Global Journal of Advanced Research
on Classical and Modern Geometries
ISSN: 2284-5569, Vol. 14, (2025), Issue 2, pp.208-217

A SURPRISING COLLINEARITY ASSOCIATED WITH THE NAPOLEON'S TRIANGLES

UTSAB SARKAR

ABSTRACT. This article investigates a geometric configuration arising from the classical Napoleon triangles and explores a natural extension to it. Building on the works of Belenkiy [1] and Rigby [2], we examine particular intersections of lines associated with these constructions. Through the lens of barycentric transformations, we uncover surprising instances of collinearity, contributing new insights into the intricate structure and symmetries inherent in this classical geometric configuration.

1. INTRODUCTION

The well-known Napoleon Triangles are formed by the centers of equilateral triangles constructed on the sides of $\triangle ABC$. Suppose an equilateral triangle is constructed externally on side BC in such a way that it does not overlap with $\triangle ABC$; we denote this triangle as $\triangle A^+BC$. Similarly, we define $\triangle B^+CA$ and $\triangle C^+AB$ for the other sides of $\triangle ABC$. The centers of these externally constructed triangles form the Outer-Napoleon triangle, denoted as $\triangle O_a^+O_b^+O_c^+$.
In a similar manner, we define the Inner-Napoleon triangle, $\triangle O_a^-O_b^-O_c^-$, which is formed by the centers of equilateral triangles $\triangle A^-BC$, $\triangle B^-CA$, and $\triangle C^-AB$ that are constructed internally on the sides of $\triangle ABC$. Here, $O_a^\pm$ represents the center of $\triangle A^\pm BC$, and similar definitions hold for the other vertices.
Let M_a, M_b, M_c be the midpoints of sides BC, CA, AB of $\triangle ABC$, and let $M_a^\pm, M_b^\pm, M_c^\pm$ be the midpoints of the sides $O_b^\pm O_c^\pm, O_c^\pm O_a^\pm, O_a^\pm O_b^\pm$ of the Napoleon triangles, respectively. Next, let $X_a^\pm$ be the foot of the perpendicular from M_a to $O_b^\pm O_c^\pm$, with similar definitions for $X_b^\pm$ and $X_c^\pm$. Likewise, let Y_a^+ and Y_a^- be the feet of the perpendiculars from M_a^+ and M_a^- to BC, respectively, with corresponding definitions for $Y_b^\pm$ and $Y_c^\pm$.

With these definitions in place, we now present our first theorem.

Theorem 1.1. *The following results hold:*
1. The lines $M_aX_a^+$, $M_bX_b^+$, and $M_cX_c^+$ are concurrent at a point, defined as P^+. Similarly, the lines $M_aX_a^-$, $M_bX_b^-$, and $M_cX_c^-$ also meet at a common point, denoted as P^-.
2. The lines $M_aM_a^+$, $M_bM_b^+$, and $M_cM_c^+$ are concurrent at a point, defined as Q^+. Likewise, the lines $M_aM_a^-$, $M_bM_b^-$, and $M_cM_c^-$ meet at a common point Q^-.
3. The lines $M_a^\pm Y_a^\pm$, $M_b^\pm Y_b^\pm$, and $M_c^\pm Y_c^\pm$ are concurrent at the Nine-Point center N of $\triangle ABC$.

2020 *Mathematics Subject Classification.* 51M05, 51M04, 51M15.
Key words and phrases. Napoleon's Triangles, Nine-point center, Concurrency, Collinearity, Barycentric coordinates.

4. The points P^+, Q^+, and N are collinear, as are the points P^-, Q^-, and N.

2. A Generalization

We now extend Theorem 1.1 in the following manner. Consider the triangles $\triangle A^{\pm}BC$, $\triangle B^{\pm}CA$, and $\triangle C^{\pm}AB$ such that the angles satisfy $\theta = \angle CA^{\pm}B = \angle AB^{\pm}C = \angle BC^{\pm}A$. Additionally, let the points $O_a^{\pm}, O_b^{\pm}, O_c^{\pm}$ be the circumcenters of the respective triangles $\triangle A^{\pm}BC$, $\triangle B^{\pm}CA$, and $\triangle C^{\pm}AB$.

With these definitions, we now state the generalization in the following result.

Theorem 2.1. *The following concurrency and collinearity results hold:*
1. The lines $M_aX_a^{\pm}$, $M_bX_b^{\pm}$, and $M_cX_c^{\pm}$ are concurrent at the points $P^{\pm}$, which have barycentric coordinates:

$$\Big(\left(2S^2 + a^2 S_\phi\right)\left(S^2 + S_{A\phi}\right) : \left(2S^2 + b^2 S_\phi\right)\left(S^2 + S_{B\phi}\right) : \left(2S^2 + c^2 S_\phi\right)\left(S^2 + S_{C\phi}\right)\Big).$$

2. The lines $M_aM_a^{\pm}$, $M_bM_b^{\pm}$, and $M_cM_c^{\pm}$ are concurrent at the points $Q^{\pm}$, whose barycentric coordinates are given by:

$$\Big(\left(S_\phi + S_A\right)\left(2S_\phi + a^2\right) : \left(S_\phi + S_B\right)\left(2S_\phi + b^2\right) : \left(S_\phi + S_C\right)\left(2S_\phi + c^2\right)\Big).$$

3. The lines $M_a^{\pm}Y_a^{\pm}$, $M_b^{\pm}Y_b^{\pm}$, and $M_c^{\pm}Y_c^{\pm}$ are concurrent at the Nine-Point center N of $\triangle ABC$, with barycentric coordinates:

$$\left(S^2 + S_{BC} : S^2 + S_{CA} : S^2 + S_{AB}\right).$$

4. The sets of points $\{P^+, Q^+, N\}$ and $\{P^-, Q^-, N\}$ are collinear. The equation of the line of collinearity is given by:

$$\left(S_B - S_C\right)\left(2S_A(S_{\phi\phi} + S^2) + S_\phi(S_{AA} - 2S_{BC} + 3S^2)\right) X$$

$$+\left(S_C - S_A\right)\left(2S_B(S_{\phi\phi} + S^2) + S_\phi(S_{BB} - 2S_{CA} + 3S^2)\right) Y$$

$$+\left(S_A - S_B\right)\left(2S_C(S_{\phi\phi} + S^2) + S_\phi(S_{CC} - 2S_{AB} + 3S^2)\right) Z = 0.$$

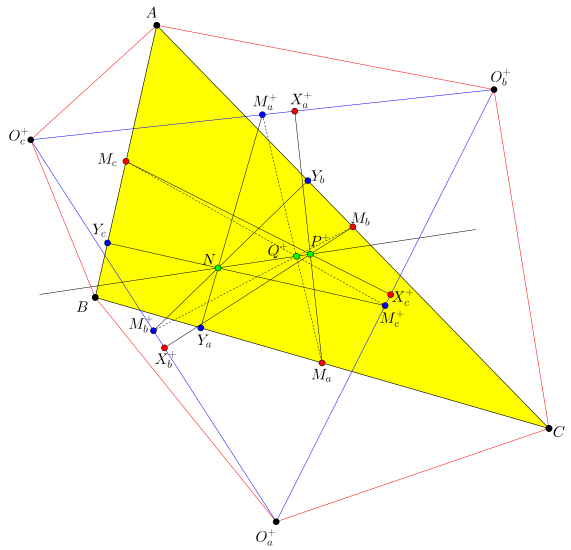
The common angle ϕ satisfies

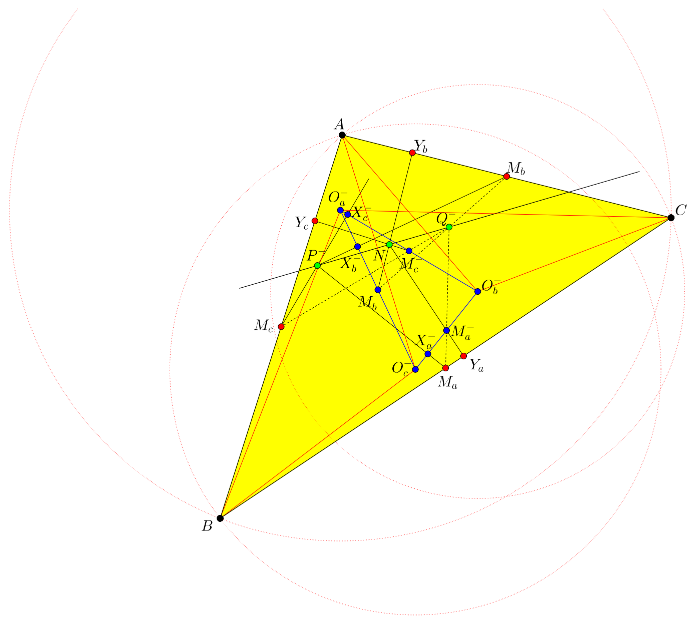
$$\phi = \angle O_a^{\pm}BC = \angle O_b^{\pm}CA = \angle O_c^{\pm}AB.$$

If $\theta \in (0, \frac{\pi}{2}]$, then $\phi = \frac{\pi}{2} - \theta$, and if $\theta \in (\frac{\pi}{2}, \pi]$, then $\phi = \frac{3\pi}{2} - \theta$. The notations follow Conway's definitions:

$$S_A = S\cot A = \frac{b^2 + c^2 - a^2}{2}, \quad S_B = S\cot B = \frac{c^2 + a^2 - b^2}{2}, \quad S_C = S\cot C = \frac{a^2 + b^2 - c^2}{2},$$

where $S = 2\,\mathrm{Area}(\triangle ABC)$.





It is evident that Theorem 1.1 is a special case of Theorem 2.1 when we set $\theta = \frac{\pi}{3}$ or $\frac{2\pi}{3}$. Consequently, we will focus on proving Theorem 2.1 directly. Before proceeding further, we first outline some well-known results that will be essential in our proof.

3. Auxiliary Lemmata

Lemma 3.1. *Let P be a point on the plane of $\triangle ABC$ such that $\measuredangle(PB, BC) = \delta, \measuredangle(BC, PC) = \gamma$, then the barycentric coordinates of P are given by*

$$\left(-a^2 : S_C + S_\gamma : S_B + S_\delta\right).$$

Proof. From $\triangle PBC$, applying the Law of Sines, we obtain:

$$PB = \frac{a\sin\gamma}{\sin(\gamma+\delta)}, \quad PC = \frac{a\sin\delta}{\sin(\gamma+\delta)}.$$

The area of $\triangle PBC$ is given by:

$$\text{Area}(\triangle PBC) = \frac{PB\cdot PC\cdot \sin(\pi-\delta-\gamma)}{2} = \frac{a^2\sin\gamma\sin\delta}{2\sin(\gamma+\delta)}.$$

Similarly, the areas of the other two sub-triangles are:

$$\text{Area}(\triangle PCA) = \frac{ab\sin\delta\sin(C+\gamma)}{2\sin(\gamma+\delta)}, \quad \text{Area}(\triangle PAB) = \frac{ac\sin\gamma\sin(B+\delta)}{2\sin(\gamma+\delta)}.$$

Thus, the barycentric coordinates of P are:

$$\begin{aligned} P &= \left(-\frac{a^2\sin\gamma\sin\delta}{2\sin(\gamma+\delta)} : \frac{ab\sin\delta\sin(C+\gamma)}{2\sin(\gamma+\delta)} : \frac{ac\sin\gamma\sin(B+\delta)}{2\sin(\gamma+\delta)}\right) \\ &= \left(-a^2 : ab\frac{\sin(C+\gamma)}{\sin\gamma} : ac\frac{\sin(B+\delta)}{\sin\delta}\right) \\ &= \left(-a^2 : ab(\sin C\cot\gamma + \cos C) : ac(\sin B\cot\delta + \cos B)\right) \\ &= \left(-a^2 : S_C + S_\gamma : S_B + S_\delta\right). \end{aligned}$$

Lemma 3.2. *Let* $\vec{\boldsymbol{u}} = x_1\boldsymbol{A} + y_1\boldsymbol{B} + z_1\boldsymbol{C}$ *,* $\vec{\boldsymbol{v}} = x_2\boldsymbol{A} + y_2\boldsymbol{B} + z_2\boldsymbol{C}$ *are two vectors such that* $x_1 + y_1 + z_1 = x_2 + y_2 + z_2 = 0$*. Then* $\vec{\boldsymbol{u}} \perp \vec{\boldsymbol{v}}$ *if and only if*

$$a^2(y_1z_2 + y_2z_1) + b^2(z_1x_2 + z_2x_1) + c^2(x_1y_2 + x_2y_1) = 0$$

Proof. Note $\vec{\boldsymbol{u}} \perp \vec{\boldsymbol{v}}$ if and only if

$$(x_1\boldsymbol{A} + y_1\boldsymbol{B} + z_1\boldsymbol{C}) \cdot (x_2\boldsymbol{A} + y_2\boldsymbol{B} + z_2\boldsymbol{C}) = 0$$

Now shifting the origin to the point A, observe the dot product reduces to

$$(y_1\boldsymbol{B} + z_1\boldsymbol{C}) \cdot (y_2\boldsymbol{B} + z_2\boldsymbol{C}) = 0.$$

Upon expanding, we obtain $y_1y_2\boldsymbol{B}\cdot\boldsymbol{B} + z_1z_2\boldsymbol{C}\cdot\boldsymbol{C} + (y_1z_2 + y_2z_1)\boldsymbol{B}\cdot\boldsymbol{C} = 0$, which is equivalent to $y_1y_2c^2 + z_1z_2b^2 + (y_1z_2 + y_2z_1)\frac{b^2+c^2-a^2}{2} = 0$. Now employing $x_1 + y_1 + z_1 = x_2 + y_2 + z_2 = 0$, one gets

$$a^2(y_1z_2 + y_2z_1) + b^2(z_1x_2 + z_2x_1) + c^2(x_1y_2 + x_2y_1) = 0.$$

Lemma 3.3. *Let* L_i *be the line* $p_ix + q_iy + r_iz = 0$*. Then for* $i = 1, 2, 3$ *the lines* L_1, L_2, L_3 *are concurrent at a point if and only if*

$$\begin{vmatrix} p_1 & q_1 & r_1 \\ p_2 & q_2 & r_2 \\ p_3 & q_3 & r_3 \end{vmatrix} = 0$$

The intersection point of the lines L_i, L_j *has barycentric coordinates*

$$(q_ir_j - q_jr_i : r_ip_j - r_jp_i : p_iq_j - p_jq_i)$$

Proof. See [3].

We are now ready to prove our main result.

4. Proof of Theorem 2.1

Working with directed angles modulo π. Define θ as follows:

$$\theta = \measuredangle CA^{\pm}B = \frac{1}{2}\measuredangle CO_a^{\pm}B = \frac{\pi}{2} - \measuredangle O_a^{\pm}BC = \frac{\pi}{2} - \phi.$$

Since $O_a^{\pm}$ is the circumcenter of $\triangle A^{\pm}BC$, the triangle $\triangle O_a^{\pm}BC$ is isosceles. The same argument applies to the other points, meaning that we can equivalently analyze similar isosceles triangles, positioned similarly on the sides of $\triangle ABC$, each with an oriented base angle of ϕ. Using Lemma 3.1, the coordinates of the circumcenters are:

$$O_a^{\pm} = \left(-a^2 : S_C + S_\phi : S_B + S_\phi\right),$$

$$O_b^{\pm} = \left(S_C + S_\phi : -b^2 : S_A + S_\phi\right),$$

$$O_c^{\pm} = \left(S_B + S_\phi : S_A + S_\phi : -c^2\right).$$

Now, consider the vector between two of these circumcenters:

$$\overrightarrow{O_b^{\pm}O_c^{\pm}} = O_b^{\pm} - O_c^{\pm} = \frac{1}{2S_\phi}\left(S_C - S_B, -b^2 - S_A - S_\phi, c^2 + S_A + S_\phi\right).$$

Next, let $(u : v : w)$ be a point on $M_aX_a^{\pm}$ such that $u + v + w = k$. Then,

$$\overrightarrow{X_a^{\pm}M_a} = X_a^{\pm} - M_a = \frac{1}{2k}(2u, 2v - k, 2w - k) = \frac{1}{2k}(2u, -u + v - w, -u - v + w).$$

Since $\overrightarrow{X_a^{\pm}M_a} \perp \overrightarrow{O_b^{\pm}O_c^{\pm}}$ and both vectors have coordinate sums equal to zero, Lemma 3.2 implies that

$$\begin{aligned}
&a^2\Big((-u+v-w)(2S_A+S_B+S_\phi) - (-u-v+w)(2S_A+S_C+S_\phi)\Big)\\
&+ b^2\Big(2u(2S_A+S_B+S_\phi) + (-u-v+w)(S_C-S_B)\Big)\\
&+ c^2\Big(-2u(2S_A+S_C+S_\phi) + (-u+v-w)(S_C-S_B)\Big) = 0
\end{aligned}$$

Using $a^2 = S_B + S_C$ and similarly for b^2, c^2, we obtain

$$\begin{aligned}
&(S_B+S_C)\Big((-u+v-w)(2S_A+S_B+S_\phi) - (-u-v+w)(2S_A+S_C+S_\phi)\Big)\\
&+ (S_C+S_A)\Big(2u(2S_A+S_B+S_\phi) + (-u-v+w)(S_C-S_B)\Big)\\
&+ (S_A+S_B)\Big(-2u(2S_A+S_C+S_\phi) + (-u+v-w)(S_C-S_B)\Big) = 0
\end{aligned}$$

A simplification yields

$$u(S_C - S_B)S_\phi + (v - w)(2S_AS_B + 2S_BS_C + 2S_CS_A + S_\phi(S_B + S_C)) = 0$$

Using $S_AS_B + S_BS_C + S_CS_A = S^2$, we get,

$$\frac{S_B - S_C}{S_B + S_C + \frac{2S^2}{S_\phi}}u - v + w = 0 \tag{4.1}$$

Since $(u : v : w)$ represents any point on the line $M_aX_a^{\pm}$, equation (4.1) gives the equation of this line. By the symmetry in notation, we can similarly derive the equations for the other two lines:

$$u + \frac{S_C - S_A}{S_C + S_A + \frac{2S^2}{S_\phi}} v - w = 0 \tag{4.2}$$

defines the line $M_bX_b^{\pm}$, and

$$-u + v + \frac{S_A - S_B}{S_A + S_B + \frac{2S^2}{S_\phi}} w = 0 \tag{4.3}$$

defines the line $M_cX_c^{\pm}$. Now, define t_a as:

$$t_a = \frac{S_B - S_C}{S_B + S_C + \frac{2S^2}{S_\phi}},$$

and similarly define t_b and t_c.
According to Lemma 3.3, the three lines (4.1),(4.2) and (4.3) are concurrent if and only if

$$\begin{vmatrix} t_a & -1 & 1 \\ 1 & t_b & -1 \\ -1 & 1 & t_c \end{vmatrix} = 0 \implies t_a + t_b + t_c + t_a t_b t_c = 0 \tag{4.4}$$

An explicit calculation below confirms (4.4).
Note,

$$\begin{aligned} \sum_{cyc} \frac{S_B - S_C}{S_B + S_C + \frac{2S^2}{S_\phi}} &= \frac{\sum_{cyc} (S_B - S_C)\left(S_C + S_A + \frac{2S^2}{S_\phi}\right)\left(S_A + S_B + \frac{2S^2}{S_\phi}\right)}{\left(S_B + S_C + \frac{2S^2}{S_\phi}\right)\left(S_C + S_A + \frac{2S^2}{S_\phi}\right)\left(S_A + S_B + \frac{2S^2}{S_\phi}\right)} \\ &= \frac{\mathcal{N}}{\left(S_B + S_C + \frac{2S^2}{S_\phi}\right)\left(S_C + S_A + \frac{2S^2}{S_\phi}\right)\left(S_A + S_B + \frac{2S^2}{S_\phi}\right)} \end{aligned}$$

where,

$$\begin{aligned} \mathcal{N} &= \sum_{cyc} (S_B - S_C)(S_C + S_A)(S_A + S_B) \\ &\quad + \frac{2S^2}{S_\phi} \sum_{cyc} (2S_A + S_B + S_C)(S_B - S_C) \\ &\quad + \frac{4S^4}{S_\phi^2} \sum_{cyc} (S_B - S_C) \\ &= -(S_B - S_C)(S_C - S_A)(S_A - S_B) \end{aligned}$$

Using Lemma 3.3, from (4.1) and (4.2), we see, $P^{\pm} = (1 - t_b : 1 + t_a : 1 + t_a t_b)$. Let $t = \frac{2S^2}{S_\phi}$. Then,

$$\begin{aligned}
1 - t_b &= 1 - \frac{S_C - S_A}{S_C + S_A + \frac{2S^2}{S_\phi}} = \frac{t + 2S_A}{t + S_C + S_A} \\
1 + t_a &= 1 - \frac{S_B - S_C}{S_B + S_C + \frac{2S^2}{S_\phi}} = \frac{t + 2S_B}{t + S_B + S_C} \\
1 + t_a t_b &= 1 - \frac{t_a + t_b + t_c}{t_c} = -\frac{t_a + t_b}{t_c} \\
&= -\frac{t + S_A + S_B}{S_A - S_B}\left(\frac{S_B - S_C}{t + S_B + S_C} + \frac{S_C - S_A}{t + S_C + S_A}\right) \\
&= \tfrac{t + S_A + S_B}{S_B - S_A} \cdot \tfrac{(S_B - S_C)(t + S_C + S_A) + (S_C - S_A)(t + S_B + S_C)}{(t + S_B + S_C)(t + S_C + S_A)} \\
&= \frac{(t + 2S_C)(t + S_A + S_B)}{(t + S_B + S_C)(t + S_C + S_A)}
\end{aligned}$$

Therefore, one obtains

$$\begin{aligned}
P^{\pm} &= \Big((t + 2S_A)(t + S_B + S_C) : (t + 2S_B)(t + S_C + S_A) \\
&\qquad : (t + 2S_C)(t + S_A + S_B)\Big) \\
&= \Big((S^2 + S_{A\phi})(2S^2 + a^2 S_\phi) : (S^2 + S_{B\phi})(2S^2 + b^2 S_\phi) \\
&\qquad : (S^2 + S_{C\phi})(2S^2 + c^2 S_\phi)\Big)
\end{aligned} \tag{4.5}$$

The mid-point of $O_b^{\pm} O_c^{\pm}$ is $M_a^{\pm} = \frac{O_b^{\pm} + O_c^{\pm}}{2} = \frac{1}{4S_\phi}\big(2S_\phi + S_B + S_C : S_\phi - S_C : S_\phi - S_B\big)$. Then the equation of the line $M_a M_a^{\pm}$ is

$$\begin{vmatrix} 0 & 1 & 1 \\ 2S_\phi + S_B + S_C & S_\phi - S_C & S_\phi - S_B \\ x & y & z \end{vmatrix} = 0$$

Transforming $C_2' = C_2 - C_3$,

$$\begin{vmatrix} 0 & 0 & 1 \\ 2S_\phi + S_B + S_C & S_B - S_C & S_\phi - S_B \\ x & y - z & z \end{vmatrix} = 0$$

Which gives,

$$\begin{aligned}
& x(S_B - S_C) - (y - z)(2S_\phi + S_B + S_C) = 0 \\
\implies & \frac{S_B - S_C}{S_B + S_C + 2S_\phi} x - y + z = 0
\end{aligned} \tag{4.6}$$

The obvious symmetry shows the equations of the lines $M_bM_b^{\pm}$, $M_cM_c^{\pm}$ are

$$x+\frac{S_C-S_A}{S_C+S_A+2S_\phi}y-z=0 \tag{4.7}$$

$$-x+y+\frac{S_A-S_B}{S_A+S_B+2S_\phi}z=0 \tag{4.8}$$

Lets define $t'=2S_\phi, t'_a=\frac{S_B-S_C}{S_B+S_C+2S_\phi}, t'_b=\frac{S_C-S_A}{S_C+S_A+2S_\phi}, t'_c=\frac{S_A-S_B}{S_A+S_B+2S_\phi}$, and therefore, as before the lines (4.6),(4.7) and (4.8) are concurrent at a point $Q^{\pm}$ if and only if

$$\begin{vmatrix} t'_a & -1 & 1 \\ 1 & t'_b & -1 \\ -1 & 1 & t'_c \end{vmatrix} = t'_a+t'_b+t'_c+t'_at'_bt'_c=0 \tag{4.9}$$

But here we observe that the condition (4.9) inherently resembles (4.4), thus holds true. Therefore,

$$\begin{aligned} Q^{\pm} &= \left(1-t'_b : 1+t'_a : 1+t'_at'_b\right) \\ 1-t'_b &= 1-\frac{S_C-S_A}{S_C+S_A+2S_\phi} = \frac{t'+2S_A}{t'+S_C+S_A} \\ 1+t'_a &= 1+\frac{S_B-S_C}{S_B+S_C+2S_\phi} = \frac{t'+2S_B}{t'+S_A+S_B} \\ 1+t'_at'_b &= \frac{(t'+2S_C)(t'+S_A+S_B)}{(t'+S_B+S_C)(t'+S_C+S_A)} \end{aligned}$$

Employing $t'=2S_\phi$, we obtain

$$Q^{\pm}=\Big((S_\phi+S_A)(2S_\phi+a^2):(S_\phi+S_B)(2S_\phi+b^2):(S_\phi+S_C)(2S_\phi+c^2)\Big) \tag{4.10}$$

Note that $\overrightarrow{M_a^{\pm}M_b}=M_a^{\pm}-M_b=\frac{1}{4S_\phi}\Big(a^2+2S_\phi : S_\phi-S_C : S_\phi-S_B\Big)-\Big(\frac{1}{2}:0:\frac{1}{2}\Big)=\frac{1}{4S_\phi}(a^2 : S_\phi-S_C : -S_\phi-S_B)$. Again, $\overrightarrow{M_a^{\pm}M_c}=M_a^{\pm}-M_c=\frac{1}{4S_\phi}\Big(a^2+2S_\phi : S_\phi-S_C : S_\phi-S_B\Big)-\Big(\frac{1}{2}:\frac{1}{2}:0\Big)=\frac{1}{4S_\phi}(a^2:-S_\phi-S_C:S_\phi-S_B)$.

Now shifting the origin to the point A, we see

$$\begin{aligned} &16S_\phi^2\left|\left|\overrightarrow{M_a^{\pm}M_c}\right|\right|^2 \\ &= (a^2:-S_\phi-S_C:S_\phi-S_B)\cdot(a^2:-S_\phi-S_C:S_\phi-S_B) \\ &= \big((-S_\phi-S_C)\boldsymbol{B}+(S_\phi-S_B)\boldsymbol{C}\big)\cdot\big((-S_\phi-S_C)\boldsymbol{B}+(S_\phi-S_B)\boldsymbol{C}\big) \\ &= (S_\phi+S_C)^2\boldsymbol{B}\cdot\boldsymbol{B}+(S_\phi-S_B)^2\boldsymbol{C}\cdot\boldsymbol{C}-2(S_\phi+S_C)(S_\phi-S_B)\boldsymbol{B}\cdot\boldsymbol{C} \\ &= (S_\phi+S_C)^2(S_A+S_B)+(S_\phi-S_B)^2(S_A+S_C)-2(S_\phi+S_C)(S_\phi-S_B)S_A \\ &= (S_B+S_C)(S_\phi^2+S^2) \end{aligned}$$

Similarly,

$$\begin{aligned}
&16S_\phi^2\left|\left|\overrightarrow{M_a^{\pm}M_b}\right|\right|^2 \\
&= ((S_\phi - S_C)\boldsymbol{B} + (-S_\phi - S_B)\boldsymbol{C})\cdot((S_\phi - S_C)\boldsymbol{B} + (-S_\phi - S_B)\boldsymbol{C}) \\
&= (S_B + S_C)(S_\phi^2 + S^2)
\end{aligned}$$

Thus, we conclude $||\overrightarrow{M_a^{\pm}M_b}|| = ||\overrightarrow{M_a^{\pm}M_c}||$, meaning that $\triangle M_a^{\pm}M_bM_c$ is isosceles. Additionally, since $M_a^{\pm}Y_a^{\pm}$ is perpendicular to BC and $BC \parallel M_bM_c$, it follows that $M_a^{\pm}Y_a^{\pm} \perp M_bM_c$. Given that $\triangle M_a^{\pm}M_bM_c$ is isosceles, this implies that $M_a^{\pm}Y_a^{\pm}$ serves as the perpendicular bisector of M_bM_c.
Furthermore, this establishes that $Y_a^+ \equiv Y_a^- = Y_a$ (say), which coincides on BC, and that $M_a^+M_a^-$ is perpendicular to BC at Y_a. By similar reasoning, we can demonstrate that $M_b^+M_b^-$ is the perpendicular bisector of M_cM_a and that $M_c^+M_c^-$ is the perpendicular bisector of M_aM_b. Consequently, the point N, defined as the intersection $M_a^+M_a^- \cap M_b^+M_b^- \cap M_c^+M_c^-$, is the circumcenter of $\triangle M_aM_bM_c$ and thus corresponds to the Nine-point center of $\triangle ABC$.
For the final step, to establish that the points $P^{\pm}, Q^{\pm}, N$ are collinear, it suffices to compute the following determinant. After an extensive calculation, we find that:

$$\begin{aligned}
&\begin{vmatrix}
\left(2S^2 + a^2S_\phi\right)\left(S^2 + S_{A\phi}\right) & \left(2S^2 + b^2S_\phi\right)\left(S^2 + S_{B\phi}\right) & \left(2S^2 + c^2S_\phi\right)\left(S^2 + S_{C\phi}\right) \\
\left(S_\phi + S_A\right)\left(2S_\phi + a^2\right) & \left(S_\phi + S_B\right)\left(2S_\phi + b^2\right) & \left(S_\phi + S_C\right)\left(2S_\phi + c^2\right) \\
S^2 + S_{BC} & S^2 + S_{CA} & S^2 + S_{AB}
\end{vmatrix} \\
&= S_\phi\left(S^2 - S_\phi^2\right)\left(S_A - S_B\right)\left(S_B - S_C\right)\left(S_C - S_A\right)\left(-S_{AB} - S_{BC} - S_{CA} + S^2\right) \\
&= 0
\end{aligned}$$

also the equation of the line passing through $P^{\pm}, Q^{\pm}$ and N is

$$\begin{vmatrix}
X & Y & Z \\
\left(S_\phi + S_A\right)\left(2S_\phi + a^2\right) & \left(S_\phi + S_B\right)\left(2S_\phi + b^2\right) & \left(S_\phi + S_C\right)\left(2S_\phi + c^2\right) \\
S^2 + S_{BC} & S^2 + S_{CA} & S^2 + S_{AB}
\end{vmatrix} = 0$$

or equivalently,

$$\begin{aligned}
&\left(S_B - S_C\right)\left(2S_A(S_{\phi\phi} + S^2) + S_\phi(S_{AA} - 2S_{BC} + 3S^2)\right)X \\
&+ \left(S_C - S_A\right)\left(2S_B(S_{\phi\phi} + S^2) + S_\phi(S_{BB} - 2S_{CA} + 3S^2)\right)Y \\
&+ \left(S_A - S_B\right)\left(2S_C(S_{\phi\phi} + S^2) + S_\phi(S_{CC} - 2S_{AB} + 3S^2)\right)Z = 0 \qquad (4.11)
\end{aligned}$$

as desired.

REFERENCES

[1] Belenkiy, I. *New features of Napoleon's triangles*, Journal of Geometry **66 1999**, 17-26.

[2] Rigby, J.F. *Napoleon Revisited*, Journal of Geometry **33 1988**, 129-146.

[3] P. Yiu, *Introduction to the Geometry of the Triangle*, Florida Atlantic University Lecture Notes, 2001; with corrections, **2013**.

DEPARTMENT OF MATHEMATICS
IIT BOMBAY
POWAI, 400076 MUMBAI, INDIA.
Email address: reachutsab@gmail.com