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TOTALLY OMBILIC AND TOTALLY GEODESIC HYPERSURFACES OF A SILVER-RIEMANN MANIFOLDS

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ABSTRACT. In this work, we introduce a new structure which is a special case of a metallic structure on a Riemannian manifold, called silver structure. Based on the case of a structure induced on a hypersurface by a metallic Riemannian structure, we determine some properties of a structure Σ induced on a hypersurface by a silver-Riemann structure (g, Θ). Moreover we analyze the totally umbilical and geodesic hypersurfaces to complete by an example of a totally umbilical hypersurface on a 6-dimensional Euclidean space.

1. INTRODUCTION

The golden number (or golden proportion, or divine proportion) is a proportion, initially defined in geometry as the single ratio L_1/L_2 between two lengths L_1 and L_2 such as the ratio of the sum $L_1 + L_2$ of the two lengths on the larger "L_1" is equal to that of the larger "L_1" on the smaller "L_2". A divine proportion, a formula capable of delivering the beauty found in nature. Gold number supporters argue that this relationship is aesthetic because it is ubiquitous in the natural world. Aristote said that the nature of art is to imitate nature...

In 1997 a significant generalization of this number was introduced by Vera Winitzkyde Spinadel in [9] and is called the family of metallic numbers or metallic proportions. The metallic number is the positive real number given by

$$\delta_{p,q} = \frac{p + \sqrt{p^2 + 4q}}{2},$$

which is solution of the quadratic equation $x^2 - px - q = 0$ with p, q of natural integers. Thus, through this generalization we can observe other metals than gold, such as silver, nickel, copper, bronze, etc. Members of this metal family have important mathematical properties, for example the silver number was used to describe fractal geometry [2].

However, a silver Riemannian manifold is a Riemannian manifold with a non-zero (1.1) tensor field that verifies the relation $\Theta^2 = 2\Theta + I$ where I is an identical transformation. The structure Θ is inspired by the quadratic equation $x^2 - 2x - 1 = 0$ and the positive

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real $\theta_{2,1} = \delta_{2,1} = 1+\sqrt{2}$ which is the solution of this equation and called silver number. Nowadays it is undoubtedly that hypersurfaces occupy several fields of application especially in physics, which gives all the importance to their studies.
This work is divided into four parts. The first part is devoted to this introduction. A second part quive somme preliminaries wchich bring together the fundamental elements for the continuation of this work and a brief overview on the silver structure. The third part determines the properties of a structure induced.The fourth and last one give somme characterization of totally umbilical and totally geodesic hypersurface..

2. Preliminary

A (1.1)-tensor field, of a smmooth manifold M, satisfying the equation:

$$\Theta^2 = 2\Theta + I, \tag{2.1}$$

is called silver structure.
Moreover, if M is endowed with a Riemannian metric g, then the triplet (M, g, Θ) is said to be silver-Riemannian manifold if g is Θ-compatible. Otherwise if :

$$g(\Theta X, Y) = g(X, \Theta Y) \Leftrightarrow g(\Theta X, \Theta Y) = 2g(X, \Theta Y) + g(X, Y), \tag{2.2}$$

whith $X, Y \in \chi(M)$.
One can notice the existence of a relationship between a silver structure with other structures of the same categories as shown by the following propositions.

Proposition 2.1. *Any product structure induces two silver structure:*

$$\Theta_1 \;=\; I + (\theta_{2,1} - 1)F_1 \quad \textit{and} \quad \Theta_2 = I - (\theta_{2,1} - 1)F_2, \tag{2.3}$$

Inversely, a silver structure Θ induces two product structure :

$$F_1 = \Big(\frac{1}{\theta_{2,1}-1}\Theta_1 - \frac{1}{\theta_{2,1}-1}I\Big) \quad \textit{and} \quad F_2 = \Big(-\frac{1}{\theta_{2,1}-1}\Theta_2 + \frac{1}{\Theta_{2,1}-1}I\Big). \tag{2.4}$$

Hence the two projections associated to the tangent space decomposition by a product structure are given by:

$$p_1 = \frac{\theta_{2,1}}{2\theta_{2,1}-2}I - \frac{\Theta_1}{2\theta_{2,1}-2} \qquad \text{and} \qquad p_2 = \frac{\theta_{2,1}-2}{2\theta_{2,1}-2}I + \frac{\Theta_2}{2\theta_{2,1}-2}. \tag{2.5}$$

See [4] for more details.

Proposition 2.2. [4]

(1) *The eigen values of Θ are the real numbers $\theta_{2,1}$ and $2 - \theta_{1,2}$.*
(2) *The silver structure Θ is an isopmorphism of the tangent space of M at any point x in M.*
(3) *The silver structure Θ in invertible with inverse map $\tilde{\Theta}$ satisfying:*

$$2\tilde{\Theta}^2 + \tilde{\Theta} - I = 0.$$

3. Hypersuface of a silver- Riemann manifold

By analogie to Riemannian submanifold, one can construct objects of a hypersurface H since a hypersurface is a submanifold with codmimension 1.
Now let H be a hypersurface isometrically immersed in a locally silver-Riemannian $n+1$-manifold M ($\nabla\Theta = 0$) with $n \in \mathbb{N}$. Hence we have the following decomposition :

$$T_xH \oplus T_xH^{\perp} \tag{3.1}$$

with T_xH the tangent space to the hypersurface H at a point $x \in H$ and $T_xH^{\perp}$ normal space of T_xH.
It's well know that there is a natural unique connection ∇ of M satisfying the Gauss and Weingarten formulas:

$$\overline{\nabla}_X Y = \nabla_X Y + h_1(X,Y), \tag{3.2}$$
$$\overline{\nabla}_X N_1 = -(A_1)_{N_1}X, \tag{3.3}$$

with $(A_1)_{N_1}$ the Weingarten operator $(ie\ :\ (\nabla^H_{N_1}X)^{\top})$, ∇^H is induced connection by ∇ on H and $h(X,Y) = g((A_1)_{N_1}X, Y)$.
In the following we will consider $(A_1)_{N_1} = A_1$.
Let i be an immersion $H \longrightarrow M$. The induce metric $g_{|H}$ on H by the Riemanian metric g is giving by :

$$g_{|H}(X,Y) = g(i_*X, i_*Y) \quad \forall X, Y \in \chi(H)$$

with i_* is the differantial immersion of i. Let consider a local base $\{N_1\}$ of the normal tangent space $T_xH^{\perp}$ because the hypersurface H is codimension 1. The two silver vectors are respectively given via decomposition (3.1), by:

$$\Theta(i_*X) = i_*(PX) + u_1N_1, \tag{3.4}$$
$$\Theta(N_1) = i_*(\xi_1) + a_{11}N_1, \tag{3.5}$$

where P is a (1.1)-tensor field induce on the hypersurface$H$, $\xi_1 \in \chi(TH)$, u_1 is a 1-forme on H and a_{11} is a smooth real function on H.
By taking $X = (i_*X)$ the formulas (3.4) and (3.5) become :

$$\Theta(X) = PX + u_1N_1, \tag{3.6}$$
$$\Theta(N_1) = \xi_1 + a_{11}N_1. \tag{3.7}$$

The induce structure on the hypersurface by the silver-Riemann structure (g,Θ) is called Σ-structure on H if it is giving by the quintuple:

$$\Sigma = (P, g, \xi_1, u_1, a_{11}). \tag{3.8}$$

Hence, one has the following proposition.

Proposition 3.1. *The Σ-structure induce on the hypersurface H by a silver-Riemann structure (g, Θ) verifies the folowing relations:*

$$\begin{aligned} P^2X &= 2P(X) + X - u_1(X)\xi_1, && (3.9)\\ u_1(PX) &= 2u_1(X) - a_{11}u_1(X), && (3.10)\\ u_1(\xi_1) &= 1 + 2a_{11} - a_{11}^2, && (3.11)\\ u_1(X) &= g(X, \xi_1), && (3.12)\\ P(\xi_1) &= 2\xi_1 - a_{11}\xi_1, && (3.13)\\ g(PX, Y) &= g(X, PY), && (3.14)\\ g(PX, PY) &= 2g(X, PY) + g(X, Y) + u_1(X)u_1(Y). && (3.15) \end{aligned}$$

Proof. For all $X, Y \in \chi(TH)$, by applying Θ to the relation (3.6), one has:

$$\Theta^2(X) = \Theta(PX) + u_1(X)\Theta(N_1).$$

On the one hand, one has:

$$\Theta^2(X) = 2\Theta(X) + X = 2(PX + u_1N_1) + X = 2\Theta(X) + X = 2PX + X + 2u_1(X)N_1.$$

On the other hand one has:

$$\begin{aligned} \Theta(PX) + u_1(X)\Theta(N_1) &= P^2X + u_1(PX)N_1 + \xi_1u_1(X) + u_1(X)a_{11}N_1\\ &= P^2X + \xi_1u_1(X) + u_1(PX)N_1 + u_1(X)a_{11}N_1. \end{aligned}$$

Equalizing these two relations and identifying the normal and the tangent part, one obtain the two first relations.
Now let's calculate

$$\Theta^2(N_1) = \Theta\xi_1 + a_{11}\Theta(N_1).$$

On the one hand, one has:

$$\Theta^2(N_1) = 2\Theta(N_1) + N_1 = 2(\xi_1 + a_{11}N_1) + N_1 = 2\xi_1 + 2a_{11}N_1 + N_1.$$

On the other hand, one has:

$$\Theta\xi_1 + a_{11}\Theta(N_1) = P(\xi_1) + u_1(\xi_1) + a_{11}(\xi_1 + a_{11}N_1).$$

Equalizing these two relations and following the normal and the tangent we obtain the relation (3.11) and (3.13).
The relation (3.14) is determinated by a direct calculation using the relation (2.2).
At last, from (2.2) it follows:

$$g(\Theta X, \Theta Y) = 2g(X, \Theta Y) + g(X, Y).$$

On the one hand, one has:

$$\begin{aligned} g(\Theta X, \Theta Y) &= g(PX + u_1(X)N_1, PY + u_1(Y)N_1)\\ &= g(PX, PY) + g(u_1(X)N_1, PY) + g(PX, u_1(Y)N_1) + g(u_1(X)N_1, u_1(Y)N_1). \end{aligned}$$

On the other hand, one has:

$$\begin{aligned} 2g(X, \Theta Y) + g(X, Y) &= 2g(X+, PY + u_1(Y)N_1) + g(X, Y)\\ &= 2g(X, PY) + 2g(X, u_1(Y)N_1) + g(X, Y). \end{aligned}$$

□

On a silver-Riemann manifold (M, g, Θ), the silver structure Θ is said to be parallel to the Levi-Civita connection $\overline{\nabla}$ associated to the Riemann metric g if $\overline{\nabla}\Theta = 0$, for all $X \in \chi(TM)$. For this class of siver-Riemann manifold, one has the following proposition.

Proposition 3.2. *Let H be a hypersurface of a silver-Riemann manifold (M, g, Θ) such that Θ is parallel to the associated Levi-Civita connection $\overline{\nabla}$. Then the elements of the induce structure Σ on H satisfy:*

$$(\nabla_X P)(Y) = g(A_1X, Y)\xi_1 + u_1(Y)A_1X, \tag{3.16}$$

$$(\nabla_X u_1)(Y) = g(A_1X, PY) + a_{11}g(A_1X, Y), \tag{3.17}$$

$$\nabla_X \xi_1 = -P(A_1X) + a_{11}A_1X, \tag{3.18}$$

$$X(a_{11}) = -2u_1(A_1X) = -2g(A_1X, \xi_1) = -2g(A_1X, \xi_1) = -2g(X, A_1\xi_1) \tag{3.19}$$

for all $X, Y \in \chi(TH)$.

Proof. For all $X, Y \in \chi(TH)$, one has:

$$(\overline{\nabla}_X\Theta)Y = \overline{\nabla}_X\Theta Y - \Theta\overline{\nabla}_X Y = 0 \Longrightarrow \overline{\nabla}_X\Theta Y = \Theta\overline{\nabla}_X Y.$$

On the one hand, one has:

$$\begin{aligned}\overline{\nabla}_X\Theta Y &= \nabla_X(PY + u_1(Y)N_1) = \nabla_X PY + \nabla_X u_1(Y)N_1 \\ &= \nabla_X PY + h_1(X, PY) - X(u_1(Y))N_1 + u_1(Y)(A_1X).\end{aligned}$$

On the other hand, one has:

$$\Theta\overline{\nabla}_X Y = \Theta(\nabla_X) + h(X, Y)\Theta N_1 = P(\nabla_X) + h(X, Y)\xi_1 + a_{11}h(X, Y)N_1.$$

Taking the normal and tangential part and using the fact that $h(X, Y) = g(A_1X, Y)$ we get the first two relations.

$$(\overline{\nabla}_X\Theta)N_1 = \overline{\nabla}_X\Theta N_1 - \overline{\nabla}_X N_1 = 0 \Longrightarrow \overline{\nabla}_X\Theta N_1 = \Theta\overline{\nabla}_X N_1.$$

On the one hand, one has:

$$\overline{\nabla}_X\Theta N_1 = \nabla_X(\xi_1 + a_{11}N_1) = h_1(X, Y) + \nabla_X\xi_1 + X(a_{11})N_1 - a_{11}A_1X$$

On the other hand, one has:

$$\Theta\overline{\nabla}_X N_1 = \Theta(-A_1X) = -PA_1X - u_1(A_1X)N_1$$

Taking the normal and tangential part and using the fact that $h(X, Y) = g(A_1X, Y)$ and $u_1(A_1X) = g(A_1X, \xi)$ one get the last two relations. $\square$

4. Somes properties of totally ombilical and totally geodesic hypersurface

A hypersurface is said to be totally ombilical if the Weingarten operator satisfies $A_1 = \lambda I$. In this case, one has the following theorem.

Theorem 4.1. *If the hypersurface H is totally ombilical in a silver-Riemann manifold (M, g, Θ) with $(\overline{\nabla}\Theta = 0)$ and Σ the induce structure on H, then the elements of Σ satisfy the following properties:*

$$\begin{aligned} (\nabla_X P)(Y) &= \lambda\big[g(X,Y)\xi_1 + g(Y,\xi_1)\big], & (4.1)\\ (\nabla_X u)(Y) &= \lambda\big[a_{11}g(X,Y) - g(X,PY)\big], & (4.2)\\ (\nabla_X u)\xi &= \lambda(a_{11}X - P(X)), & (4.3)\\ \nabla_\xi \xi &= \lambda(2a_{11}-1)\xi, & (4.4)\\ X(a_{11}) &= -2\lambda g(X,\xi). & (4.5) \end{aligned}$$

Proof. For all $X, Y \in \chi(H)$ one has respectively the following results :

$$\begin{aligned} (\nabla_X P)Y &= g(\lambda X, Y)\xi_1 + u_1(Y)\lambda X \\ &= \lambda\big(g(\lambda X, Y)\xi_1 + g(Y,\xi_1)\big). \end{aligned}$$

And

$$\begin{aligned} (\nabla_X u)Y &= -g(\lambda X, PY) + a_{11}g(\lambda X, Y) \\ &= \lambda(a_{11}g(X,Y) - g(X,PY)). \\ \nabla_X \xi_1 &= -P(\lambda X) + a_{11}\lambda X \\ &= \lambda(a_{11}X - PX). \\ \nabla_{\xi_1}\xi_1 &= -P(\lambda\xi_1) + a_{11}\lambda\xi_1 \\ &= -\lambda(P\xi_1) + a_{11}\xi_1 \\ &= a_{11}\xi_1\lambda - \lambda(2\xi_1 - a_{11}\xi_1) \\ &= -2\lambda(a_{11} - I)\xi_1. \\ X(a_{11}) &= -2\lambda g(X,\xi_1). \end{aligned}$$

□

Corollary 4.1. *Let H be a totally ombilical hypersurface of a silver-Riemann manifold (M, g, Θ) with $(\overline{\nabla}\Theta = 0)$, for all $X, Y \in \chi(M)$. One has :*

$$\begin{aligned} (\nabla_X P)(\xi_1) &= \lambda(1 + 2a_{11} - a_{11}^2)X, & (4.6)\\ (\nabla_{\xi_1} P)(X) &= 2\lambda g(X,\xi_1)\xi_1, & (4.7)\\ (\nabla_X u)\xi &= 2a_{11}\lambda g(X,\xi) - 2\lambda g(X,\xi) = 2\lambda(a_{11}-1)g(X,\xi). & (4.8) \end{aligned}$$

Proof. The hypersuface H being totally umbilical then according to (3.3) and (3.16), one has:

$$\begin{aligned}(\nabla_X P)(\xi_1) &= g(A_1X,\xi_1)\xi_1 + u_1(\xi_1)A_1X\\ (\nabla_X P)(Y) &= g(-\nabla_X N_1,\xi_1)\xi_1 + \lambda u_1(\xi_1)X\\ &= \lambda u_1(\xi_1)X.\end{aligned}$$

As $u_1(\xi_1) = 1 - 2a_{11} + a_{11}^2$,then :

$$(\nabla_X P)(\xi_1) = \lambda(1 + 2a_{11} - a_{11}^2)X.$$

For $Y = \xi$, one has:

$$\begin{aligned}(\nabla_{\xi_1} P)(Y) &= g(A_1\xi_1, Y)\xi_1 + u_1(Y)A_1\xi_1\\ &= \lambda g(\xi, Y)\xi_1 + \lambda g(\xi_1, Y)\xi_1.\end{aligned}$$

For $Y = X$ and as the metric g is symetric, one has:

$$\begin{aligned}(\nabla_{\xi_1} P)(X) &= g(A_1\xi_1, X)\xi_1 + u_1(X)A_1\xi_1\\ &= \lambda g(\xi, X)\xi_1 + \lambda g(\xi_1, X)\xi_1\\ &= 2\lambda g(\xi_1, X)\xi_1.\\ (\nabla_X u_1)Y &= a_{11}g(\lambda X, Y) - g(\lambda X, PY).\end{aligned}$$

For $Y = \xi_1$ and $P\xi_1 = (2 - a_{11})\xi_1$, one has :

$$\begin{aligned}(\nabla_X u_1)\xi_1 &= a_{11}g(\lambda X, \xi_1) - g(\lambda X, P\xi_1)\\ &= a_{11}g(\lambda X, \xi_1) - g(\lambda X, (2 - a_{11})\xi_1)\\ &= 2\lambda(a_{11}g(X, \xi_1) - g(\lambda X, \xi_1)).\end{aligned}$$

□

Proposition 4.1. *Let Σ be the induce structure on a totally ombilic hypersurface H of a silver-Riemann manifold (M, g, Θ) with $(\overline{\nabla}\Theta = 0)$, for all $X \in \chi(M)$. Then one has :*

$$\nabla_X u_1 = 0 \Longleftrightarrow \nabla_X \xi_1 = 0. \tag{4.9}$$

Proof. Suppose that $(\nabla_X u)Y = 0$. Then:

$$(\nabla_X u)Y = a_{11}g(A_1X, Y) - g(A_1X, PY) = 0.$$

As the tensor P is g-compatible, then:

$$g((PA_1 - A_1a_{11})X, Y) = 0.$$

As $\nabla_X\xi_1 = -P(A_1X) + A_1a_{11}X$, then:

$$g(\nabla_X\xi_1, Y) = 0 \Longrightarrow \nabla_X\xi_1 = 0 \text{ because } Y \neq 0.$$

Conversely, suppose that $\nabla_X\xi_1 = 0$.Then:

$$\begin{aligned}g(\nabla_X\xi_1, Y) = 0 \Longrightarrow g((PA_1 - A_1a_{11})X, Y) = 0 &\Longrightarrow a_{11}g(A_1X, Y) - g(A_1X, PY) = 0\\ &\Longrightarrow \nabla_X u_1 = 0.\end{aligned}$$

□

Proposition 4.2. *If H is a totally ombilical hypersurface of a silver-Riemann manifold (M, g, Θ), then the 1-forme u_1 is a closed form.*

Proof. The hypersurface H being totally ombilical then:

$$(\nabla_X u_1)Y \quad = \quad \lambda(a_{11}g(X,Y) - g(X,PY))$$

and

$$(\nabla_Y u_1)X \quad = \quad \lambda(a_{11}g(Y,X) - g(Y,PX)).$$

As $du_1(X,Y) = (\nabla_X u_1)Y - (\nabla_Y u_1)X$ and g is symetric and P-compatible, then:

$$(\nabla_X u_1)Y - (\nabla_Y u_1)X = \lambda(a_{11}g(X,Y) - g(X,PY)) + \lambda(-a_{11}g(Y,X) + g(Y,PX)).$$

□

Proposition 4.3. *Let H be a totally ombilical hypersurface of a silver-Riemann manifold (M, g, Θ) with $\overline{\nabla}\Theta = 0$ and ξ_1 a Killing vector field. If $a_{11} \neq \theta_{2,1}$ then the Weingarten operator A_1 is rank 1 , with eigen vector ξ_1 and eigen value η given by:*

$$\eta = \frac{\xi_1(a_{11})}{2(a_{11}^2 - 2a_{11} - 1)}. \tag{4.10}$$

Proof. The hypersurface H being totally geodesic and the tensor P and A_1 commut, then one has:

$$\begin{aligned} P^2(A_1X) = a_{11}^2(A_1X) &\Longrightarrow (2P(A_1X) + (A_1X) - u_1(A_1X)\xi_1) = a_{11}^2(A_1X) \\ &\Longrightarrow (2a_{11}(A_1X) + (A_1X) - u_1(A_1X)\xi_1) = a_{11}^2(A_1X) \\ &\Longrightarrow u_1(A_1X)\xi_1) = a_{11}^2(A_1X) - 2a_{11}(A_1X) - (A_1X) \\ &\Longrightarrow -u_1(A_1X)\xi_1) = (a_{11}^2 - 2a_{11} - 1)(A_1X). \end{aligned}$$

As $X(a_{11}) = 2u_1(A_1X)$, then:

$$\frac{X(a_{11})}{2}\xi_1 = (a_{11}^2 - a_{11} - 1)A_1X \Longrightarrow A_1X = \frac{X(a_{11})}{2(a_{11}^2 - a_{11} - 1)}\xi_1,$$

at last, just take $X = \xi_1$ and one has the value of η. □

Proposition 4.4. *Let H be a hypersurface of a silver-Riemann manifold (M, g, Θ) and $\xi_1 \neq 0$ a Killing verctor field. The hypersurface is totally geodesic if and only if for all $X \in \chi(M)$*

$$\nabla_X P = 0. \tag{4.11}$$

Proof. A hypersurface is totally geodesic if the operateur $A_1 = 0$. Then:

$$(\nabla_X P)(Y) \quad = \quad g(A_1X, Y)\xi_1 + u_1(Y)A_1X \Longrightarrow (\nabla_X P)Y = 0.$$

Suppose $g(A_1X, Y)\xi_1 + u_1(Y)A_1X = 0$, then one has the following conditions:

- If the vectors A_1X and ξ_1 are lineary independant, then there is a scalar α such that $A_1X = \alpha\xi_1$ and one has:

  $g(\alpha\xi_1, Y)\xi_1 + u_1(Y)\alpha\xi_1 = 0 \Longrightarrow 2\alpha g(\xi_1, Y)\xi_1 = 0 \Longrightarrow g(\xi_1, Y)\xi_1 = 0.$

  Then for $Y = \xi$ one has:

  $$g(\xi_1, \xi_1)\xi_1 = 0 \Longrightarrow \xi_1 = 0.$$

  This is contradictory.

- If the vectors A_1X and ξ_1 are free, then one has:

  $$g(A_1X, Y) = 0,$$

  then $A_1 = 0$, which implies that the hypersurface is totally goedesic on M.

□

Proposition 4.5. *Let H be a hypersurface of a silver-Riemann manifold (M, g, Θ) with $\nabla\Theta = 0$. Then ξ_1 is a Killing vector field with respect to the metric g on the hypersurface H if and only if*

$$2a_{11}A_1 = PA_1 + A_1P, \tag{4.12}$$

where A_1 is the Weingarten operator on H.

Proof. The vector field ξ_1 is said to be Killing if and only if:

$$(L_{\xi_1}g)(Y, Z) = 0 \quad \forall Y, Z \in \chi(H).$$

$$\begin{aligned}
L_{\xi_1}g(Y,Z) - g(L_{\xi_1}Y, Z) - g(Y, L_{\xi_1}Z) &= 0 \\
\xi_1 g(Y,Z) - g([\xi_1, Y], Z) - g(Y, [\xi_1, Z]) &= 0 \\
g(\nabla_\xi Y, Z) + g(Y, \nabla_\xi Z) - g(\nabla_{\xi_1}Y - \nabla_Y\xi_1, Z) - g(Y, \nabla_{\xi_1}Z - \nabla_Z\xi_1) &= 0 \\
g(\nabla_Y\xi_1, Z) + g(Y, \nabla_Z\xi) &= 0.
\end{aligned}$$

As $\nabla_X\xi_1 = -P(A_1X) + a_{11}(A_1X)$ one has:

$$\begin{aligned}
g(-P(A_1Y) + a_{11}(A_1Y), Z) + g(Y, -P(A_1Z) + a_{11}(A_1Z)) &= 0 \\
g((-PA_1 + a_{11}A_1)Y, Z) + g((-PA_1 + a_{11}A_1)Y, Z) &= 0 \\
g((-PA_1 + a_{11}A_1 - A_1P + a_{11}A_1)Y, Z) &= 0 \\
g((-PA_1 - A_1P + 2a_{11}A_1)Y, Z) &= 0.
\end{aligned}$$

□

5. Example

Let consider the ambient space $\mathbb{E}^6 = \mathbb{E}^{2\times 2+2} = \mathbb{E}^2 \times \mathbb{E}^2 \times \mathbb{E}^2$, a point of $\mathbb{E}^6$ is a 6-uplet giving by :

$$(x^1, x^2, y^1, y^2, z^1, z^2) = (x^i, y^i, z^j),$$

with $i = j = 1, 2$. The tangent space to $\mathbb{E}^6$ at a point x is isomorphic to $\mathbb{E}^6$. Let

$$\begin{array}{rcl}
\Theta : \ \mathbb{E}^6 & \to & \mathbb{E}^6 \\
(x^i, y^i, z^j) & \longmapsto & \Theta(x^i, y^i, z^j) = \big(\theta_{2,1}x^i, \theta_{2,1}y^i, (2-\theta_{2,1})z^j\big).
\end{array}$$

be an application. Then Θ is a silver structure d. Indeed,

$$\Theta^2(x^i, y^i, z^j) \quad = \quad \big(\theta_{2,1}^2 x^i, \theta_{2,1}^2 y^i, (2-\theta_{2,1})^2 z^j\big).$$

As $\theta_{2,1}$ and $2 - \theta_{2,1}$ are solutions of the equation $x^2 = 2x + 1$ then

$$\theta_{2,1}^2 \quad = \quad 1 + 2\theta_{2,1}, \quad (2-\theta_{2,1})^2 = 1 + (2-\theta_{2,1}),$$

and

$$\begin{aligned}\Theta^2(x^i,y^i,z^j) &= \big(\theta_{2,1}x^i,\theta_{2,1}^2y^i,(2-\theta_{2,1})^2z^j\big)\\ &= \big((1+2\theta_{2,1})x^i,(1+2\theta_{2,1})y^i,(1+(2-\theta_{2,1}))z^j\big)\\ &= (x^i,y^i,z^j)+2\big(\theta_{2,1}x^i,\theta_{2,1}y^i,(2-\theta_{2,1})z^j\big)\\ &= 2\Theta(x^i,y^i,z^j)+(x^i,y^i,z^j).\end{aligned}$$

Hence

$$\Theta^2=2\Theta+I.$$

Let $<>$ be a Riemannian metric on $\mathbb{E}^6$. Then one has:

$$\begin{aligned}<\Theta(x^i,y^i,z^j),(x_1^i,y_1^i,z_1^j)> &= <(\theta_{2,1}x^i,\theta_{2,1}y^i,(2-\theta_{2,1})z^j),(x_1^i,y_1^i,z_1^j)>\\ &= \theta_{2,1}x^ix_1^i+\theta_{2,1}y^iy_1^i+(2-\theta_{2,1})z^jz_1^j\\ &= x^i\theta_{2,1}x_1^i+y^i\theta_{2,1}y_1^i+z^j(2-\theta_{2,1})z_1^j\\ &= <(x^i,y^i,z^j),(\theta_{2,1}x_1^i,\theta_{2,1}y_1^i,(2-\theta_{2,1})z_1^j)>\\ &= <(x^i,y^i,z^j),\Theta(x_1^i,y_1^i,z_1^j)>.\end{aligned}$$

The triplet $(\mathbb{E}^6,<>,\Theta)$ is a silver-Riemann manifold.
In the space $\mathbb{E}^6$ one obtain the hypersurface

$$S^5(R)=\{(x^i,y^i,z^j)\in\mathbb{E}^6,\ \sum_{i=1}^{2}(x^i)^2+\sum_{i=1}^{2}(y^i)^2+\sum_{i=1}^{2}(z^j)^2=R^2\}$$

with $R=r^2+r_3^2$ where $r^2=r_1^2+r_2^2$ and

$$\sum_{i=1}^{2}(x^i)^2 = r_1^2,\quad \sum_{i=1}^{2}(y^i)^2=r_2^2,\quad \sum_{i=1}^{2}(z^j)^2=r_3^2.$$

Note that the only normal field of $S^5(R)$ is the vector

$$N_1=\frac{1}{R}(x^i,y^i,z^j),$$

and in addition

$$\Theta(N_1)=\frac{1}{R}(\theta_{2,1}x^i,\theta_{2,1}y^i,(2-\theta_{2,1})z^j).$$

For the tangent vector field X on $S^5(R)$ one can set:

$$X=(X^i,Y^i,Z^j).$$

As a result, we have:

$$\begin{aligned}2\sum_{i=1}(x^i)X^i+2\sum_{i=1}^{2}(y^i)Y^i+2\sum_{i=1}^{2}(z^j)Z^j &= 0\\ \sum_{i=1}(x^i)X^i+\sum_{i=1}^{2}(y^i)Y^i+\sum_{i=1}^{2}(z^j)Z^j &= 0.\end{aligned}$$

Thus, the vectors $\Theta(N_1)$ and $\Theta(N_1)$ can be writ into tangential and normal components on $T_{(x,y,z)}S^5(R)$:

$$\begin{aligned} \Theta(N_1) &= \xi_1 + A_1 N \\ \Theta(X^i, Y^i, Z^j) &= P(X^i, Y^i, Z^j) + u_1(X^i, Y^i, Z^j), \end{aligned}$$

where A_1 is a smooth real function on $S(R)^5$. Using

$$\begin{aligned} A_1 &= < \Theta(N_1), N_1 > \\ \xi_1 &= \Theta(N_1) - A_1 N_1 \end{aligned}$$

and

$$u_1 =< u_1(X^i, Y^i, Z^j), \xi_1 > .$$

$$P(X^i, Y^i, Z^j) = \Theta(X^i, Y^i, Z^j) - u_1(X^i, Y^i, Z^j),$$

thus, the elements of the induce structure Σ on the hypersurface $S(R)^5$ by the structure $(\Theta, <>)$ on $\mathbb{E}^6$ are

$$\begin{aligned} A_1 &= \frac{\theta_{2,1} r^2 + (2 - \theta_{2,1}) r_3^2}{R^2}, \\ \xi_1 &= \frac{2\theta_{2,1} - 2}{R^3} (r_3^2 x^i, r_3^2 y^i, -r^2 z^j), \\ u_1(X) &= \frac{1}{R} (\theta_{2,1} \sum_1^2 (x^i X^i + y^i Y^j) + (2 - \theta_{2,1}) \sum_{j=1}^2 z^j Z^j), \\ P(X) &= (\theta_{2,1} X^i - \frac{1}{R} u(X) x^i, \theta_{2,1} Y^i - \frac{1}{R} u_1(X) y^i, (2 - \phi) Z^j - \frac{1}{R} u_1 u(X) z^j). \end{aligned}$$

Indeed,

$$\begin{aligned} A_1 &= < \Theta(N_1), N_1 >, \\ < \Theta(N_1), N_1 > &= < \xi + A_1 N_1, N_1 >, \\ < \Theta(N_1), N_1 > &= < A_1 N_1, N_1 >, \\ < \Theta(\frac{1}{R}(x^i, y^i, z^j), \frac{1}{R}(x^i, y^i, z^j) > &= < A_1 N_1, N_1 >, \\ \frac{1}{R^2} < \Theta(x^i, y^i, z^j), (x^i, y^i, z^j) > &= < A_1 N_1, N_1 >, \\ \frac{1}{R^2} < (\theta_{2,1} x^i, \theta_{2,1} y^i, (2 - \theta_{2,1}) z^j), (x^i, y^i, z^j) > &= < A_1 N_1, N_1 >, \\ \frac{1}{R^2} \Big(\theta_{2,1} \sum_{i=1}^2 (x^i)^2 + \theta_{2,1} \sum_{i=1}^2 (y^i)^2 + (2 - \theta_{2,1}) \sum_{i=1}^2 (z^j)^2 \Big) &= \frac{A_1}{R^2} < (x^i, y^i, z^j), (x^i, y^i, z^j) >, \\ \Big(\theta_{2,1} \sum_{i=1}^2 (x^i)^2 + \theta_{2,1} \sum_{i=1}^2 (y^i)^2 + (2 - \theta_{2,1}) \sum_{i=1}^2 (z^j)^2 \Big) &= \Big(\sum_{i=1}^2 (x^i)^2 + \sum_{i=1}^2 (y^i)^2 + \sum_{i=1}^2 (z^j)^2 \Big). \end{aligned}$$

As $\sum_{i=1}^{2}(x^i)^2 = r_1^2, \sum_{i=1}^{2}(y^i)^2 = r_2^2, \sum_{i=1}^{2}(z^j)^2 = r_3^2$. then one has:

$$\begin{aligned}
\Big(\theta_{2,1}r_1^2 + \theta_{2,1}r_2^2 + (2-\theta_{2,1})r_3^2\Big) &= A_1\Big(r_1^2 + r_2^2 + r_3^2\Big),\\
\Big(\theta_{2,1}r^2 + (2-\theta_{2,1})r_3^2\Big) &= A_1\Big(R^2\Big) \Longrightarrow A_1 = \frac{\Big(\theta_{2,1}r^2 + (2-\theta_{2,1})r_3^2\Big)}{R^2}.
\end{aligned}$$

Then

$$\begin{aligned}
\xi_1 &= \Theta(N_1) - A_1N_1,\\
&= \frac{1}{R}\Big(\theta_{2,1}x^i, \theta_{2,1}y^i, (2-\theta_{2,1})z^j\Big) - \frac{A_1}{R}(x^i, y^i, z^j)\\
&= \frac{1}{R}\Big[\Big((\theta_{2,1}-A_1)x^i, (\theta_{2,1}-A_1)y^i, (A_1-(2-\theta_{2,1})z^j\Big)\Big]\\
&= \frac{1}{R}\Big[\Big(\Big(\theta_{2,1} - \frac{(\theta_{2,1}r^2 + (2-\theta_{2,1})r_3^2)}{R^2}\Big)x^i, \Big(\theta_{2,1} - \frac{(\theta_{2,1}r^2 + (2-\theta_{2,1})r_3^2}{R^2}\Big)y^i, \Big((2-\theta_{2,1})\\
&- \frac{(\theta_{2,1}r^2 + (2-\theta_{2,1})r_3^2)}{R^2}\Big)z^j\Big)\Big].
\end{aligned}$$

Thus

$$\begin{aligned}
u_1(X) &= u_1(X^i, Y^i, Z^j) =< (X^i, Y^i, Z^j), \xi >\\
&= < (X^i, Y^i, Z^j), \frac{1}{R}\Big[\Big(\Big(\theta_{2,1} - \frac{(\theta_{2,1}r^2 + (2-\theta_{2,1})r_3^2)}{R^2}\Big)x^i\\
&, \quad \Big(\theta_{2,1} - \frac{(\theta_{2,1}r^2 + (2-\theta_{2,1})r_3^2}{R^2}\Big)y^i, \Big((2-\theta_{2,1}) - \frac{(\theta_{2,1}r^2 + (2-\phi)r_3^2)}{R^2}\Big)z^j\Big)\Big] >\\
&= \frac{\theta_{2,1} - A_1}{R}\Big(\sum_{i=1}^{2} x^iX^i + y^iY^i\Big) + \frac{(2-\theta_{2,1}) - A_1}{R}\sum_{j=1}^{2} z^jZ^j.
\end{aligned}$$

As a consequence

$$\begin{aligned}
P(X) &= (X^i, Y^i, Z^i) = \Theta(X^i, Y^i, Z^i) + u_1(X^i, Y^i, Z^i)N_1,\\
&= (\theta_{2,1}X^i, \theta_{2,1}Y^i, (2-\theta_{2,1})Z^i) - \frac{u_1(X)}{R}(x^i, y^i, z^j),\\
&= \Big(\theta_{2,1}X^i - \frac{u_1(X)}{R}x^i, \theta_{2,1}Y^i - \frac{u_1(X)}{R}y^i, (2-\theta_{2,1})Z^i - \frac{u_1(X)}{R}z^j)\Big).
\end{aligned}$$

In conclusion $\Sigma = (P, <>, \xi_1, u_1, a_{11})$ is an induce structure on $S(R)^5$ by the silver structure Θ of $\mathbb{E}^6$.
At last $S(R)^5$ is a totally ombilic in $\mathbb{E}^6$.

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