mathscr{C}, \mathcal{N}_-)$ and δ (Fig. 9).

Proof. Let $d_1, d'_1, d_2, d'_2, d_3, d'_3$ denote the radical axes of the pairs of circles $(\mathcal{V}_+, \mathcal{N}_-)$, $(\mathcal{N}_+, \mathcal{V}_-)$, $(\mathscr{C}, \mathcal{V}_+)$, $(\mathscr{C}, \mathcal{N}_+)$, $(\mathscr{C}, \mathcal{V}_-)$, $(\mathscr{C}, \mathcal{N}_-)$, respectively (Fig. 9). They intersect the Euler line at points X, X', Y, Y', Z, Z', respectively.
Then, statements (i)-(iii) are written as follows: (i') $XO_9 = X'O_9$, (ii') $2YO_9 = Y'O_9$, (iii') $2ZO_9 = Z'O_9$. We will only prove statement (i'); the others are shown in the same way.

We use the notation $p(P,\mathscr{C})$ for the power of P with respect to circle $\mathscr{C}$. We have:

$$X \in d_1 \Longrightarrow p(X,\mathscr{V}_+) = p(X,\mathscr{N}_-) \Longrightarrow XM^2 - \frac{l_+^2}{9} = XG^2 - \frac{l_-^2}{9}$$

(for the center and radius of the circle $\mathscr{V}_+$, Corollaries 2.2 and 2.4 were taken into account), from where

$$XM^2 - XG^2 = \frac{l_+^2}{9} - \frac{l_-^2}{9}.$$

Similarly, from $X' \in d_1'$ we find

$$X'G^2 - X'M^2 = \frac{l_+^2}{9} - \frac{l_-^2}{9}.$$

Then, we obtain:

$$XM^2 - XG^2 = X'G^2 - X'M^2.$$

Therefore,

$$(XO_9 + O_9M)^2 - (XO_9 - O_9G)^2 = \left(X'O_9 + O_9G\right)^2 - \left(X'O_9 - O_9M\right)^2.$$

Taking into account the equality $O_9M = O_9G$ and doing the calculations, we are led to $XO_9 = X'O_9$.
Thus, the proposition is proven. □

Remark 4.1. The distances of points X, X' etc. can be calculated based on the formulas in [6], #45.

Proposition 4.4. *The radical center of any three (or more) circles chosen from $\mathscr{C}$, $\mathscr{V}_+$, $\mathscr{V}_-$, $\mathscr{N}_+$, $\mathscr{N}_-$ is the point of the line at infinity in the direction δ.*

Proof. It follows immediately from Propositions 4.1 and 4.2. □

Remark 4.2. In [8], this point is referred as X(523); it has barycentric coordinates ($b^2 - c^2$, $c^2 - a^2$, $a^2 - b^2$) and is isogonal conjugate of the focus of Kiepert parabola.

5. Centers and axes of perspective

In the first part of this section we consider only the triangles ABC, $V_+^a V_+^b V_+^c$ and $V_-^a V_-^b V_-^c$.

Proposition 5.1. *(i) H is the common center of perspective of the following pairs of triangles: $(ABC, V_+^a V_+^b V_+^c)$, $(ABC, V_-^a V_-^b V_-^c)$, $(V_+^a V_+^b V_+^c, V_-^a V_-^b V_-^c)$;*
(ii) the three axes of perspective of these pairs are concurrent.

Proof. (i) The statement results directly from the fact that the points A, H, V_+^a, V_-^a and their analogous points are collinear (Proposition 2.1 and 2.2).
(ii) According to [6], #377, from (i) it follows that the three axes of perspective of triangle are concurrent. □

In [2], the circumradii of triangles AB_2C_1 and AB_1C_2 were denoted r_+^a and r_-^a respectively; the radii r_+^b, r_+^c and r_-^b, r_-^c were defined cyclically. The following lemma represents part of Corollary 4.1 in [2], which we will need next.

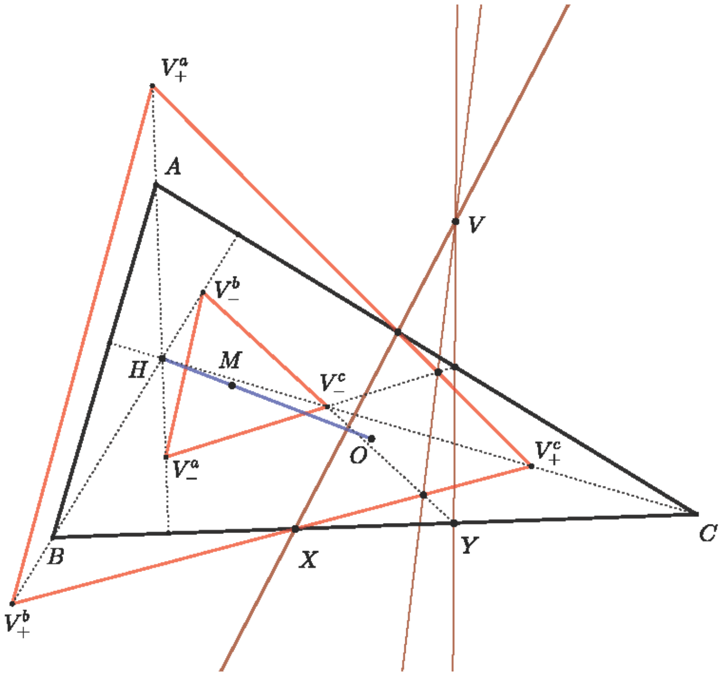

Figure 10. Axes of perspective of triangles $ABC, V_+^a V_+^b V_+^c, V_-^a V_-^b V_-^c$

Lemma 5.1. *The collinear points A, H, V_+^a and V_-^a have the properties:*
(i) $V_-^a H = r_-^a$, $V_-^a H = r_+^a$;
(ii) V_+^a, V_-^a are isotomic with respect to the segment AH.
Analogous results are valid relative to vertices B and C.

Proposition 5.2. *The perspective axes of the pairs $(ABC, V_+^a V_+^b V_+^c)$ and $(ABC, V_-^a V_-^b V_-^c)$ are isotomic transversal with respect to the triangle ABC.*

Proof. Let X and Y denote the points where side BC is intersected by these axes (Fig. 10). Let's show that $BX = CY$.
We note that X and Y are also the points where BC is intersected by the lines $V_+^b V_+^c$ and respectively $V_-^b V_-^c$. Applying Menelaus theorem to triangle HBC and transversal $V_+^b V_+^c$, we obtain:

$$\frac{\overline{XB}}{\overline{XC}} \cdot \frac{\overline{V_+^c C}}{\overline{V_+^c H}} \cdot \frac{\overline{V_+^b H}}{\overline{V_+^b B}} = 1.$$

Applying Menelaus theorem again to the same triangle and the transversal $V_-^b V_-^c$, we obtain:

$$\frac{\overline{YB}}{\overline{YC}} \cdot \frac{\overline{V_-^c C}}{\overline{V_-^c H}} \cdot \frac{\overline{V_-^b H}}{\overline{V_-^b B}} = 1.$$

Taking into account Lemma 5.1, it follows that

$$\frac{\overline{XB}}{\overline{XC}} = \frac{r_+^b r_-^c}{r_+^c r_-^b} \quad \text{and} \quad \frac{\overline{YB}}{\overline{YC}} = \frac{r_+^c r_-^b}{r_+^b r_-^c}.$$

So, we have:

$$\frac{\overline{XB}}{\overline{XC}} = \frac{\overline{YC}}{\overline{YB}}.$$

As a result,

$$\frac{\overline{XB}}{\overline{BC}} = \frac{\overline{YC}}{-\overline{BC}} \quad \text{i.e.} \quad \overline{XB} = -\overline{YC},$$

from where $BX = CY$.
We proceed in the same way with sides CA and AB. $\square$

Now, let's add to the three triangles considered above the Napoleon triangles. It is well known that O is the perspective center of triangles $N_+^a N_+^b N_+^c$ and $N_-^a N_-^b N_-^c$. Propositions 3.6 and 3.7 of [2] give us new perspectivity properties of these triangles in connection with the points H_+ and H_-. More, about the pairs formed by a Napoleon triangle and a vertex triangle we have:

Proposition 5.3. *(i) O_9 is the common center of perspective of the following pairs $(N_+^a N_+^b N_+^c$, $V_+^a V_+^b V_+^c)$ and $(N_-^a N_-^b N_-^c$, $V_-^a V_-^b V_-^c)$;*
(ii) the line at infinity is the common axis of perspective of these pairs.

Proof. (i) By Proposition 2.3, the lines $N_+^a V_+^a$, $N_+^b V_+^b$, $N_+^c V_+^c$ are concurrent at O_9, and we have: $N_+^a N_+^b \parallel V_+^a V_+^b$, $N_+^b N_+^c \parallel V_+^b V_+^c$, $N_+^c N_+^a \parallel V_+^c V_+^a$ (as well as the relations that are obtained from these by changing "+" to "−". $\square$

Finally, it is immediately clear that the triangles $N_+^a N_+^b N_+^c$ and $V_-^a V_-^b V_-^c$, as well as $N_-^a N_-^b N_-^c$ and $V_+^a V_+^b V_+^c$, are not in perspective.

6. Ortology centers of Napoleon and vertex triangles

The results in Section 2 are very useful in what follows. In particular, the fact that triangles $V_+^a V_+^b V_+^c$ and $N_+^a N_+^b N_+^c$, as well as $V_-^a V_-^b V_-^c$ and $N_-^a N_-^b N_-^c$, are symmetric with respect to O_9 plays an important role in what follows.

Proposition 6.1. *(i) Triangles $V_+^a V_+^b V_+^c$ and $N_+^a N_+^b N_+^c$ are orthologic with the centers M and G.*
(ii) Triangles $V_-^a V_-^b V_-^c$ and $N_-^a N_-^b N_-^c$ are orthologic with the same centers, M and G.

Proof. (i) Since $V_+^a V_+^b V_+^c$ and $N_+^a N_+^b N_+^c$ are equilateral and have parallel sides, the perpendicular from vertex V_+^a to side $N_+^b N_+^c$ is also perpendicular to side $V_+^b V_+^c$ and, therefore, passes through the center M of triangle $V_+^a V_+^b V_+^c$. Similarly, the perpendiculars from V_+^b and V_+^c to sides $N_+^c N_+^a$ and respectively $N_+^a N_+^b$ pass through M. So, triangles $V_+^a V_+^b V_+^c$ and $N_+^a N_+^b N_+^c$ are orthologic at M.
With the same arguments it is shown that the triangles $N_+^a N_+^b N_+^c$ and $V_+^a V_+^b V_+^c$ are orthologic at G.
(ii) Similar. $\square$

In the next step, we need certain properties of Napoleon triangles (vertex triangles).

Lemma 6.1. *We have (Fig. 11):*
(i) $AF_+ \perp N_+^b N_+^c$ and $AF_- \perp N_-^b N_-^c$,
(i') $AF_+ \perp V_+^b V_+^c$ and $AF_- \perp V_-^b V_-^c$,

(ii) $F_-N_+^a \parallel N_-^bN_-^c \parallel V_-^bV_-^c$ *and* $F_+N_-^a \parallel N_+^bN_+^c \parallel V_+^bV_+^c$*,*
and their analogues.

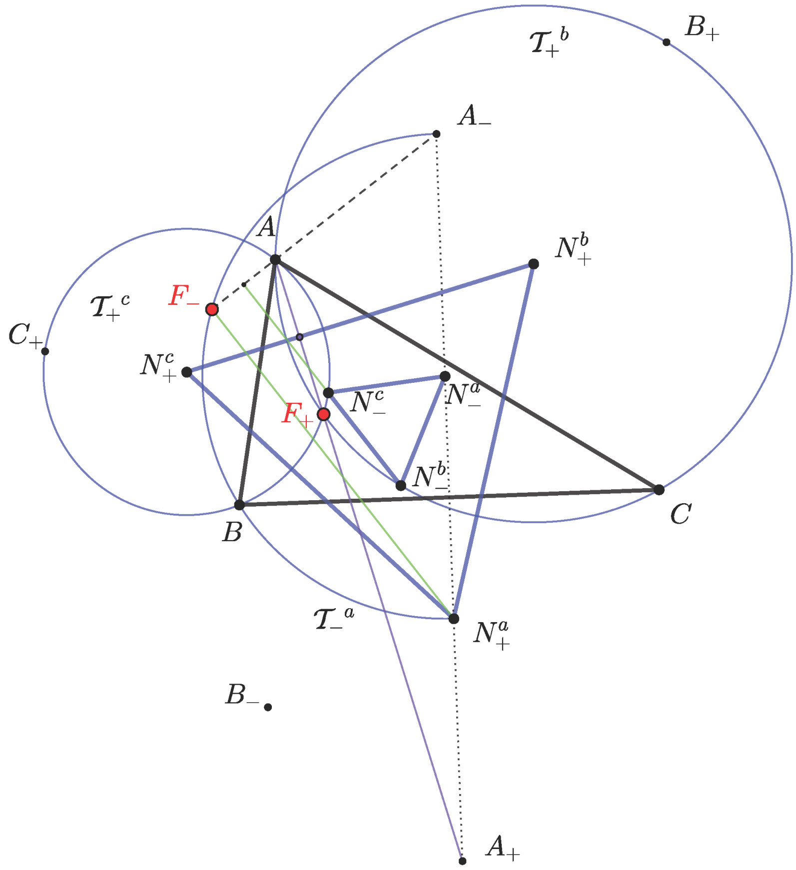

Figure 11. Properties that interconnect triangles ABC, $N_+^aN_+^bN_+^c$ and $V_+^aV_+^bV_+^c$

Proof. (i) AF_+ and $N_+^bN_+^c$ are the common chord and the line of centers of the Torricelli circles $\mathcal{T}_+{}^b, \mathcal{T}_+{}^c$. Therefore, $AF_+ \perp N_+^bN_+^c$ (Fig. 11). The second part is obtained with the same arguments relative to the circles $\mathcal{T}_-{}^b$ and $\mathcal{T}_-{}^c$.
(i') From (i) and the fact that the triangles $N_+^aN_+^bN_+^c$ and $V_+^aV_+^bV_+^c$ (or $N_-^aN_-^bN_-^c$ and $V_-^aV_-^bV_-^c$) have corresponding parallel sides.
(ii) Let us show that $F_-N_+^a \parallel N_-^bN_-^c$. We note that the angle $\widehat{N_+^aF_-A_-}$ is inscribed in a semicircle of the Torricelli circle $\mathcal{T}_-{}^a$ (Fig. 11); so it is a right angle, i.e $A_-F_- \perp N_+^aF_-$ or $AF_- \perp N_+^aF_-$. Then, from this and $AF_- \perp N_+^aF_-$ and $AF_- \perp N_-^bN_-^c$ we deduce $F_-N_+^a \parallel N_-^bN_-^c$. Etc.etc. □

Proposition 6.2. *(i) Triangles* $V_+^aV_+^bV_+^c$ *and* ABC *are orthologic with the centers* H *and* F_+*.*
(ii) Triangles $V_-^aV_-^bV_-^c$ *and* ABC *are orthologic with the centers* H *and* F_-*.*

Proof. (i) Since V_+^a, A, H, and the analogs of these points are collinear, it follows that the triangle $V_+^aV_+^bV_+^c$ is orthologic with triangle ABC at H.

As is known, in turn the triangle ABC is orthologic with $V_+^a V_+^b V_+^c$, but their center of orthology may differ from H. So, it remains to show that the perpendiculars from vertices A, B, C to the corresponding sides $V_+^b V_+^c$, $V_+^c V_+^a$, $V_+^a V_+^b$ are concurrent in F_+. We will only verify that $AF_+ \perp V_+^b V_+^c$ (or, equivalently, $AF_+ \perp .N_+^b N_+^c$) holds. But, according to Lemma 6.1,(i') and (i), these relations are true. Finally, F_+ is the orthology center of triangles ABC and $V_+^a V_+^b V_+^c$.
(ii) We proceed in the same way. □

We will see that the orthologic centers of the pairs formed by triangles chosen from the triangles $N_+^a N_+^b N_+^c$, $N_-^a N_-^b N_-^c$, $V_+^a V_+^b V_+^c$, $V_-^a V_-^b V_-^c$ are obtained by simple constructions from the Fermat points F_+ and F_- of the triangle ABC. For this, we introduce four new points $\Phi_+, \Phi_-, \Phi_+^*, \Phi_-^*$ as follows: Φ_+, Φ_- are antipodes of points F_+ and respectively F_- in the circles $\mathcal{N}_-$ and $\mathcal{N}_+$ on which they lie (i.e. $\Phi_+ = \mathcal{S}_G(F_+), \Phi_- = \mathcal{S}_G(F_-)$) and Φ_+^*, Φ_-^* are the symmetrical points of the points Φ_+ and respectively Φ_- with respect to O_9 (i.e. $\Phi_+^* = \mathcal{S}_{O_9}(\Phi_+), \Phi_-^* = \mathcal{S}_{O_9}(\Phi_-)$). (Fig. 12)

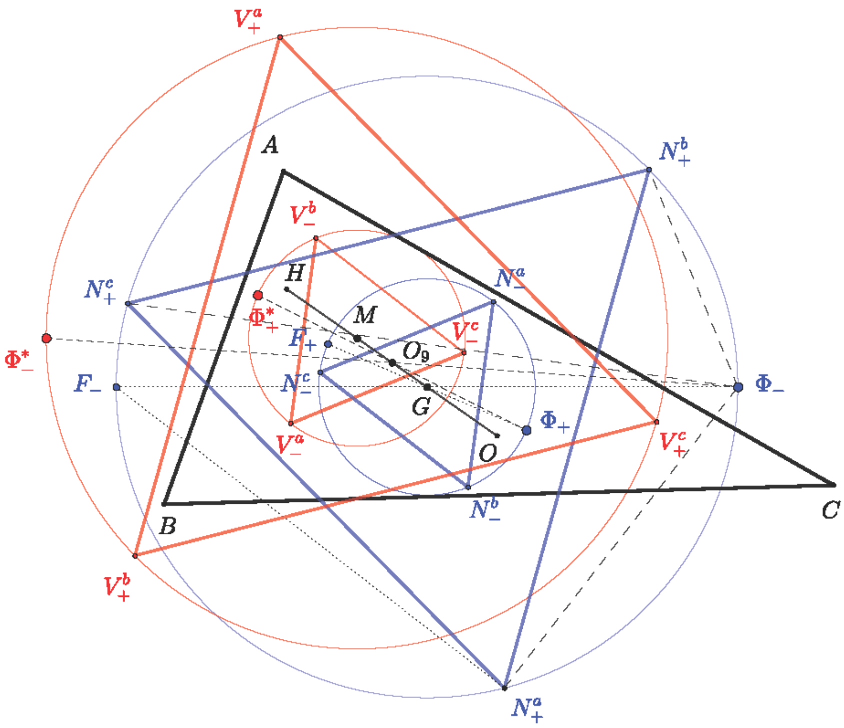

Figure 12. The orthology centers $\Phi_+, \Phi_-, \Phi_+^*, \Phi_-^*$

Proposition 6.3. *(i) Triangles $N_+^a N_+^b N_+^c$ and $N_-^a N_-^b N_-^c$ are orthologic with the centers Φ_- and Φ_+.*
(ii) Triangles $N_+^a N_+^b N_+^c$ and $V_-^a V_-^b V_-^c$ are orthologic with the centers Φ_- and Φ_+^* (Fig. 12).

Proof. (i) First, let's show that $N_+^a N_+^b N_+^c$ and $N_-^a N_-^b N_-^c$ are orthologic at Φ_-. The perpendiculars from N_+^a and N_+^b to the corresponding sides of the equilateral triangle $N_-^a N_-^b N_-^c$ intersect at a point X and we have $\widehat{N_+^a X N_+^b} = \dfrac{\pi}{3}$. The quadrilateral $X N_+^b N_+^c N_+^a$ is cyclic, so $X \in \mathcal{N}_+$. It is

also shown that the perpendiculars from N_+^b and N_+^c on the corresponding sides of the triangle $N_-^a N_-^b N_-^c$ intersect at a point $Y \in \mathcal{N}_+$. It is clear that the points X and Y coincide. Therefore, X is the orthology center of the triangles $N_+^a N_+^b N_+^c$ and $N_-^a N_-^b N_-^c$. From Lemma 6.1(ii), we have that $F_- N_+^a \parallel N_-^b N_-^c$. As $N_-^b N_-^c \perp N_+^a X$, it follows that $F_- N_+^a \perp N_+^a X$. Then, the triangle $F_- N_+^a X$ is right-angled in N_+^a and, consequently, $F_- X$ is a diameter of the circle $\mathcal{N}_+$. Therefore, $X = \mathcal{S}_G(F_-)$, .i.e. $X = \Phi_-$.
(ii) From (i) and the fact that the corresponding sides of triangles $V_-^a V_-^b V_-^c$ and $N_-^a N_-^b N_-^c$ are parallel, it follows that triangles $N_+^a N_+^b N_+^c$ and $V_-^a V_-^b V_-^c$ are orthologic at Φ_-.
On the other hand, from (i) and the fact that $V_-^a V_-^b V_-^c = \mathcal{S}_{O_9}\left(N_-^a N_-^b N_-^c\right)$, Proposition 2.3 (ii), it follows that triangles $V_-^a V_-^b V_-^c$ and $N_+^a N_+^b N_+^c$ are orthologic at $\Phi_+^* = \mathcal{S}_{O_9}(\Phi_+)$. □

Propositions 6.4-6.5 below are obtained immediately from Proposition 2.3 by appropriately using Proposition 2.3. In fact, some statements are repeated and are considered only for the sake of completeness.

Proposition 6.4. *(i) Triangles $N_-^a N_-^b N_-^c$ and $N_+^a N_+^b N_+^c$ are orthologic with the centers Φ_+ and Φ_-.*
(ii) Triangles $N_-^a N_-^b N_-^c$ and $V_+^a V_+^b V_+^c$ are orthologic with the centers Φ_+ and Φ_-^*.

Proposition 6.5. *(i) Triangles $V_+^a V_+^b V_+^c$ and $N_-^a N_-^b N_-^c$ are orthologic with the centers Φ_-^* and Φ_+.*
(ii) Triangles $V_+^a V_+^b V_+^c$ and $V_-^a V_-^b V_-^c$ are orthologic with the centers Φ_-^* and Φ_+^*.

Proposition 6.6. *(i) Triangles $V_-^a V_-^b V_-^c$ and $N_+^a N_+^b N_+^c$ are orthologic with the centers Φ_+^* and Φ_-.*
(ii) Triangles $V_-^a V_-^b V_-^c$ and $V_+^a V_+^b V_+^c$ are orthologic with the centers Φ_+^* and Φ_-^*.

Finally, we summarize the previous results in the following table.

	$ABC)$	$N_+^a N_+^b N_+^c)$	$N_-^a N_-^b N_-^c)$	$V_+^a V_+^b V_+^c)$	$V_-^a V_-^b V_-^c)$
$(ABC$	$-$	F_+	F_-	F_+	F_-
$(N_+^a N_+^b N_+^c$	O	$-$	Φ_-	G	Φ_-
$(N_-^a N_-^b N_-^c$	O	Φ_+	$-$	Φ_+	G
$(V_+^a V_+^b V_+^c$	H	M	Φ_-^*	$-$	Φ_-^*
$(V_-^a V_-^b V_-^c$	H	Φ_+^*	M	Φ_+^*	$-$

The parentheses surrounding the triangles in the first row and column suggest the position of the triangle in the pair being considered.

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