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h-RICCI-BOURGUIGNON SOLITONS ON THE $\mathbb{H}^2 \times \mathbb{R}$ LIE GROUP

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ABSTRACT. In this paper, we study the h-Ricci-Bourguignon soliton on the three-dimensional Lie group $\mathbb{H}^2 \times \mathbb{R}$, equipped with left-invariant Riemannian metric.

1. INTRODUCTION AND MOTIVATIONS

In [4], Hamilton presents the idea of Ricci-solitons, which is a natural extension of Einstein metrics. A pseudo-Riemannian metric g on a smooth manifold M is called a Ricci-solitons if there is a smooth vector field X on M that satisfies the equation:

$$Ric + \frac{1}{2}\mathcal{L}_X g = \lambda g, \tag{1.1}$$

where $\mathcal{L}_X$ denotes the Lie derivative in the direction of X, Ric denotes the Ricci tensor and λ is a real number. It is argued that a Ricci soliton is shrinking, steady, or expanding, depending on whether $\lambda > 0$, $\lambda = 0$, or $\lambda < 0$. Furthermore, for some potential function F, a Ricci soliton (M, g) is a gradient Ricci soliton if it admits a vector field X satisfying $X = gradF$.

Since Ricci-solitons are the flow's fixed points, describing them can be thought of as the first step in comprehending the Ricci flow. They are also crucial for comprehending Ricci flow singularities. Type I singularity models correspond to shrinking solitons, type I models to steady Ricci solitons, and type III models to expanding Ricci solitons under the right circumstances. For more details about Ricci-soliton see [1, 6, 8]. In the paper [5], the author consider the three-dimensional Lie group denoted by $\mathbb{H}^2 \times \mathbb{R}$, equipped with left-invariant Riemannian metric. He prove the existence of non-trivial (i.e., not Einstein) Ricci solitons on this three-dimensional Lie group. Moreover, He show that there are not gradient Ricci solitons.

Motivated by [5],in this work we extend the notion of Ricci soliton in the Lie group $\mathbb{H}^2 \times \mathbb{R}$. We show the existence of the h-Ricci-Bourguignon soliton in this Lie group under some condition, where h is a smooth function.

The paper is organized as follow: in Section 2, we recall some basic notions about Ricci-Bourguignon soliton. We give the geometic model of the considered Lie group and their structures. In Section 3, we state our main results.

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2. Preliminaries

In this section we recall some basic notions for proving our main results.

2.1. **Ricci-Bourguignon soliton on a Riemanian manifold.** We refer the reader to [2, 3] for more informations.

Definition 2.1. *Let (M^n, g) be a Riemannian manifold. It is said to be a h-Ricci-Bourguignon soliton if there exist a vector field X, two real constants λ, ρ and a smooth function $h \neq 0$ such that:*

$$\operatorname{Ric} + \frac{h}{2}\mathcal{L}_X g = (\lambda + \rho \operatorname{Scal})g, \tag{2.1}$$

where Scal *is the scalar curvature of the metric g. We then denote this soliton by the triple (M^n, g, X, h, λ).*

Therefore, if $X = \nabla\zeta$ is the gradient of a function $\zeta \in C^\infty(M)$, then we obtain an *h-gradient-Ricci-Bourguignon solution*. Thus, the equation (2.1) becomes:

$$\operatorname{Ric} + h\nabla^2\zeta = (\lambda + \rho \operatorname{Scal})g, \tag{2.2}$$

where $\nabla^2\zeta$ denotes the hessian of ζ. So, for $\rho = \frac{1}{2}$, we obtain a *h-gradient Einstein soliton*, and for $\rho = \frac{1}{n}$, we obtain a *nearly traceless h-gradient-Ricci soliton*.
If the function h is equal to 1, then we have a Ricci-Bourguignon soliton. Furthermore, if there exists a Perelman potential, then we refer to it as a gradient-Ricci-Bourguignon soliton.

2.2. **Geometry of the Lie group $\mathbb{H}^2 \times \mathbb{R}$.** Let $\mathbb{H}^2$ be the Poincaré half plane with the metric

$$g_{\mathbb{H}^2} = \frac{dx^2 + dy^2}{y^2}, \quad y > 0.$$

The space $\mathbb{H}^2$ with the group structure induced by the composition of eigen affine maps is a Lie group, and the metric $g_{\mathbb{H}^2}$ is left-invariant.
Therefore, the Riemannian product $\mathbb{H}^2 \times \mathbb{R}$ is a Lie group with the operation

$$(x, y, z) \cdot (a, b, c) = (x + a, yb, z + c),$$

and the left-invariant product metric is given by

$$g = \frac{dx^2 + dy^2}{y^2} + dz^2. \tag{2.3}$$

With respect to the metric g, an orthonormal basis of left-invariant vector fields on $\mathbb{H}^2 \times \mathbb{R}$ is given by

$$e_1 = y\frac{\partial}{\partial x}, \quad e_2 = y\frac{\partial}{\partial y}, \quad e_3 = \frac{\partial}{\partial z}.$$

And the non-zero Lie brackets of these vector fields are

$$[e_1, e_2] = e_1, \quad [e_2, e_3] = 0, \quad [e_3, e_1] = 0.$$

The Levi-Civita connection (noted here as D) associated with this basis is given by

$$\begin{array}{lll} D_{e_1}e_1 = e_2, & D_{e_1}e_2 = -e_1, & D_{e_1}e_3 = 0, \\ D_{e_2}e_1 = 0, & D_{e_2}e_2 = 0, & D_{e_2}e_3 = 0, \\ D_{e_3}e_1 = 0, & D_{e_3}e_2 = 0, & D_{e_3}e_3 = 0. \end{array}$$

The non-zero components of the curvature tensor are

$$R(e_1, e_2)e_1 = e_2, \quad R(e_1, e_2)e_2 = -e_1.$$

And the components of the Ricci tensor are given by

$$\mathrm{Ric}(e_1, e_1) = \mathrm{Ric}(e_2, e_2) = -1, \quad \mathrm{Ric}(e_3, e_3) = 0.$$

By easy computation one can get that the scalar curvature is constant and is

$$\mathrm{Scal} = \mathrm{tr}(\mathrm{Ric}) = -2.$$

3. MAINS RESULTS

In this section, we give the conditions for a vector field to be a special vector field in the Lie group $\mathbb{H}^2 \times \mathbb{R}$. We begin by this following proposition.

Proposition 3.1. *Let $X(x, y, z) = \xi_1(x, y, z)\partial x + \xi_2(x, y, z)\partial y + \xi_3(x, y, z)\partial z \in \mathcal{X}(\mathbb{H}^2 \times \mathbb{R})$ be a vector. If the triple $(\mathbb{H}^2 \times \mathbb{R}, g, X)$ is an h-Ricci Bourguignon soliton, then we have*

$$\begin{cases} \xi_1(x, y, z) = -(\lambda - 2\rho + 1)e^{-y} \int \left(e^y \partial x \left(\int \frac{dy}{h(x,y,z)}\right)\right) dy - \partial x \varphi_2(x, z) + e^{-y}\varphi_1(x, z) \\ \xi_2(x, y, z) = (\lambda - 2\rho + 1) \int \frac{dy}{h(x,y,z)} + \varphi_2(x, z) \\ \xi_3(x, y, z) = (\lambda - 2\rho) \int \frac{dz}{h(x,y,z)} + \varphi_3(x, y) \end{cases}$$

and

$$\begin{cases} h(x, y, z)(\xi_2(x, y, z) - \partial x \xi_1(x, y, z)) = 2\rho - \lambda - 1 \\ \partial z \xi_1(x, y, z) + \partial x \xi_3(x, y, z) = 0 \\ \partial z \varphi_2(x, z) + \partial y \varphi_3(x, y) = (2\rho - \lambda - 1)\partial z \left(\int \frac{dy}{h(x,y,z)}\right) + (2\rho - \lambda)\partial y \left(\int \frac{dz}{h(x,y,z)}\right) \end{cases}$$

where $\varphi_i : \mathbb{R}^2 \to \mathbb{R}$ is a smooth function, for all $i = 1; 2; 3$.

Proof. The Lie derivative of g with respect to X is given by:

$$\begin{cases} (\mathcal{L}_X g)(e_1, e_1) & = -2(\xi_2 - \partial x \xi_1), \\ (\mathcal{L}_X g)(e_1, e_2) & = \xi_1 + \partial y \xi_1 + \partial x \xi_2, \\ (\mathcal{L}_X g)(e_1, e_3) & = \partial z \xi_1 + \partial x \xi_3, \\ (\mathcal{L}_X g)(e_2, e_2) & = 2\partial y \xi_2, \\ (\mathcal{L}_X g)(e_2, e_3) & = \partial z \xi_2 + \partial y \xi_3, \\ (\mathcal{L}_X g)(e_3, e_3) & = 2\partial z \xi_3. \end{cases} \tag{3.1}$$

That give the following system:

$$\begin{cases} -h(x,y,z)(\xi_2(x,y,z) - \partial x\xi_1(x,y,z)) = \lambda - 2\rho + 1 \quad (L_1), \\ \xi_1(x,y,z) + \partial y\xi_1(x,y,z) + \partial x\xi_2(x,y,z) = 0, \quad (L_2) \\ \partial z\xi_1(x,y,z) + \partial x\xi_3(x,y,z) = 0 \quad (L_3), \\ h(x,y,z)\partial y\xi_2(x,y,z) = \lambda - 2\rho + 1 \quad (L_4), \\ \partial z\xi_2(x,y,z) + \partial y\xi_3(x,y,z) = 0 \quad (L_5), \\ h(x,y,z)\partial z\xi_3(x,y,z) = \lambda - 2\rho \quad (L_6). \end{cases} \tag{3.2}$$

The line (L_6) of the sytem (3.2) gives

$$\xi_3(x,y,z) = (\lambda - 2\rho)\int \frac{dz}{h(x,y,z)} + \varphi_3(x,y)$$

and the line (L_4) gives

$$\xi_2(x,y,z) = (\lambda - 2\rho + 1)\int \frac{dy}{h(x,y,z)} + \varphi_2(x,z)$$

The line (L_2) imply that

$$\partial y\xi_1(x,y,z) + \xi_1(x,y,z) = -(\lambda - 2\rho + 1)\partial x\left(\int \frac{dy}{h(x,y,z)}\right) - \partial x\varphi_2(x,z)$$

$\xi_1^0(x,y,z) = u(x,z)e^{-y}$ is a homogeneous solution. Suppose that $\xi_1^1(x,y,z) = k(x,y,z)e^{-y}$ is a particular solution, then we have

$$\partial yk(x,y,z) = -(\lambda - 2\rho + 1)e^y\partial x\left(\int \frac{dy}{h(x,y,z)}\right) - e^y\partial x\varphi_2(x,z)$$

$$k(x,y,z) = -(\lambda - 2\rho + 1)\int \left(e^y\partial x\left(\int \frac{dy}{h(x,y,z)}\right)\right)dy - e^y\partial x\varphi_2(x,z) + v(x,z),$$

so we get

$$\xi_1(x,y,z) = -(\lambda - 2\rho + 1)e^{-y}\int \left(e^y\partial x\left(\int \frac{dy}{h(x,y,z)}\right)\right)dy - \partial x\varphi_2(x,z) + e^{-y}\varphi_1(x,z)$$

where $\varphi_1(x,z) = u(x,z) + v(x,z)$.

The line $(L_5) \Rightarrow \partial z\varphi_2(x,z) + \partial y\varphi_3(x,y) = (2\rho - \lambda - 1)\partial z\left(\int \frac{dy}{h(x,y,z)}\right) + (2\rho - \lambda)\partial y\left(\int \frac{dz}{h(x,y,z)}\right)$.

□

Proposition 3.2. *Let $X(x,y,z) = \xi_1(x,y,z)\partial x + \xi_2(x,y,z)\partial y + f(z)\partial z$ be a vector field in $\mathcal{X}(\mathbb{H}^2 \times \mathbb{R})$ where f is a smooth function $\mathbb{R}$ such that $z \in \mathbb{R}$, $f(z) \neq 0$ and $\lambda = 2\rho - 1$. If $(\mathbb{H}^2 \times \mathbb{R}, g, X)$ is a h-Ricci Bourguignon soliton, then we have*

$$X(x,y,z) = \left(ae^{-y} + b\sin(x) - c\cos(x) + d\right)\partial x + \left(b\cos(x) + c\sin(x)\right)\partial y + f(z)\partial z$$

and

$$h(x,y,z) = \frac{-1}{f'(z)}$$

where a, b, c, d are constants.

Proof. The system (3.2) becomes

$$\begin{cases} \xi_2(x,y,z) - \partial x\xi_1(x,y,z) = 0 \quad (S_1), \\ \xi_1(x,y,z) + \partial y\xi_1(x,y,z) + \partial x\xi_2(x,y,z) = 0, \quad (S_2) \\ \partial z\xi_1(x,y,z) = 0 \quad (S_3), \\ \partial y\xi_2(x,y,z) = 0 \quad (S_4), \\ \partial z\xi_2(x,y,z) = 0 \quad (S_5), \\ h(x,y,z)f'(z) = -1 \quad (S_6). \end{cases}$$

The line (S_3) implies that $\partial z\xi_1(x,y,z) = 0$, so we get

$$\xi_1(x,y,z) = \zeta(x,y).$$

The lines (S_2) and (S_4) imply that

$$\partial_{yy}\xi_1(x,y,z) + \partial y\xi_1(x,y,z) = 0$$

and the previous result directly implies that ;

$$\partial_{yy}\zeta(x,y) + \partial y\zeta(x,y) = 0$$

and then we have

$$\partial_{yy}\zeta(x,y) + \partial y\zeta(x,y) = 0 \Rightarrow \partial y\zeta_2(x,y) = a(x)e^{-y} \Rightarrow \zeta(x,y) = a(x)e^{-y} + b(x).$$

Using the lines (S_4) and (S_1), we get

$$\partial y\partial x\xi_1(x,y,z) = 0 \Rightarrow -e^{-y}\partial xa(x) = 0 \Rightarrow \xi_1(x,y,z) = c_2e^{-y} + b(x)$$

The lines (S_6) and (S_5) imply also that $\partial_{zz}\xi_2(x,y,z) = 0 \Rightarrow \xi_2(x,y,z) = k_1(x,y)z + k_2(x,y)$.
The line (S_4) implies that

$$\begin{cases} k(x,y) = \phi_1(x) \\ k_2(x,y) = \phi_2(x). \end{cases}$$

Using (S_2), (S_1) and the fact that $\partial x\partial y\xi_1(x,y,z) = 0$, we have directely

$$\begin{cases} \phi_1(x) + \phi_1''(x) = 0 \\ \phi_2(x) + \phi_2''(x) = 0 \end{cases}$$

and the characteristic equation of these second order linear differential equations is

$$r^2 + 1 = 0 \Rightarrow \begin{cases} \phi_1(x) = x_0\cos(x) + x_1\sin(x) \\ \phi_2(x) = x_2\cos(x) + x_3\sin(x) \end{cases} \Rightarrow \xi_2(x,y,z) = x_0z\cos(x) + x_1\sin(x)z + x_2\cos(x) + x_3$$

The line (S_5) implies that

$$\xi_2(x,y,z) = x_2\cos(x) + x_3\sin(x),$$

so using the line (S_1), we get

$$x_2\cos(x) + x_3\sin(x) = b'(x)$$

and by identification we have

$$b(x) = x_0\sin(x) - x_1\cos(x) + c_1$$

$$\xi_1(x,y,z) = c_2e^{-y} + x_2\sin(x) - x_3\cos(x) + c_1.$$

And the line (S_6) shows that $h(x,y,z) = \frac{-1}{f'(z)}$. □

Theorem 3.1. *Consider the three-dimensional Lie group $\mathbb{H}^2 \times \mathbb{R}$, with the left-invariant Riemannian metric g, defined by the equation* (2.3). *Then we have*

(1) $(\mathbb{H}^2 \times \mathbb{R}, g, X)$ is a shrinking Ricci-Bourguignon soliton if $\rho > \frac{1}{2}$;
(2) $(\mathbb{H}^2 \times \mathbb{R}, g, X)$ is a steady Ricci-Bourguinion soliton if $\rho = \frac{1}{2}$;
(3) $(\mathbb{H}^2 \times \mathbb{R}, g, X)$ is an expanding Ricci-Bourguignon soliton if $\rho < \frac{1}{2}$.

Moreover we have

$$X = \left(c_1 \sin(x) - c_2 \cos(x) + ae^{-y}\right) \partial x + \left(c_1 \cos(x) + c_2 \sin(x)\right) \partial y + (-z + b)\, \partial z$$

where c_1, c_2, a, b are constants.

Proof. Just take $h(x,y,z) = 1$ in the Proposition 3.1. Then we get

$$\begin{cases} \xi_1(x,y,z) = -\partial x \varphi_2(x,z) + e^{-y}\varphi_1(x,z) \quad (T_1) \\ \xi_2(x,y,z) = (\lambda - 2\rho + 1)y + \varphi_2(x,z) \quad (T_2) \\ \xi_3(x,y,z) = (\lambda - 2\rho)z + \varphi_3(x,y) \quad (T_3) \\ \xi_2(x,y,z) - \partial x \xi_1(x,y,z) = 2\rho - \lambda - 1 \quad (T_4) \\ \partial z \xi_1(x,y,z) + \partial x \xi_3(x,y,z) = 0 \quad (T_5) \\ \partial z \varphi_2(x,z) + \partial y \varphi_3(x,y) = 0 \quad (T_6) \end{cases}$$

The line (T_4) implies that

$$\varphi_2(x,z) + \partial_{xx}\varphi_2(x,z) = e^{-y}\partial x \varphi_1(x,z) + (2\rho - \lambda - 1)(y+1)$$

This equation is true if

$$\begin{cases} \partial x \varphi_1(x,z) = 0 \\ \lambda = 2\rho - 1 \end{cases}$$

so there exists a smooth function f in $\mathbb{R}$ such that $\varphi_1(x,z) = f(z)$.
The line (T_5) implies that

$$-\partial z \partial x \varphi_2(x,z) + e^{-y} f'(z) + \partial x \varphi_3(x,y) = 0$$

this equation is true only if $f'(z) = c \in \mathbb{R}$ and $\partial z \partial x \varphi_2(x,z) = 0$

$$\begin{cases} f(z) = cz + a \Rightarrow \varphi_3(x,y) = cxe^{-y} + \eta(y) \\ \partial z \varphi_2(x,z) = \psi_1(x) \Rightarrow \varphi_2(x,z) = \psi_1(x)z + \psi_2(x) \end{cases}$$

The line (T_6) becomes

$$\psi_1(x) - cxe^{-y} + \eta'(y) = 0$$

this equation is true only if $\psi_1(x) = c = \eta'(y) = 0$. So we get

$$\begin{cases} \varphi_2(x,z) = \psi_2(x) \\ \varphi_3(x,y) = b \in \mathbb{R} \end{cases}$$

Moreover $\psi_2(x) + \partial_{xx}\psi_2(x) = 0$

$$r^2 + 1 = 0$$

the solutions are $r_1 = i \in \mathbb{C}$ and $r_2 = -i \in \mathbb{C}$ then

$$\psi_2(x) = c_1 \cos(x) + c_2 \sin(x).$$

Finally

$$\begin{cases} \xi_1(x,y,z) = c_1\sin(x) - c_2\cos(x) + ae^{-y} \\ \xi_2(x,y,z) = c_1\cos(x) + c_2\sin(x) \\ \xi_3(x,y,z) = -z + b. \end{cases}$$

□

If we take $f(z) = -e^z$ in the Proposition 3.2 then by a first-order Taylor expansion in the neighborhood of $z = 0$, the triple $(\mathbb{H}^2 \times \mathbb{R}, g, X)$ becomes Ricci-Bourguignon soliton.

Proposition 3.3. *Consider the three-dimensional Lie group $(\mathbb{H}^2 \times \mathbb{R}, g)$, with the left-invariant Riemannian metric defined by the expression* (2.3). *This group is gradient-Ricci Bourguignon soliton only if $X(x,y,z) = (-z+b)\partial_z$ where b is constant.*

Proof. Let's reason by contradiction. Assume that there exists a smooth function ζ on $\mathbb{H}^2 \times \mathbb{R}$ such that $X(x,y,z) = \mathrm{grad}(\zeta(x,y,z))$. The expression of the gradient is given by

$$\mathrm{grad}\,\zeta(x,y,z) = y^2\partial x\zeta(x,y,z)\,\partial_x + y^2\partial y\zeta(x,y,z)\,\partial_y + \partial z\zeta(x,y,z)\partial_z.$$

By identification we have

$$\begin{cases} \partial x\zeta(x,y,z) = \frac{c_1\sin(x)-c_2\cos(x)+ae^{-y}}{y^2} \quad (H_1) \\ \partial y\zeta(x,y,z) = \frac{c_1\cos(x)+c_2\sin(x)}{y^2} \quad (H_2) \\ \partial z\zeta(x,y,z) = -z+b \quad (H_3) \end{cases}$$

$(H_3) \Rightarrow \zeta(x,y,z) = \frac{-1}{2}z^2 + bz + \zeta_3(x,y)$.
$(H_2) \Rightarrow \partial y\zeta_3(x,y) = \frac{c_1\cos(x)+c_2\sin(x)}{y^2} \Rightarrow \zeta_3(x,y) = \frac{-c_1\cos(x)-c_2\sin(x)}{y} + \zeta_2(x)$ so we have

$$\zeta(x,y,z) = \frac{-c_1\cos(x) - c_2\sin(x)}{y} - \frac{1}{2}z^2 + bz + \zeta_2(x)$$

$(H_1) \Rightarrow \frac{c_1\sin(x)-c_2\cos(x)+ae^{-y}}{y^2} = \frac{c_1\sin(x)-c_2\cos(x)}{y} + \zeta_2'(x)$ and then we have

$$\zeta_2'(x) = \frac{(1-y)(c_1\sin(x) - c_2\cos(x))}{y^2} + \frac{ae^{-y}}{y^2}.$$

This is only true if $c_1 = c_2 = a = 0$. □

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