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A NOTE ON APOLLONIAN GASKET

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Abstract. We solved the seven problems regarding Apollonian gasket configuration proposed by Dao Thanh Oai on April 21, 2018.

1. Introduction and Statements of the Problems

Apollonian gasket is a certain fractal structure concerning families of circles with prescribed tangency conditions. There is a vast literature in the study on Apollonian gasket with different emphases, such as the density of the circle packing, its dynamics, or its arithmetic and number theoretic aspects [3],[5],[4],[6],[2]. In the proposed problem set by Dao [1], it involves the geometric aspects such as concyclicity, collinearity, concurrency or tangency related to the points, lines or circles derived from the configurations. Our solution to Dao's problems is guided by the viewpoint of Felix Klein in its Erlangen program, namely geometry is the study of invariants under some suitable transformation groups. The transformation group we are employing is the group of Möbius transformations or fractional linear transformations.

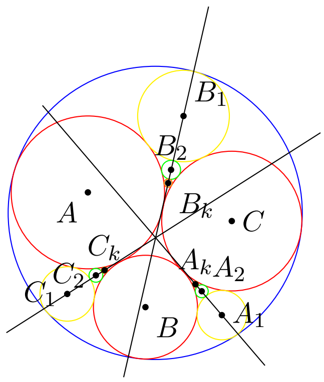

Figure 1

Statement of the Problems. Starting with triangle ABC, construct mutually tangent circles $(A), (B)$ and (C) with outer Soddy circle (D) [7]. Let (A_1) be the circle tangent

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to $(B), (C)$ and (D) (in the figures (D) is the exterior enclosing circle) and defined inductively (A_k) such that (A_k) is the inner Soddy circle of $(B), (C)$ and (A_{k-1}) for $k \geq 2$. Define $(B_k), (C_k), k \geq 1$ similarly (see Figure 1). Furthermore, let (A) be tangent to $(B_k), (C_k)$ at $A_{b_k}, A_{c_k}, k \geq 1$ respectively. Define $B_{c_k}, B_{a_k}, C_{a_k}, C_{b_k}, k \geq 1$ similarly (see Figure 2). Lastly let (A_k) be tangent to (A_{k+1}) at $T_{a_k}, \geq 1$ and define $T_{b_k}, T_{c_k}, k \geq 1$ similarly (see Figure 3). Then one has the following results.

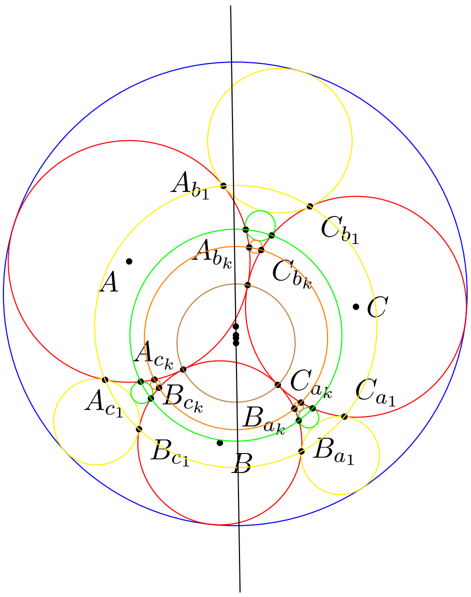

Figure 2

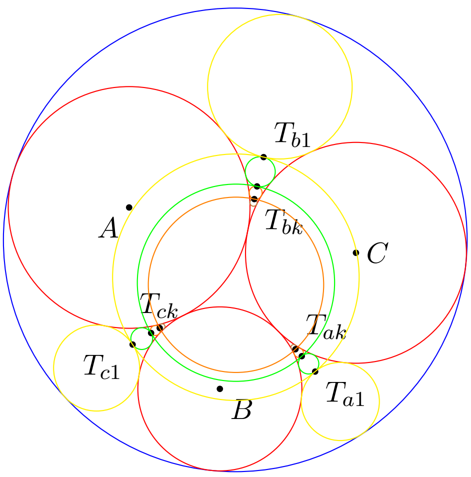

Figure 3

Problem 1. The three lines A_jA_k, B_jB_k, C_jC_k are concurrent for any $j \neq k$, with $j \geq 1, k \geq 1$.

Problem 2. The three lines AA_k, BB_k, CC_k are concurrent for any $k \geq 1$.

Problem 3. The six points $A_{b_k}, A_{c_k}, B_{c_k}, B_{a_k}, C_{a_k}, C_{b_k}$ are concyclic for any $k \geq 1$ and the centers of these circles are collinear.

Problem 4. The three lines $T_{a_j}T_{a_k}, T_{b_j}T_{b_k}, T_{c_j}T_{c_k}$ are concurrent for any $j \neq k$ with $j \geq 1, k \geq 1$.

Problem 5. The circle $(T_{a_k}T_{b_k}T_{c_k})$ is tangent to the six circles $(A_k), (A_{k+1}), (B_k), (B_{k+1}), (C_k), (C_{k+1})$ for any $k \geq 1$.

Problem 6. The three lines $AT_{a_k}, BT_{b_k}, CT_{c_k}$ are concurrent for any $k \geq 1$.

Problem 7. The three lines $A_jT_{a_k}, B_jT_{b_k}, C_jT_{c_k}$ are concurrent for any $j \geq 1, k \geq 1$.

2. Preliminaries and Strategies of Solutions

We start by mentioning the key formula relating the curvatures of the mutually tangent circles:

Lemma 1. (Descartes' formula [8],[3]) One has the following formula:

$$2(k_1^2 + k_2^2 + k_3^2 + k_4^2) = (k_1 + k_2 + k_3 + k_4)^2,$$

where the curvature for enclosing circle should take negative sign, for example in the above setup, let's denote the radii of the circles $(A), (B), (C)$ and (D) by r_A, r_B, r_C and r_C. Then Descartes' formula gives that

$$2\left(\frac{1}{r_A^2} + \frac{1}{r_B^2} + \frac{1}{r_C^2} + \frac{1}{r_D^2}\right) = \left(\frac{1}{r_A} + \frac{1}{r_B} + \frac{1}{r_C} - \frac{1}{r_D}\right)^2.$$

The following lemma is not directly used in the proof, but is useful in producing the pictures.

Lemma 2. Writing the centers of the circles as vectors or complex numbers z_j, one has

$$\frac{\sum_{j=1}^4 k_j^2 z_j}{\sum_{j=1}^4 k_j^2} = \frac{\sum_{j=1}^4 k_j z_j}{\sum_{j=1}^4 k_j}.$$

Proof. By Lemma 1, one has $2\sum k_j^2 = (\sum k_j)^2$. By (2.3) of [3], one has

$$2\sum_{j=1}^4 k_j^2 z_j = 2\sum_{j=1}^4 k_j(k_j z_j) = (\sum_{j=1}^4 k_j)(\sum_{j=1}^4 k_j z_j).$$

Dividing the latter equality by the former, one has

$$\frac{\sum_{j=1}^4 k_j^2 z_j}{\sum_{j=1}^4 k_j^2} = \frac{\sum_{j=1}^4 k_j z_j}{\sum_{j=1}^4 k_j},$$

as required.

In the configuration of Figure 1, assume that $\alpha_1 \geq \alpha$ and $\beta\gamma - (\beta+\gamma)\delta \geq 0$. Then Descartes' formula gives that

$$\alpha_1 = \beta + \gamma - \delta + 2\sqrt{\beta\gamma - (\beta+\gamma)\delta}$$

and

$$\alpha_{k+1} = \beta + \gamma + \alpha_k + 2\sqrt{\beta\gamma + (\beta+\gamma)\alpha_k}, \qquad k \geq 1. \tag{1}$$

Extrapolating, it is natural to put $\alpha_0 = -\delta$. Aiming to solve the recursive relation (1) in the form of $\alpha_k = a \cdot k^2 + b \cdot k + c$, we arrive at the following

Proposition 1. Assuming $\alpha_1 \geq \alpha$ (resp. $\beta_1 \geq \beta$, or $\gamma_1 \geq \gamma$), $\beta\gamma - (\beta+\gamma)\delta \geq 0$ and putting $\alpha_0 = -\delta$ (resp. $\beta_0 = \delta$ or $\gamma_0 = -\delta$), one has

$$\alpha_k = (\beta+\gamma)\cdot k^2 + 2\sqrt{\beta\gamma - (\beta+\gamma)\delta}\cdot k - \delta, \qquad k \geq 0.$$

(resp.

$$\beta_k = (\alpha+\gamma)\cdot k^2 + 2\sqrt{\alpha\gamma - (\alpha+\gamma)\delta}\cdot k - \delta,$$

or

$$\gamma_k = (\alpha+\beta)\cdot k^2 + 2\sqrt{\alpha\beta - (\alpha+\beta)\delta}\cdot k - \delta.)$$

Proof. We leave the straightforward verification to the reader.

Lemma 3. Given two circles of center z_j and radius r_j, $j = 1, 2$, the point of tangency is in a form of weighted average given by

$$\operatorname{avg}(z_1, z_2, r_1, r_2) := \frac{r_2}{r_1 + r_2} z_1 + \frac{r_1}{r_1 + r_2} z_2,$$

where the radius r_j of the enclosing circle takes negative sign.

Lemma 4. The general Möbius transformation is given by $\mu(z) = \frac{az+b}{cz+d}$, where $a, b, c, d \in \mathbb{C}$ and $ad - bc \neq 0$. For computation with real coordinates, let $a = a_1 + ia_2, b = b_1 + ib_2, c = c_1 + ic_2, d = d_1 + id_2$, with $a_1, a_2, b_1, b_2, c_1, c_2, d_1, d_2 \in \mathbb{R}$. Then the mapping is given by

$$(x, y) \mapsto \left(\frac{U}{W}, \frac{V}{W}\right),$$

where

$$U = (a_1 x - a_2 y + b_1)(c_1 x - c_2 y + d_1) + (a_1 y + a_2 x + b_2)(c_1 y + c_2 x + d_2)$$

$$V = (a_1 y + a_2 x + b_2)(c_1 x - c_2 y + d_1) - (a_1 x - a_2 y + b_1)(c_1 y + c_2 x + d_2)$$

and

$$W = (c_1^2 + c_2^2)(x^2 + y^2) + (d_1^2 + d_2^2) + 2[(c_1 d_1 + c_2 d_2)x - (c_2 d_1 - c_1 d_2)y].$$

Proof. Straightforward computation.

Remark 1. It is well known that a Möbius transformation preserves the class of circles or lines, tangency, and coaxial circles.

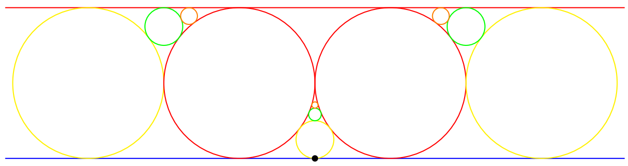

Figure 4

Strategy of Solutions. Our strategy is to apply a suitable Möbius transformation that maps the configuration in Figure 1 to a band Apollonian gasket. Specifically we map the point of tangency of (D) and (A) to ∞, the point of tangency of (A_1) and (D) to $(0,0)$, together with scaling or rotation such that the final band model looks like Figure 4, where the two parallel lines correspond to circle (A) and circle (D), and circles $(B),(C),(C_1),(B_1)$ are tangent to the two parallel lines and of radius 1. We then make an explicit computation of all the relevant points needed to prove the Problems. However we need to be aware of what properties are being preserved under a Möbius transformation; for example one of the troubles that we face is that the center of the circles is not in general preserved under a Möbius transformation. Similarly collinearity is not in general preserved under a Möbius transformation. We will address these problems when we come to the proofs. For now, let's do the basic computation to obtain all the points of interest in our problem for the band Apollonian gasket.

Unless there is confusion, we keep the same notations for the corresponding points of interest in both the original model and the band model of the Apollonian gaskets. After the mapping into the band Apollonian gasket, we have that $B=(-1,1), C=(1,1), B_1=(3,1), C_1=(-3,1)$. By direct computation or by the limiting case of Proposition 1, we see that the circle (B_k) or (C_k) has curvature given by $\beta_k=\gamma_k=k^2, k\geq 1$. Similarly the circle (A_k) has curvature given by $\alpha_k=2k(k+1)$. For example, in Proposition 1, we have $\beta=\gamma=1$ and $\delta=0$, hence

$$\alpha_k=(1+1)k^2+2\sqrt{1\cdot 1-(1+1)\cdot 0}\cdot k-0=2k(k+1).$$

To obtain the points of tangency, we have the following observations:

Observation 1. The coordinates of T_{a_k}'s are given by

$$T_{a_k}=\left(0,\sum_{j=1}^{k}\frac{2}{\alpha_j}\right), \alpha_j=2j(j+1).$$

Observation 2. Using the length of common tangent $d = 2\sqrt{r_1 r_2}$, one easily see that the coordinates of B_{k+1} $(k \geq 1)$ is given by

$$\left(3 - \sum_{j=1}^{k} \frac{2}{\sqrt{\beta_{j+1}\beta_j}}, 2 - \frac{1}{(k+1)^2}\right), \beta_j = j^2.$$

Since $B_1 = (3, 1)$, by telescopic sum method one gets a closed form for B_k as follows

$$B_k = \left(\frac{k+2}{k}, 2 - \frac{1}{k^2}\right), \quad k \geq 1.$$

Observation 3.

$$T_{b_k} = \operatorname{avg}(B_k, B_{k+1}, r_k, r_{k+1}) = \frac{r_k}{r_{k+1} + r_k} B_{k+1} + \frac{r_{k+1}}{r_{k+1} + r_k} B_k,$$

where $r_k = 1/k^2, r_{k+1} = 1/(k+1)^2$ are the radii of B_k, B_{k+1}, respectively.

Using the above observations and by telescopic sum method, one arrives at the following formulas:

$$T_{a_k} = \left(0, \frac{k}{k+1}\right)$$

$$T_{b_k} = \left(\frac{2k^2 + 6k + 3}{2k^2 + 2k + 1}, \frac{4k(k+1)}{2k^2 + 2k + 1}\right)$$

$$T_{c_k} = \left(-\frac{2k^2 + 6k + 3}{2k^2 + 2k + 1}, \frac{4k(k+1)}{2k^2 + 2k + 1}\right) \text{ (by symmetry)}$$

$$B_k = \left(\frac{k+2}{k}, 2 - \frac{1}{k^2}\right)$$

Similarly the fact that $A_1 = \left(0, \frac{1}{\alpha_1}\right)$ and $A_k = \left(0, \frac{1}{\alpha_k} + \sum_{j=1}^{k-1} \frac{1}{\alpha_j}\right), k \geq 2$ shows that

$$A_k = \left(0, \frac{2k^2 - 1}{2k(k+1)}\right), k \geq 1.$$

Furthermore, we have

$$A_{bk} = \left(\frac{k+2}{k}, 2\right)$$

(by geometric observation, A_{b_k} has the same x-coordinate as that of B_k)

$$C_{bk} = \operatorname{avg}\left((1,1), B_k, 1, \frac{1}{k^2}\right)$$
$$= \left(\frac{(k+1)^2}{k^2+1}, \frac{2k^2}{k^2+1}\right)$$

$$C_{ak} = \operatorname{avg}\left(A_k, (1,1), \frac{1}{2k(k+1)}, 1\right)$$
$$= \left(\frac{1}{2k^2 + 2k + 1}, \frac{2k^2}{2k^2 + 2k + 1}\right)$$

Now we are ready for

3. Carrying out the Proofs

We will prove the original problem set [1] in the order of # 5, # 3, # 4, # 1, # 2, # 6 and # 7.

Proof of Problem 5.

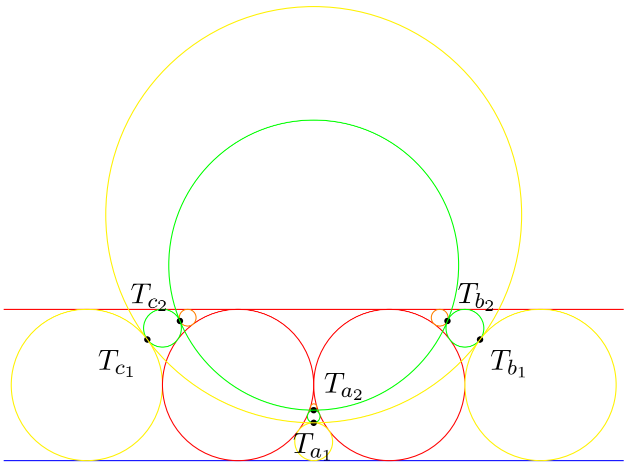

Figure 5

By definition it is clear that B_kB_{k+1} is perpendicular to the common tangent line through T_{b_k} of the circles. Let Z_k be the intersection of B_kB_{k+1} with the y-axis (the line with equation $x = 0$). Then by direct computation, we have that

$$Z_k = \left(0, \frac{4k^2 + 6k + 3}{2k(k+1)}\right).$$

Proposition 2. Z_k is the center of the circle through T_{a_k}, T_{b_k} and T_{c_k} that is tangent to the six circles $(A_{k+1}), (A_k), (B_{k+1}), (B_k), (C_{k+1})$, and (C_k).

Proof. Let d be the distance function. By symmetry, it suffices to check that $d(Z_k, T_{a_k}) = d(Z_k, T_{b_k})$ since the tangency condition follows from construction. But by direct computation, one has

$$d(Z_k, T_{a_k}) = \frac{2k^2 + 6k + 3}{2k(k+1)}$$

and

$$d(Z_k, T_{b_k}) = \frac{2k^2 + 6k + 3}{2k(k+1)},$$

hence the result. □

Now from Proposition 2, the result of Problem 5 follows by an inverse Möbius transformation mapping the band Apollonian gasket to the original one, since a Möbius transformation preserves tangency. □

Remark 2. The family of circles are coaxial given by the equation

$$x^2 + y^2 - \frac{4k^2+6k+3}{k(k+1)}y + 3 = 0$$

passing through the common points

$$(\pm i\sqrt{3}, 0).$$

Proof of Problem 3.

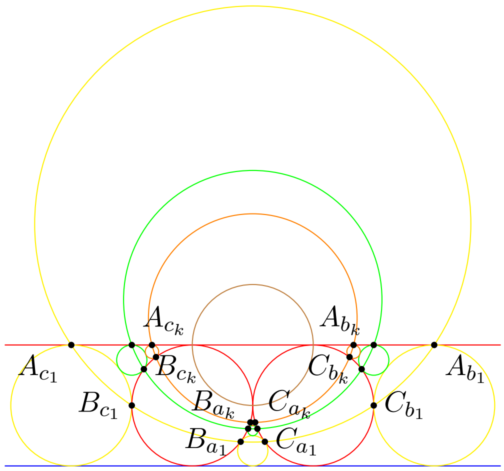

Figure 6

Let O_k be the intersection of the perpendicular bisector of $\overline{A_{b_k}C_{b_k}}$ and the y-axis (with equation $x = 0$). Then by direct computation, we have

$$O_k = \left(0, \frac{2k^2+k+1}{k^2}\right).$$

We first prove the following result.

Proposition 3. The circle S_k centered at O_k of radius $d(O_k, A_{b_k})$ passes through the six points $A_{b_k}, A_{c_k}, B_{c_k}, B_{a_k}, C_{a_k}$, and C_{b_k}. Furthermore, $S_k, k \geq 1$ is a coaxial family of circles.

Proof. By direct computation, S_k has equation given by $f(x,y) = 0$ with

$$f(x,y) = x^2 + y^2 - \frac{4k^2+2k+2}{k^2}y + 3 = 0.$$

It is directly verified that

$$f(A_{b_k}) = f(C_{b_k}) = f(C_{a_k}) = 0$$

and by symmetry also

$$f(A_{c_k}) = f(B_{c_k}) = f(B_{a_k}) = 0.$$

It remains to prove that $S_k, k \geq 1$ is a coaxial family. For this, note that $g(k) := f(x, y)$ has leading coefficient 1. By direct computation, we have that

$$g(n) - g(m) = -\frac{2(m-n)(mn+m+n)y}{m^2n^2},$$

which shows that $g(k)$ forms a family of coaxial circles with axis $y = 0$. In fact, over $\mathbb{C}$ and for $n \neq m$, we have that $g(n)$ and $g(m)$ have common intersections $(\pm i\sqrt{3}, 0)$ or $f(\pm i\sqrt{3}, 0) = 0$. □

Now to prove the result for the original Apollonian gasket, we apply an inverse Möbius transformation. This shows that the six corresponding points are concyclic. Furthermore, since the family of coaxial circles are mapped back to a coaxial family of circles, their centers are collinear, as required. □

Proof of Problem 4.

First of all, we can easily prove the case for the band Apollonian gasket using the coordinates of $T_{a_k}, T_{b_k}, T_{c_k}$ derived from the previous observations. By direct computation, the lines $T_{a_j}T_{a_k}, T_{b_j}T_{b_k}, T_{c_j}T_{c_k}$ are concurrent at the point

$$\left(0, \frac{4jk+3j+3k+3}{2jk+j+k}\right).$$

Then we can just apply Lemma 4 to map the points derived above to the general cases. However the direct application of Lemma 4 would result in messy formulas, so we now describe an alternative to reduce the computation time. Recall that any Möbius transformation is of the form

$$z \mapsto \frac{az+b}{cz+d},$$

where $ad - bc \neq 0$. The case when $c = 0$ is composed of a homothety/rotation, and a translation. The result after this transformation is obvious, provided that the initial band Apollonian case is verified. It suffices to consider the case when $c = 1$. Then the corresponding Möbius transformation can be written as

$$z \mapsto \frac{az+b}{z+d} = a - \frac{ad-b}{z+d},$$

whence barring homotheties, rotations and translations, one only needs to consider the inversion of the form $z \mapsto \frac{1}{z+d}, d = d_1 + id_2, z = x + iy$ where the corresponding transformation can be represented by the Cartesian coordinates as follows:

$$\mu : (x, y) \mapsto \left(\frac{x+d_1}{(x+d_1)^2+(y+d_2)^2}, \frac{-(y+d_2)}{(x+d_1)^2+(y+d_2)^2}\right), \tag{2}$$

where all the variables involved are now real.

Applying the above transformation in terms of coordinates, we easily computed the following:

$$T'_{a_k} = \mu(T_{a_k}) = \left(\frac{U_1}{W_1}, \frac{V_1}{W_1}\right),$$

where

$$U_1 = d_1(k+1)^2$$
$$V_1 = -(d_2k + d_2 + k)(k+1)$$

and

$$W_1 = (d_1^2 + d_2^2 + 2d_2 + 1)k^2 + 2(d_1^2 + d_2^2 + d_2)k + d_1^2 + d_2^2.$$

$$T'_{b_k} = \mu(T_{b_k}) = \left(\frac{U_2}{W_2}, \frac{V_2}{W_2}\right),$$

where

$$U_2 = 2(d_1+1)k^2 + 2(d_1+3)k + d_1 + 3,$$
$$V_2 = -(2(d_2+2)k^2 + 2(d_2+2)k + d_2)$$

and

$$W_2 = 2(d_1^2 + d_2^2 + 2d_1 + 4d_2 + 5)k^2 + 2(d_1^2 + d_2^2 + 6d_1 + 4d_2 + 9)k + d_1^2 + d_2^2 + 6d_1 + 9.$$

$$T'_{c_k} = \mu(T_{c_k}) = \left(\frac{U_3}{W_3}, \frac{U_3}{W_3}\right),$$

where

$$U_3 = 2(d_1-1)k^2 + 2(d_1-3)k + d_1 - 3,$$
$$V_3 = -(2(d_2+2)k^2 + 2(d_2+2)k + d_2),$$

and

$$W_3 = 2(d_1^2 + d_2^2 - 2d_1 + 4d_2 + 5)k^2 + 2(d_1^2 + d_2^2 - 6d_1 + 4d_2 + 9)k + d_1^2 + d_2^2 - 6d_1 + 9.$$

By direct computation, one has that lines $T'_{a_j}T'_{a_k}$ and $T'_{b_j}T'_{b_k}$ intersect at

$$\left(\frac{U_4}{W_4}, \frac{V_4}{W_4}\right),$$

where

$$U_4 = (2jk + j + k)d_1,$$
$$V_4 = -(2(d_2+2)jk + (d_2+3)j + (d_2+3)k + 3)$$

and

$$W_4 = 2(d_1^2 + d_2^2 + 4d_2 + 3)jk + (d_1^2 + d_2^2 + 6d_2 + 3)j + (d_1^2 + d_2^2 + 6d_2 + 3)k + 6d_2.$$

Since the same is true for the intersection of lines $T'_{a_j}T'_{a_k}$ and $T'_{c_j}T'_{c_k}$, it follows that the three lines $T'_{a_j}T'_{a_k}, T'_{b_j}T'_{b_k}$ and $T'_{c_j}T'_{c_k}$ are concurrent. $\square$

Proof of Problem 1, 2, 6 and 7.

Despite the similarity between the statement of Problem 4 and either of Problems 1, 2, 6 and 7, the problem types are very different in that Problems 1, 2, 6 and 7 involve the center of circles, but center of circles are in general not invariant under a Möbius transformation. Fortunately this difficulty can be resolved by working with sets of three points characterizing the circles for which we need to compute the centers. More precisely, we have the following correspondence in the band Apollonian gasket, where $(P) \longleftrightarrow (X, Y, Z)$ means that the circle (P) with center P is determined by three points X, Y, Z on it:

$$(A) \longleftrightarrow (\infty, A_{b_1}, A_{c_1}) \text{ or } (\infty, (-1,2), (1,2))$$

$$(B) \longleftrightarrow ((-1,0), B_{c_1}, B_{a_1}) \text{ or } ((-1,0), (-2,1), (0,1))$$

$$(C) \longleftrightarrow ((1,0), C_{a_1}, C_{b_1}) \text{ or } ((1,0), (0,1)), (2,1))$$

$$(A_1) \longleftrightarrow (T_{a_1}, B_{a_1}, C_{a_1})$$

$$(A_i) \longleftrightarrow (T_{a_i}, B_{a_i}, C_{a_i})$$

$$(B_1) \longleftrightarrow (T_{b_1}, C_{b_1}(2,1), A_{b_1}(3,2))$$

$$(B_i) \longleftrightarrow (T_{b_i}, C_{b_i}, A_{b_i})$$

$$(C_1) \longleftrightarrow (T_{c_1}, A_{c_1}, B_{c_1})$$

$$(C_i) \longleftrightarrow (T_{c_i}, A_{c_i}, B_{c_i})$$

For example, after a Möbius transformation μ, circle (A) will be transformed to the circle determined by $\mu(\infty), \mu(A_{b_1})$, and $\mu(A_{c_1})$. We denote the center of this circle by A', so that (A') and $\mu((A))$ represent the same circle, but $A' \neq \mu(A)$ in general. Similarly, we define B', C' and $A'_k, B'_k, C'_k, k \geq 1$. Since we have done the explicit computation for all the points in the band Apollonian gaskets, and by the sets of three points characterizing the circles for which we know how to find the centers, we can solve Problems 1, 2, 6, and 7 by straightforward computation, hence the proof. (We provide the details in each of the problems below. The SAGE [9] codes for our computation will be available upon request.) □

For the following details, we are applying the Möbius transformation given in (2).

Problem 1.

Band Apollonian gasket case:

$$A_k = \left(0, \frac{2k^2-1}{2k(k+1)}\right),$$

$$B_k = \left(\frac{k+2}{k}, \frac{2k^2-1}{k^2}\right),$$

$$C_k = \left(-\frac{k+2}{k}, \frac{2k^2-1}{k^2}\right),$$

and A_jA_k, B_jB_k, C_jC_k are concurrent at
$$\left(0, \frac{4jk+j+k+2}{2jk}\right).$$
General case: After Möbius transformation of (2),
$$A'_k = \left(\frac{U_5}{W_5}, \frac{V_5}{W_5}\right),$$
where
$$U_5 = 2d_1k(k+1),$$
$$V_5 = -(2(d_2+1)k^2 + 2d_2k - 1),$$
and
$$W_5 = 2((d_1^2+d_2^2+2d_2+1)k^2 + (d_1^2+d_2^2-1)k - d_2).$$
Similarly,
$$B'_k = \left(\frac{U'_5}{W'_5}, \frac{V'_5}{W'_5}\right),$$
where
$$U'_5 = k(d_1k+k+2)$$
$$V'_5 = -((d_2+2)k^2 - 1)$$
$$W'_5 = (d_1^2+d_2^2+2d_1+4d_2+5)k^2 + 4d_1k + 4k - 2d_2$$
and
$$C'_k = \left(\frac{U''_5}{W''_5}, \frac{V''_5}{W''_5}\right),$$
where
$$U''_5 = k(d_1k-k-2)$$
$$V''_5 = -((d_2+2)k^2 - 1)$$
and
$$W''_5 = (d_1^2+d_2^2-2d_1+4d_2+5)k^2 - 4d_1k + 4k - 2d_2.$$
By direct computation, the intersection of lines $A'_jA'_k$ and $B'_jB'_k$ is given by
$$\left(\frac{U_6}{W_6}, \frac{V_6}{W_6}\right),$$
where
$$U_6 = 2d_1jk,$$
$$V_6 = -(2(d_2+2)jk + j + k + 2)$$
and
$$W_6 = 2((d_1^2+d_2^2+4d_2+3)jk + d_2j + d_2k + 2d_2).$$
Since the same is true for the intersection of lines $A'_jA'_k$ and $C'_jC'_k$, the three lines $A'_jA'_k, B'_jB'_k$ and $C'_jC'_k$ are concurrent.

Problem 2.

Band Apollonian gasket case:
$$A = (0, 5/4), B = (-1, 1), C = (1, 1)$$

and AA_k, BB_k, CC_k are concurrent at

$$\left(0, \frac{3k-1}{2k}\right).$$

General case:

$$A' = \left(0, -\frac{1}{2(d_2+2)}\right)$$

$$B' = \left(\frac{d_1-1}{d_1^2+d_2^2-2d_1+2d_2+1}, -\frac{d_2+1}{d_1^2+d_2^2-2d_1+2d_2+1}\right)$$

$$C' = \left(\frac{d_1+1}{d_1^2+d_2^2+2d_1+2d_2+1}, -\frac{d_2+1}{d_1^2+d_2^2+2d_1+2d_2+1}\right)$$

Common intersection of $A'A'_k, B'B'_k, C'C'_k$:

$$\left(\frac{d_1 k}{(d_1^2+d_2^2+3d_2+3)k-d_2}, -\frac{(2d_2+3)k-1}{2((d_1^2+d_2^2+3d_2+3)k-d_2)}\right).$$

Problem 6.

Band Apollonian gasket case: $AT_{a_k}, BT_{b_k}, CT_{c_k}$ are concurrent at

$$\left(0, \frac{6k^2+10k+3}{4(k+1)^2}\right).$$

General case: Common intersection of $A'T'_{a_k}, B'T'_{b_k}$, and $C'T'_{c_k}$ is

$$\left(\frac{U_7}{W_7}, \frac{V_7}{W_7}\right),$$

where

$$U_7 = 4d_1(k+1)^2,$$

$$V_7 = -(2(2d_2+3)k^2+2(4d_2+5)k+4d_2+3),$$

and

$$W_7 = 2(2(d_1^2+d_2^2+3d_2+3)k^2+2(2d_1^2+2d_2^2+5d_2+6)k+2d_1^2+2d_2^2+3d_2+6).$$

Problem 7.

Band Apollonian gasket case: $A_jT_{a_k}, B_jT_{b_k}, C_jT_{c_k}$ are concurrent at

$$\left(0, \frac{8j^2k-8jk^2+6j^2-8jk-2k^2-6k-3}{2j(2jk-2k^2+j-2k-1)}\right).$$

General case: Common intersection of $A'_jT'_{a_k}, B'_jT'_{b_k}$, and $C'_jT'_{c_k}$ is

$$\left(\frac{U_8}{W_8}, \frac{V_8}{W_8}\right),$$

where

$$U_8 = 2d_1 j(2jk-2k^2+j-2k-1),$$

$$V_8 = -(4d_2j^2k-4d_2jk^2+2d_2j^2-4d_2jk+8j^2k-8jk^2-2d_2j+6j^2-8jk-2k^2-6k-3),$$

and

$$W_8 = 2(2d_1^2j^2k+2d_2^2j^2k-2d_1^2jk^2-2d_2^2jk^2+d_1^2j^2+d_2^2j^2-2d_1^2jk-2d_2^2jk+8d_2j^2k-8d_2jk^2 \\ -d_1^2j-d_2^2j+6d_2j^2-8d_2jk+6j^2k-2d_2k^2-6jk^2+3j^2-6d_2k-6jk-3d_2-3j).$$

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