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KILLING VECTOR FIELDS ON FOUR-DIMENSIONAL STRICT WALKER METRICS

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ABSTRACT. In this paper, we explore the existence and characterization of Killing vector fields on four-dimensional strict Walker manifolds. We derive necessary conditions under which a vector field qualifies as a Killing field within these specific pseudo-Riemannian structures. We also give a remark on the existence and characterization of Ricci solitons on four-dimensional strict Walker manifolds

1. INTRODUCTION AND MOTIVATIONS

A pseudo-Riemannian metric g on a four-dimensional manifold M is called a Walker metric if there exists a two dimensional null distribution on M, that is parallel with respect to the Levi-Civita connection of g. This type of metrics was introduced by Walker [13] who demonstrated that such metrics have a local canonical form dependent on three smooth functions. Various curvature properties of special classes of Walker metrics have been studied in [2, 3, 6, 7] where several examples of neutral metrics with interesting geometric properties are provided. Many examples of Walker structures have emerged, proving to be significant in both differential geometry and general relativity .

In [1], the authors study magnetic curves on Walker manifolds in order to classify the corresponding Killing magnetic curves. By imposing conditions for the existence of Killing vector fields on a 3-dimensional Walker manifold, they obtain classifications of these fields. As a result, they prove that the coordinate vector field ∂_x is Killing on the 3-dimensional Walker manifold if and only if the Walker manifold is strict. In [5], the authors study Killing vector fields on a family of Walker manifolds of dimension four. Motivated by the works [1, 5] , we investigate Killing vector fields on four-dimensionnal strict Walker manifolds. As a result, we derive certain conditions that restrict the existence of such vector fields on these spaces. The problem of the existence of Killing vector fields in pseudo-Riemannian manifolds has been analyzed by many authors (both physicists and mathematicians) with different points of view and by using several techniques [8, 11, 12].

The structure of the paper is as follows. In section 2, we present the strict Walker metric considered in this study. In section 3, we provide a characterization of Killing vector

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fields on four-dimensional strict Walker manifolds, along will illustrative examples. Finally, we give a observation on the existence of Ricci solitons on four-dimensional strict Walker manifolds.

2. On the strict Walker metrics

A four-dimensional pseudo-Riemannian manifold (M, g) of signature $(2, 2)$ is said to be a Walker manifold if it admits a parallel totally isotropic 2-plane field. Such a manifold is locally isometric to (U, g) where U is an open subset of $\mathbb{R}^4$ and the metric g is given with respect to the coordinate vector field by

$$g(\partial_1, \partial_3) = g(\partial_2, \partial_4) = 1 \text{ and } g(\partial_i, \partial_j) = g_{ij}(x_1, x_2, x_3, x_4) \text{ for } i, j = 3, 4,$$

where $\partial_i := \frac{\partial}{\partial x_i}$. One says that M is a strict Walker manifold if $g(\partial_i, \partial_j) = g_{ij}(x_3, x_4)$ for $i, j = 3, 4$. In this paper, we consider the family of metrics $g_{a,b,c}$ given by

$$\begin{aligned} g_{a,b,c} &= 2(dx_1 \circ dx_3 + dx_2 \circ dx_4) + a(x_3, x_4)dx_3 \circ dx_3 \\ &\quad + b(x_3, x_4)dx_4 \circ dx_4 + 2c(x_3, x_4)dx_3 \circ dx_4, \end{aligned} \tag{2.1}$$

where a, b and c are functions of (x_3, x_4). The couple $(U, g_{a,b,c})$ is called a four-dimensional strict Walker manifold and its matrix form is given by

$$(g_{a,b,c})_{ij} = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & a & c \\ 0 & 1 & c & b \end{pmatrix}. \tag{2.2}$$

Next, we denote by $f_i := \frac{\partial f(x_3, x_3)}{\partial x_i}$ and $f_{ij} := \frac{\partial^2 f(x_3, x_3)}{\partial x_i \partial_j}$. A straightforward calculation from (2.2), shows that the non-zero components of the Levi-Civita connection of the metric (2.1) are given by:

$$\begin{aligned} \nabla_{\partial_3}\partial_3 &= \frac{1}{2}a_3\partial_1 + \frac{1}{2}(2c_3 - a_4)\partial_2; \\ \nabla_{\partial_3}\partial_4 &= \frac{1}{2}a_4\partial_1 + \frac{1}{2}b_3\partial_2; \\ \nabla_{\partial_4}\partial_4 &= \frac{1}{2}(2c_4 - b_3)\partial_1 + \frac{1}{2}b_4\partial_2. \end{aligned}$$

The non-zero components of the curvature tensor of $(U, g_{a,b,c})$ are :

$$R(\partial_3, \partial_4)\partial_3 = \frac{1}{2}(a_{44} + b_{33} - 2c_{34})\partial_2, \text{and } R(\partial_3, \partial_4)\partial_4 = -\frac{1}{2}(a_{44} + b_{33} - 2c_{34})\partial_1.$$

The non-zero component of the $(0, 4)$-curvature tensor of the metric (2.1) is given by:

$$R_{3434} = \frac{1}{2}(a_{44} + b_{33} - 2c_{34}).$$

Conditions for a four-dimensional strict Walker manifold to be locally symmetric are given [9]. Also, in [10], conditions for a four-dimensional strict Walker manifold to be locally conformally flat and locally conformally symmetric are given.

We find that the Ricci tensor and the scalar curvature of $(U, g_{a,b,c})$ vanish. More precisely, we have:

$$\rho_{ij} = 0 \text{ and } \tau = 0 \ \forall i, j = 1, 2, 3, 4.$$

The non-zero components of the Einstein tensor $G_{ij} = \rho_{ij} - \frac{\tau}{4} g_{ij}$ are given by $G_{ij} = 0 \ \forall i, j = 1, 2, 3, 4$.

Proposition 2.1. *Let $(M, g_{a,b,c})$ be a strict four-dimensional Walker manifolds. Then, $(M, g_{a,b,c})$ is Ricci flat and so it is Einstein.*

We say that M is geodesically complete if all geodesics exist for all time. We say that M exhibits Ricci blowup if there exists a geodesic γ defined for $t \in [0, T)$ with $T < \infty$ and if $\lim_{t \to T} |\rho(\dot{\gamma}, \dot{\gamma})| = \infty$. Clearly, if M exhibits Ricci blowup, then it is geodesically incomplete and it can not be isometrically embedded in a geodesically complete manifold. Any four-dimensional strict Walker manifold is geodesically complete.

3. Killing vector fields on four-dimensional strict Walker manifolds

In this section, we focus on Killing vector fields for Walker manifolds . We examine questions regarding of existence and characterization of Killing vector fields on a four-dimensional strict Walker manifold. We find the conditions for the existence of Killing vector fields on the four-dimensional strict Walker manifold.

Recall that, a Killing vector field X on a pseudo-Riemannian manifold (M, g) is a smooth vector field that preserves the metric tensor, i.e., the Lie derivative with respect to X of the metric g vanishes:

$$\mathcal{L}_X g = 0. \tag{3.1}$$

The above definition is equivalent to

$$g(\nabla_Z X, Y) + g(Z, \nabla_Y X) = 0, \tag{3.2}$$

for any vector fields $Y, Z \in \mathfrak{X}(M)$ and where ∇ is the Levi-Civita connection of g. Note that, existence and characterization of Killing vector fields on pseudo-Riemannian metrics have been studied by several authors.

In the case of a four-dimensional strict Walker manifold, solving the Killing equation becomes a system of partial differential equations for the components of X. Because the metric has a null parallel distribution, the Killing fields tend to have a rich structure.

Next, we find the conditions for the existence of Killing vector fields on the family of four dimensional Walker manifold (2.1) and we obtain general expressions. Let

$$X = X^1 \partial_1 + X^2 \partial_2 + X^3 \partial_3 + X^4 \partial_4,$$

where $X^i, i = 1, 2, 3, 4$ are functions on x_1, x_2, x_3, x_4.

The components of the Lie derivative with respect to X of the metric (2.1) are given by :

$$(\mathcal{L}_X g_{a,b,c})(\partial_i, \partial_j) = \sum_{k=1}^{4} \Big[X^k \partial_k g_{ij} + (\partial_i X^k) g_{kj} + (\partial_j X^k) g_{ik} \Big]. \tag{3.3}$$

From (3.3), we have the following lemma :

Lemma 3.1. *The non-zero components of the Lie derivative in the direction of $X = \sum_{k=1}^{4} X^k \partial_k$ of the Walker metric (2.2) are given by :*

$$
\begin{aligned}
(\mathcal{L}_X g_{a,b,c})(\partial_1,\partial_1) &= 2\partial_1 X^3,\ (\mathcal{L}_X g_{a,b,c})(\partial_2,\partial_2) = 2\partial_2 X^4,\\
(\mathcal{L}_X g_{a,b,c})(\partial_1,\partial_2) &= \partial_1 X^4 + \partial_2 X^3,\\
(\mathcal{L}_X g_{a,b,c})(\partial_1,\partial_3) &= \partial_1 X^1 + a(\partial_1 X^3) + c(\partial_1 X^4) + \partial_3 X^3,\\
(\mathcal{L}_X g_{a,b,c})(\partial_1,\partial_4) &= \partial_1 X^2 + c(\partial_1 X^3) + b(\partial_1 X^4) + \partial_4 X^3,\\
(\mathcal{L}_X g_{a,b,c})(\partial_2,\partial_3) &= \partial_2 X^1 + a(\partial_2 X^3) + c\partial_2 X^4 + \partial_3 X^4,\\
(\mathcal{L}_X g_{a,b,c})(\partial_2,\partial_4) &= \partial_2 X^2 + c(\partial_2 X^3) + b(\partial_2 X^4) + \partial_4 X^4,\\
(\mathcal{L}_X g_{a,b,c})(\partial_3,\partial_3) &= a_3 X^3 + a_4 X^4 + 2\partial_3 X^1 + 2a\partial_3 X^3 + 2c\partial_3 X^4,\\
(\mathcal{L}_X g_{a,b,c})(\partial_3,\partial_4) &= c_3 X^3 + c_4 X^4 + \partial_3 X^2 + c\partial_3 X^3 + b\partial_3 X^4\\
&\quad + \partial_4 X^1 + a\partial_4 X^3 + c\partial_4 X^4,\\
(\mathcal{L}_X g_{a,b,c})(\partial_4,\partial_4) &= b_3 X^3 + b_4 X^4 + 2\partial_4 X^2 + 2c\partial_4 X^3 + 2b\partial_4 X^4,
\end{aligned}
$$

From the Lemma 3.1, we find the conditions for the vector field X to be Killing. Firstly, we give the form of X^3 and X^4.

Lemma 3.2. *The functions X^3 and X^4 have the following forms:*

$$X^3 = -x_2 A(x_3, x_4) + D(x_3, x_4), \tag{3.4}$$

and

$$X^4 = x_1 A(x_3, x_4) + B(x_3, x_4), \tag{3.5}$$

where A, B and D are some functions.

Proof. From the expressions of $(\mathcal{L}_X g_{a,b,c})(\partial_1,\partial_1)$ and $(\mathcal{L}_X g_{a,b,c})(\partial_2,\partial_2)$ of the above Lemma 3.1, we have :

$$X^3 = f(x_2, x_3, x_4) \quad \text{and} \quad X^4 = f(x_1, x_3, x_4). \tag{3.6}$$

From the expression $(\mathcal{L}_X g_{a,b,c})(\partial_1,\partial_2)$, we find (3.4) and (3.5). □

Secondly, we give the form of X^1. We have :

Lemma 3.3. *The function X^1 has the following form:*

$$X^1 = x_1 x_2 A_3(x_3, x_4) - x_1(cA + D_3) + E(x_2, x_3, x_4), \tag{3.7}$$

where A, D and E are some functions.

Proof. From the expression of $(\mathcal{L}_X g_{a,b,c})(\partial_1,\partial_3)$, we get (3.7). □

Thirdly, we give the form of X^2.

Lemma 3.4. *The function X^2 as the following form:*

$$X^2 = x_1 x_2 A_4(x_3, x_4) - x_1(bA + D_4) + F(x_2, x_3, x_4), \tag{3.8}$$

where A, D and F are some functions.

Proof. From the expression of $(\mathcal{L}_X g_{a,b,c})(\partial_1,\partial_4)$, we obtain (3.8). □

Form $(\mathcal{L}_X g_{a,b,c})(\partial_2, \partial_3)$ and $(\mathcal{L}_X g_{a,b,c})(\partial_2, \partial_4)$, the functions A, B, E, and F should satisfying the following partial differential equations system :

$$\begin{cases} 2A_3x_1 + E_2 - aA + B_3 &= 0, \\ 2A_4x_1 + F_2 - cA + B_4 &= 0. \end{cases} \tag{3.9}$$

Thus $A = k$, $E = akx_2 - B_3x_2 + G(x_3, x_4)$ and $F = ckx_2 - B_4x_2 + H(x_3, x_4)$.

Form $(\mathcal{L}_X g_{a,b,c})(\partial_3, \partial_3)$, $(\mathcal{L}_X g_{a,b,c})(\partial_3, \partial_4)$ and $(\mathcal{L}_X g_{a,b,c})(\partial_4, \partial_4)$, the functions B, D, G and H should satisfying the following partial differential equations system :

$$\begin{cases} ka_4 - 2kc_3 - 2D_{33} &= 0, \\ -ka_3 - 2B_{33} &= 0, \\ a_3D + a_4B + 2G_3 + 2aD_3 + 2cB_3 &= 0, \\ -kb_3 - 2D_{34} &= 0, \\ ka_4 - 2B_{34} &= 0, \\ c_3D + c_4B + H_3 + cD_3 + bB_3 + G_4 + aD_4 + cB_4 &= 0, \\ -kb_4 - 2D_{44} &= 0, \\ -kb_3 + 2kc_4 - 2B_{44} &= 0, \\ 2H_4 + 2cD_4 + 2bB_4 &= 0. \end{cases} \tag{3.10}$$

Now we can prove the following characterization of Killing vector fields on $(M, g_{a,b,c})$.

Proposition 3.1. *Let $(M, g_{a,b,c})$ be a four-dimensional strict Walker manifold , where $g_{a,b,c}$ is the metric given by (2.1). If a vector field $X = X^1\partial_1 + X^2\partial_2 + X^3\partial_3 + X^4\partial_4$ on $(M, g_{a,b,c})$ is Killing, then there exist some functions B, D, G, H, such that the following hold*

$$\begin{aligned} X^1 &= -x_1(ck + D_3) + x_2(ka - B_3) + G(x_3, x_4), \\ X^2 &= -x_1(bk + D_4) + x_2(kc - B_4) + H(x_3, x_4), \\ X^3 &= -kx_2 + D(x_3, x_4), \quad X^4 = kx_1 + B(x_3, x_4), \end{aligned}$$

where the functions B, D, G and H should satisfying the following partial differential equations system :

$$\begin{cases} ka_4 - 2kc_3 - 2D_{33} &= 0, \\ -ka_3 - 2B_{33} &= 0, \\ a_3D + a_4B + 2G_3 + 2aD_3 + 2cB_3 &= 0, \\ -kb_3 - 2D_{34} &= 0, \\ ka_4 - 2B_{34} &= 0, \\ c_3D + c_4B + H_3 + cD_3 + bB_3 + G_4 + aD_4 + cB_4 &= 0, \\ -kb_4 - 2D_{44} &= 0, \\ -kb_3 + 2kc_4 - 2B_{44} &= 0, \\ 2H_4 + 2cD_4 + 2bB_4 &= 0. \end{cases}$$

We have the following example of Killing vector field on $(M, g_{a,b;c})$.

Example 3.1. *Let us consider the functions defining the Walker metric:*

$$a = b = 1, \quad c(x_3, x_4) = e^{x_3 - x_4}.$$

The following vector field : $X = \beta\partial_1 + \beta\partial_2 + \alpha\partial_3 + \alpha\partial_4$, where $\alpha, \beta \in \mathbb{R}^*$, is a Killing vector field on the four-dimensional strict Walker manifold defined by the metric $g_{a,b,c}$.

Corollary 3.1. *The vector fields ∂_1 and ∂_2 are Killing on the four-dimensional strict Walker manifold*

Example 3.2. *The vector field $X = \alpha\partial_1 + \beta\partial_2$ where $\alpha, \beta \in \mathbb{R}$ is a Killing vector field for the metric $g_{a,b,c}$. It corresponds to a translation symmetry in the x_1 and x_2 directions.*

In this section, we have studied Killing vector fields on four-dimensional strict Walker manifolds. Starting from the local form of the metric, we derived necessary and sufficient conditions for a vector field to be Killing. Using the Lie derivative of the metric, we established a system of partial differential equations governing the components of Killing vector fields. We then provided a general expression for such fields and illustrated our results with explicit examples.

Our analysis reveals that the structure and existence of Killing vector fields are strongly influenced by the functions defining the Walker metric. In particular, we showed that both linear and more complex symmetries can be present, depending on the choice of the metric coefficients. These results enrich the understanding of symmetries in pseudo-Riemannian geometry and may contribute to applications in general relativity and geometric analysis.The number of independent Killing vector fields measures the degree of symmetry of a pseudo-Riemannian manifold. Thus, the problems of existence and characterization of Killing vector fields are important and are widely discussed by both mathematicians and physicists (see [11] and references therein).

4. Ricci soliton on four-dimensional strict Walker manifolds

Ricci solitons on strict Walker manifolds form an interesting class of solutions to the Ricci flow equation in pseudo-Riemannian geometry, particularly in neutral signature $(2, 2)$. Recall that, a Ricci soliton is a generalization of an Einstein metric. A pseudo-Riemannian manifold (M, g) with a vector field X is a Ricci soliton if it satisfies :

$$\mathcal{L}_X g + \rho = \lambda g, \tag{4.1}$$

where $\mathcal{L}_X g$ is the Lie derivative of the metric with respect to X, ρ is the is the Ricci tensor and $\lambda \in \mathbb{R}$ is a constant (positive = shrinking, zero = steady, negative = expanding).

From Proposion 2.1, four-dimensional strict Walker manifold is Ricci-flat, then the Ricci soliton equation reduces to :

$$\mathcal{L}_X g = \lambda g, \tag{4.2}$$

This has major implications.

(1) If $\lambda = 0$, then $\mathcal{L}_X g = 0$. So a four-dimensional strict Walker manifold admits a steady Ricci soliton if and only if it admits Killing vector field.
(2) If $\lambda \neq 0$, then $\mathcal{L}_X g = \lambda g$. This implies the Lie derivative of the metric with respect to X is proportional to the metric, which only happens if the metric is Einstein. However, $\rho = 0$, the Einstein constant must be zero, so $\lambda = 0$ must follow. Therefore, a four-dimensional strict Walker manifold cannot be a non-steady Ricci soliton (i.e., not shrinking or expanding).

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