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ON ORTHOLOGIC TRIANGLES

PARIS PAMFILOS

ABSTRACT. In this article we show that two triangles can be made orthologic by turning one of them to certain positions in their circumcircle. In addition, we study some structures related to the resulting configuration of the triangle's rotation.

1. INTRODUCTION

Jacob Steiner [3, I, p.155] introduced the notion of orthologic triangles observing the following symmetry property:
"Given the triangles, $\{ABC, A'B'C'\}$, if the perpendiculars from the vertices $\{A', B', C'\}$ on the sides $\{BC, CA, AB\}$ meet at a point P, then also the perpenciculars from the vertices $\{A, B, C\}$ on the sides $\{BC', C'A', A'B'\}$ meet at a point Q (see Figure 1)."

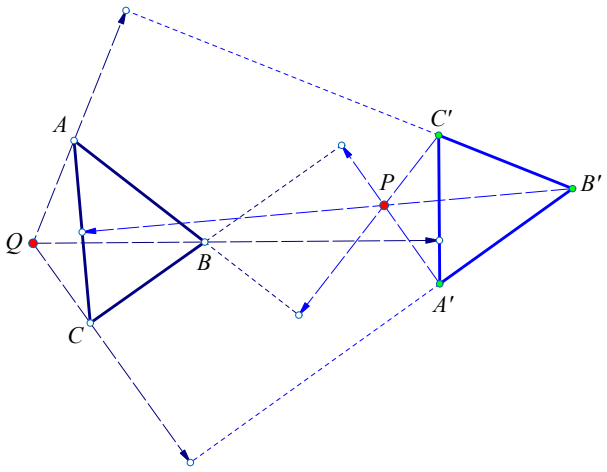

Figure 1. Orthologic triangles $\{ABC, A'B'C'\}$

Points $\{P, Q\}$ are called *"orthologic centers"* of the two triangles. The proof of this property and a nice account of several fundamental properties of this symmetric relation

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between triangles, together with a wealth of examples, can be found in the book by Patrascu and Smarandache [2]. One of the first properties proven there is the following. *"A necessary and sufficient condition for the two triangles to be orthologic is the condition:"*

$$f(A',B',C',A,B,C) \;=\; |A'B|^2-|A'C|^2+|B'C|^2-|B'A|^2+|C'A|^2-|C'B|^2 \;=\; 0\,. \tag{1.1}$$

The symmetry of the orthologic condition is reflected in the relation

$$f(A',B',C',A,B,C) \;+\; f(A,B,C,A',B',C') \;=0\,.$$

In the first section below we use condition (1.1) to show the following property.

Theorem 1.1. *Given two triangles $\{ABC, A'B'C'\}$ of the plane, there is a rotation of ABC in its circumcircle, so that in the new position $A''B''C''$ the triangle becomes orthologic to $A'B'C'$.*

In subsequent sections we discuss the geometric aspect of this phenomenon and derive some additional properties relating to an arbitrary pair of triangles.

2. Proof of the theorem

Using eventually a similarity, we may assume without loss of generality, that the circumcircle of the first triangle ABC is the unit circle of the complex plane and the vertices of the triangles are represented by the complex numbers

$$A'(a)\,,\; B'(b)\,,\; C'(c) \quad \text{and} \quad A(e^{i\alpha})\,,\; B(e^{i\alpha+\phi_1})\,,\; C(e^{i\alpha+\phi_2})\,.$$

Then, equation (1.1) takes the form:

$$\begin{gathered}
(e^{i(\alpha+\phi_1)}-a)(e^{-i(\alpha+\phi_1)}-\overline{a}) \;-\; (e^{i(\alpha+\phi_2)}-a)(e^{-i(\alpha+\phi_2)}-\overline{a}) \;+\\
(e^{i(\alpha+\phi_2)}-b)(e^{-i(\alpha+\phi_2)}-\overline{b}) \;-\; (e^{i\alpha}-b)(e^{-i\alpha}-\overline{b}) +\\
(e^{i\alpha}-c)(e^{-i\alpha}-\overline{c}) \;-\; (e^{i(\alpha+\phi_1)}-c)(e^{-i(\alpha+\phi_1)}-\overline{c}) \;=\; 0 \quad \Leftrightarrow\\
ae^{-i(\alpha+\phi_2)}+\overline{a}e^{i(\alpha+\phi_2)}-(ae^{-i(\alpha+\phi_1)}+\overline{a}e^{i(\alpha+\phi_1)})+\\
be^{-i\alpha}+\overline{b}e^{i\alpha}-(be^{-i(\alpha+\phi_2)}+\overline{b}e^{i(\alpha+\phi_2)})+\\
ce^{-i(\alpha+\phi_1)}+\overline{c}e^{i(\alpha+\phi_1)}-(ce^{-i\alpha}+\overline{c}e^{i\alpha}) \;=\; 0 \quad \Leftrightarrow\\
Re(\overline{a}(e^{i(\alpha+\phi_2)}-e^{i(\alpha+\phi_1)})+Re(\overline{b}(e^{i\alpha}-e^{i\alpha+\phi_2}))+Re(\overline{c}(e^{i(\alpha+\phi_1)}-e^{i\alpha})) \;=\; 0 \quad \Leftrightarrow\\
Re(e^{i\alpha}d) \;=\; 0\,,\\
\text{with} \quad d=\overline{a}(e^{i\phi_2}-e^{i\phi_1})+\overline{b}(1-e^{i\phi_2})+\overline{c}(e^{i\phi_1}-1) \;=\; d_1+id_2 \quad \Leftrightarrow\\
d_1\cos(\alpha)-d_2\sin(\alpha) \;=\; 0 \quad \Leftrightarrow \quad \tan\alpha=\frac{d_1}{d_2}\,.
\end{gathered}$$

Since the triangle ABC rotates in its circumcircle the angles $\{\phi_1,\phi_2\}$ and d remain constant, and α can be identified with the rotation angle. Last equation shows that there are in general two positions of the rotating triangle, corresponding to the angles α and $\alpha+\pi$ and defining two symmetric triangles w.r.t. to the center of their circumcircle. We have also proved the following.

Corollary 2.1. *Given two triangles $\{ABC, A'B'C'\}$ of the plane, there are in general two rotated positions $\{A_1B_1C_1, A_2B_2C_2\}$ of ABC in its circumcircle, so that in these positions the triangle becomes orthologic to $A'B'C'$. The two positions are symmetric w.r.t. to the center of the circumcircle of ABC.*

3. THE GEOMETRIC PICTURE

We start with a lemma concerning a certain generation of ellipses, which represents the main building block of the geometric picture.

Lemma 3.1. *For each position of the chord AB of fixed length d of the circle κ, we draw lines $\{\alpha, \beta\}$ in fixed directions $\{e_1, e_2\}$ intersecting at Y. Then, as AB varies on the circle, point Y describes an ellipse.*

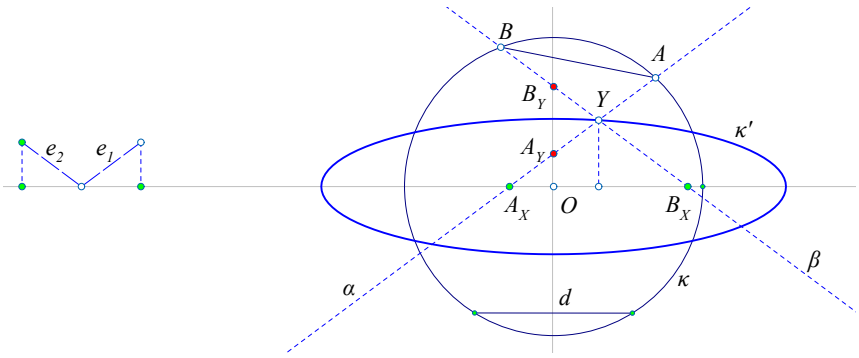

Figure 2. Ellipse described by the intersection Y of two lines

Proof. Without loss of generality we may assume that the Cartesian coordinate axes are parallel to the bisectors of the angle of the direction vectors $\{e_1, e_2\}$. We denote by $c(\phi)$ and $s(\phi)$ respectively the cosine and the sine of the angle ϕ. It is (see Figure 2):

$$e_1 = (c(\phi_1), s(\phi_1))\ ,\ \ e_2 = (c(\phi_2), s(\phi_2)))\ ,\ \ A = R(c(\phi), s(\phi))\ ,\ \ B = R(c(\phi'), s(\phi'))\ ,$$
$$\text{with}\quad \phi' = \phi + \omega\ ,\ \omega = \widehat{AOB}\ ,\ \phi_1 + \phi_2 = \pi\quad \text{and}\quad \phi_2 - \phi_1 = \delta.$$
$$\text{We abreviate further}\quad c_i = c(\phi_i)\ ,\ s_i = s(\phi_i)\quad \text{for}\quad i = 1..2.$$

Denoting by $\{A_X, A_Y\}$ the intersections of α with the axes and by $\{B_X, B_Y\}$ the intersections of β with the axes, we have for the intersection $Y(x, y)$ of $\{\alpha, \beta\}$:

$$\begin{aligned}
A_X &= \frac{R}{s_1}(c(\phi)s_1 - s(\phi)c_1) &&= \frac{R}{s_1}s(\phi_1 - \phi)\\
A_Y &= \frac{R}{c_1}(s(\phi)c_1 - c(\phi)s_1) &&= \frac{R}{c_1}s(\phi - \phi_1)\\
B_X &= \frac{R}{s_2}(c(\phi')s_2 - s(\phi')c_2) &&= \frac{R}{s_2}s(\phi_2 - \phi')\\
B_Y &= \frac{R}{c_2}(s(\phi')c_2 - c(\phi')s_2) &&= \frac{R}{c_2}s(\phi' - \phi_2)\\
x = \frac{1}{2}(A_X + B_X) &= \frac{R}{2s_1s_2}(s_2s(\phi_1 - \phi) + s_1s(\phi_2 - \phi')) &&=
\end{aligned}$$

$$\frac{R}{4s_1s_2}(c(\phi_2-(\phi_1-\phi))-c(\phi_2+(\phi_1-\phi))+c(\phi_1-(\phi_2-\phi'))-c(\phi_1+(\phi_2-\phi')))$$
$$=\frac{R}{4s_1s_2}(c(\phi+\phi_2-\phi_1)-c(\phi_2+\phi_1-\phi)+c(\phi'+\phi_1-\phi_2)-c(\phi_1+\phi_2-\phi'))$$
$$=\frac{R}{4s_1s_2}(c(\phi+\delta)+c(\phi)+c(\phi'-\delta)+c(\phi'))\quad\Rightarrow$$
$$x=\frac{R}{s_1s_2}c\left(\frac{\delta}{2}\right)c\left(\frac{\omega-\delta}{2}\right)c\left(\frac{\phi+\phi'}{2}\right).$$

Analogously we compute:

$$y=\frac{R}{c_1c_2}s\left(\frac{\delta}{2}\right)s\left(\frac{\omega-\delta}{2}\right)s\left(\frac{\phi+\phi'}{2}\right)\quad\Rightarrow\quad\frac{x^2}{a^2}+\frac{y^2}{b^2}=1,\quad\text{with}$$
$$a=\frac{R}{s_1s_2}c\left(\frac{\delta}{2}\right)c\left(\frac{\omega-\delta}{2}\right),\quad b=\frac{R}{c_1c_2}s\left(\frac{\delta}{2}\right)s\left(\frac{\omega-\delta}{2}\right).\qquad\square$$

Corollary 3.1. *For a circle κ and two directions $\{e_1,e_2\}$ in general position, the ellipse of lemma 3.1 will have its axes parallel to the bisectors $\{v_1,v_2\}$ of the angle of $\{e_1,e_2\}$ (see Figure 3).*

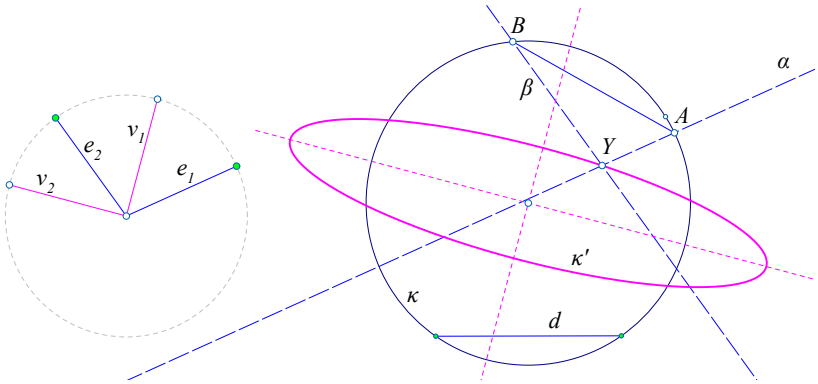

Figure 3. Ellipse described by the intersection Y of two lines in fixed directions

Next sequence of figures describes the configurations resulting by revolving the triangle ABC in its circumcircle κ and drawing from its vertices lines orthogonal to corresponding sides of the triangle $A'B'C'$. Figure 4 shows the basic operation of drawing orthogonals from the vertices $\{A,B,C\}$ of the rotating triangle to the sides $\{B'C',C'A',A'B'\}$ of the other triangle. These are parallel to the altitudes of the triangle $A'B'C'$. These lines form a triangle $A''B''C''$ homothetic to the triangles $\{A_0B_0C_0,A_1B_1C_1\}$ resulting by drawing parallels to the altitudes of $A'B'C'$ from corresponding vertices of this triangle.

As the triangle ABC revolves in its circumcircle, the corresponding triangles $A''B''C''$ vary so, that their sides remain parallel to fixed directions (the altitudes of $A'B'C'$), consequently remaining homothetic to $A_1B_1C_1$. Figure 5 shows the triangles $\{A''B''C''\}$ for various positions of the revolving ABC. It shows also a position of the triangle ABC for which the corresponding triangle $\{A''B''C''\}$ collapses to a point P, defining then an *"orthology center"* of the two triangles $\{A''B''C'',A'B'C'\}$.

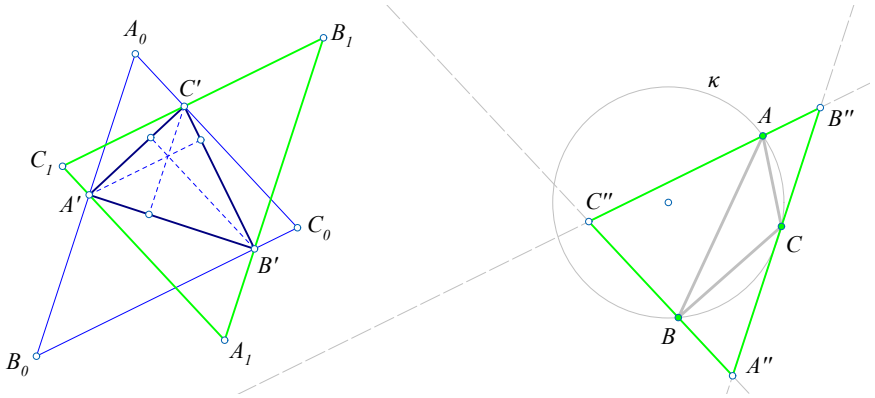

Figure 4. Drawing orthogonals from $\{A, B, C\}$ to $\{B'C', C'A', A'B'\}$

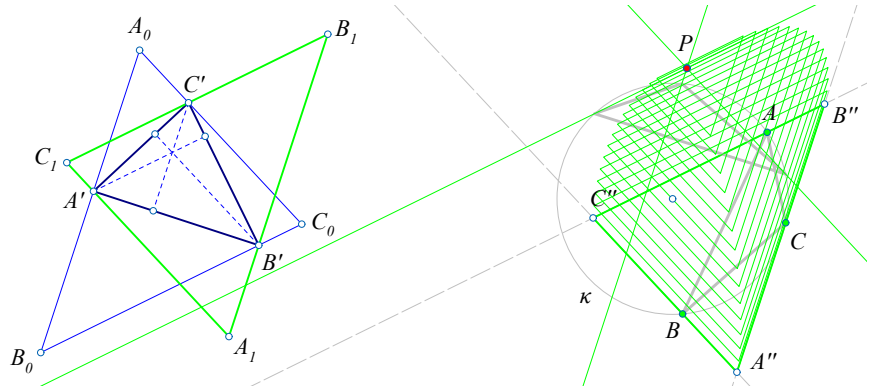

Figure 5. Triangles $\{A'', B'', C''\}$ homothetic to $\{A_1B_1C_1\}$

Theorem 3.1. *The geometric loci of the vertices of the triangles* $\{A''B''C''\}$ *as the triangle* ABC *rotates in its circumcircle* κ *are three concentric ellipses* $\{\kappa_A, \kappa_B, \kappa_C\}$ *intersecting at two real points* $\{P, Q\}$ *lying symmetrically w.r.t. to the center of the circumcircle of* ABC *(see Figure 6).*

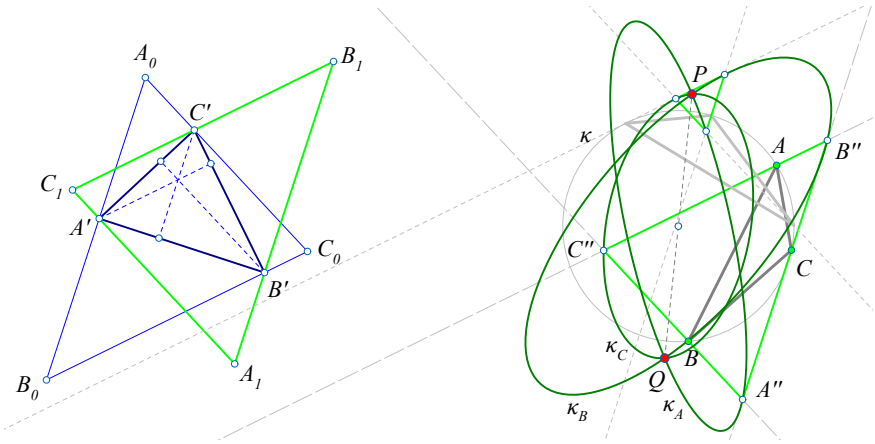

Figure 6. The ellipses-loci $\{\kappa_A, \kappa_B, \kappa_C\}$ of the varying points $\{A'', B'', C''\}$

Proof. We discuss the case of the vertex C'' and the corresponding locus κ_C. This locus results from the varying chord of constant length AB of κ, by drawing parallels to the fixed directions $\{C_1A_1, C_1B_1\}$. Thus, according to corollary 3.1, C'' describes an ellipse κ_C whose center coincides with the center of κ and with axes parallel to the bisectors of the angle $\widehat{C_1}$. Analogous arguments are valid for $\{\kappa_A, \kappa_B\}$.
The fact, that the three ellipses intersect in general at two points $\{P, Q\}$, follows from theorem 1.1, since orthologic centers correspond precisely to collapsing to a point triangles $A''B''C''$. Also the symmetry of $\{P, Q\}$ w.r.t. to the center of κ follows from the corresponding symmetry of the ellipses, in accordance also with the symmetry claimed by that theorem. □

Theorem 3.2. *The circumcenters $\{K\}$ of the varying triangles $\{A''B''C''\}$ lie on an ellipse μ with center coinciding with the center of the circumcircle κ of ABC. This ellipse passes also through the intersection points $\{P, Q\}$ of the three ellipses $\{\kappa_A, \kappa_B, \kappa_C\}$ (see Figure 7).*

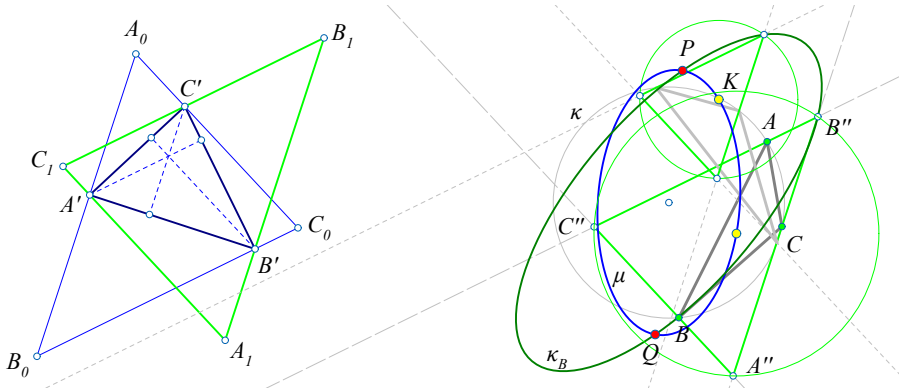

Figure 7. Ellipse μ of circumcenters of the varying triangles $\{A''B''C''\}$

Proof. That the centers $\{K\}$ of the variable homothetic triangles $\{A''B''C''\}$ move on such an ellipse follows from a general property, according to which a variable point X connected to the (variable) triangle $A''B''C''$ by constant $\{X = k_1A'' + k_2B'' + k_3C''\}$ barycentric coordinates moves on an ellipse, if the vertices of the homothetic to each other triangles $A''B''C''$ move respectively on concentric ellipses. This is immediately seen by adopting quadratic parameterizations

$$(x_i, y_i, z_i) = (f_i(t), g_i(t), h_i(t)) \ , \ \ i = 1..3 \ ,$$

for the ellipses $\{\kappa_A, \kappa_B, \kappa_C\}$. Then the points $\{X = k_1A'' + k_2B'' + k_3C''\}$ are also parameterized by quadratic functions, consequently describe a conic μ. Since the triangles $\{A''B''C''\}$ vary in a bounded domain and are in pairs symmetric w.r.t. to the circumcenter of κ, μ is an ellipse concentric with these three ellipses. The ellipse μ passes also through the points $\{P, Q\}$, which coincide with the circumcenters of the corresponding, collapsed to a point, triangles $\{A''B''C''\}$. □

4. The pivoting aspect

The existence of the two orthologic centers $\{P, Q\}$ and the corresponding two symmetric in their common circumcircle triangles $\{A''B''C''\}$, which are orthologic to $A'B'C'$, bears

a relation to Miquel's theorem and the related to it *"pivoting"* of similar to each other triangles circumscribing a given triangle ([1, I, p.457]).
In this section we discuss this relation. Figure 8 illustrates the typical pivoting configu-

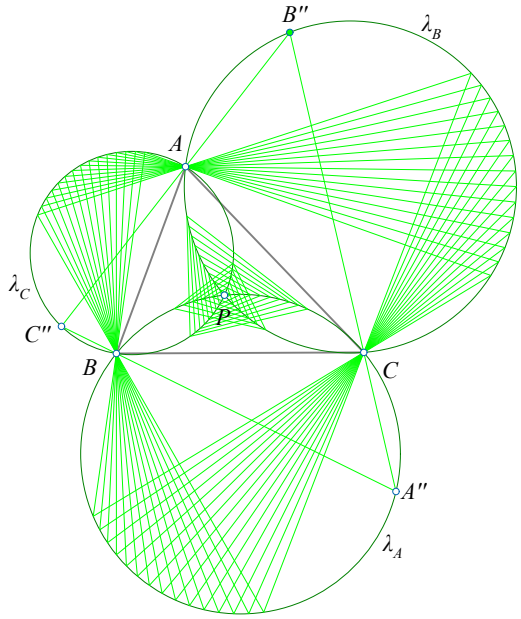

Figure 8. Similar triangles $\{A''B''C''\}$ pivoting about the fixed triangle ABC

ration. The variable triangles $\{A''B''C''\}$ circumscribe the triangle ABC, are similar to each other, and their vertices vary respectively on three fixed circles $\{\lambda_A, \lambda_B, \lambda_C\}$ intersecting at the *"pivot"* P. As the triangle $A''B''C''$ varies, each one of its vertices takes all possible positions on the corresponding circle. At the common point P of the three circles the varying triangle $A''B''C''$ collapses to that point. At every point P' distinct from P lying on such a circle, λ_B say, corresponds a unique genuine triangle $A''B''C''$ with $B'' = P'$ circumscribing the triangle ABC, as in figure 8.
The configuration of the varying, homothetic to each other triangles $\{A''B''C''\}$, of section 3 is intimately related to such a pivoting configuration. In that case it is the triangle ABC that rotates in its circumcircle and the circles $\{\alpha_0, \beta_0, \gamma_0\}$ rotate accordingly to the new positions $A_\phi B_\phi C_\phi$ of ABC, while the triangles $\{A''_\phi B''_\phi C''_\phi\}$ remain homothetic to each other, their sides remaining parallel to constant directions.
Figure 9 is created by making a copy of $ABC = A_0B_0C_0$ and the circles $\{\alpha_0, \beta_0, \gamma_0\}$ and putting it on the left next to the figure with the rotating $\{A_\phi B_\phi C_\phi\}$.
Looking on the figure on the right, as the triangle $ABC = A_0B_0C_0$ rotates in its circumcircle κ and the sides of the triangles $\{A''_\phi B''_\phi C''_\phi\}$ remain parallel to fixed directions, the angles $\{\widehat{A''_\phi}, \widehat{B''_\phi}, \widehat{C''_\phi}\}$ are constant and view the corresponding sides of fixed length of the rotated triangle $A_\phi B_\phi C_\phi$. It follows that the circles $\{\alpha_\phi = (B_\phi C_\phi A''_\phi)\}$ remain congruent to each other, the same happening with the circles $\{\beta_\phi = (C_\phi A_\phi B''_\phi)\}$ and $\{\gamma_t = (A_\phi B_\phi C''_\phi)\}$ and there is a congruence (isometry) R_ϕ, mapping the system

$$R_\phi \ : \ A_\phi B_\phi C_\phi \cup \alpha_\phi \cup \beta_\phi \cup \gamma_\phi \longrightarrow A_0B_0C_0 \cup \alpha_0 \cup \beta_0 \cup \gamma_0 \ .$$

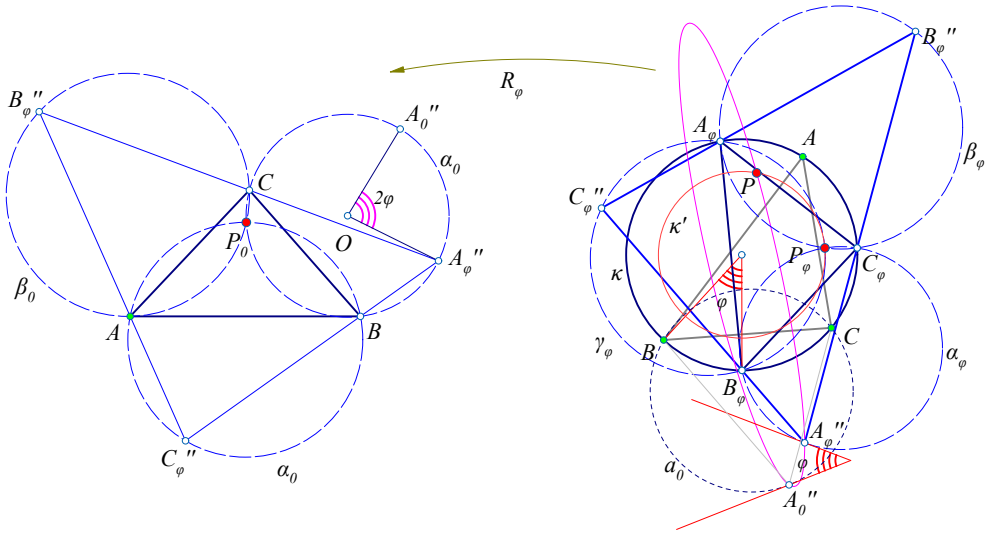

Figure 9. Relation of the rotating configuration to the Miquel one

Every point Q having a fixed position relative to ABC describes by the rotation of ABC in its circumcircle κ a circle κ_Q concentric to κ, and all the points of of κ_Q map via the congruences $\{R_\phi\}$ to the same point Q_0 of the configuration on the left. In particular, the Miquel point P_ϕ of the three circles $\{\alpha_\phi, \beta_\phi, \gamma_\phi\}$ has a fixed position relative to the rotating triangle $A_\phi B_\phi C_\phi$, and describes a concentric to κ circle κ', whose points map via the congruences $\{R_\phi\}$ to the same point P_0 on the left.
Using the congruences $\{R_\phi\}$, each one of the triangles $\{A''_\phi B''_\phi C''_\phi\}$ on the right maps to a corresponding pivoting triangle $\{A''_\phi B''_\phi C''_\phi\}$ on the left. It is easily seen, that if the triangle $ABC = A_0B_0C_0$ on the right rotates by an angle ϕ then the corresponding vertices of $\{A''_\phi B''_\phi C''_\phi\}$ on the left rotate on their circle by an angle 2ϕ. It follows that each position of $A''_\phi B''_\phi C''_\phi$ on the left corresponds to two positions of the congruent $A''_\phi B''_\phi C''_\phi$ on the right. In particular, to the Miquel point P_0 on the left, characterizing the collapsing position of the pivoting triangle $A''_\phi B''_\phi C''_\phi$, correspond two points $\{P, Q\}$ on the right, coinciding with the collapsing positions of the triangles $\{A''_\phi B''_\phi C''_\phi\}$, which also define the positions of the orthology centers. All this proves the following (see Figure 9).

Theorem 4.1. *The circle κ' passes through the orthology centers $\{P, Q\}$.*

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ESTIAS 4
IRAKLEIO 71307
GREECE.
Email address: pamfilos@uoc.gr