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DUAL UNIFORM DEFINITIONS OF CONVEX TETRAGONS

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ABSTRACT. In this article, new definitions for well-known convex tetragons are generated, as well as new convex tetragons are defined, solely based on six equalities of sums of sides, sums of angles, sums of diagonal segments, whether they are opposite or consecutive. Tetragons are also defined in multiple ways as a particularization of two other already defined tetragons. The dualities sides/angles, sides/diagonal segments and angles/diagonal segments between these definitions are made explicit.

Key Words: convex tetragon definitions, convex tetragon classification, duality
MSC 2020: 51M04

1. INTRODUCTION

The first objective of this article is to standardize the definitions of a large part of convex tetragons (Q) based on six equalities of sums of sides, sums of angles, sums of diagonal segments and whether they are opposite or consecutive. A tetragon is defined as a configuration of four points in a plane no three of which are collinear and four line segments joining these points in a cycle, where a cycle and its reverse cycle are considered the same. Four points can be cyclically ordered in three ways ($A-B-C-D$, $A-B-D-C$, $A-C-B-D$) so in tetragons we can differentiate between consecutive and opposite vertices. The diagonals of a tetragon are line segments whose ends are opposite vertices. We can differentiate between convex and non-convex tetragons. A tetragon is convex if its diagonals intersect in an inner point of the diagonal segments, otherwise it is non-convex (See Figure 1).

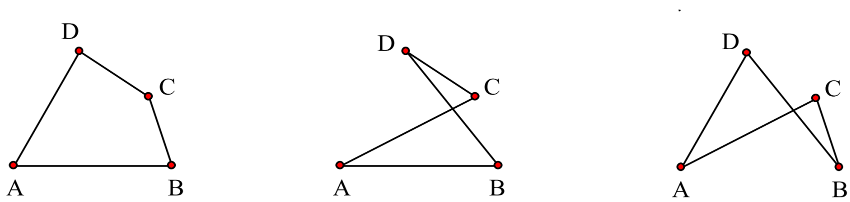

Figure 1. Convex tetragon (left) and non convex tetragons (center and right)

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In what follows we will work exclusively with convex tetragons. As notation we will use a, b, c, d for the measure of its sides, A, B, C, D for the measure of its angles and a', b', c', d' for the measure of its diagonal segments. If diagonals AC and BD intersect at O, we call OA, OB, OC, OD the diagonal segments. (See Figure 2) We will name Q as $abcd$, $ABCD$ or $a'b'c'd'$ according to the convenience of each situation.

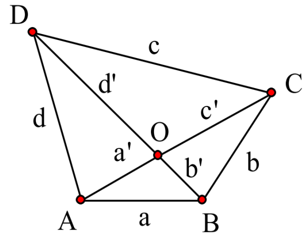

Figure 2. Notation for sides, angles and diagonal segments of the convex tetragon

2. LEVEL 1 TETRAGONS

We'll start in level 1 (Q is level 0) with the following six Qs defined in this way:

Tetragon	Definition	Definition
Tangential	Q with equal opposite sides sums	$a+c=b+d$
Extangential	Q with equal consecutive sides sums	$a+b=c+d$
Cyclic	Q with equal opposite angle sums	$A+C=B+D$
Trapezium	Q with equal consecutive angle sums	$A+B=C+D$
Equidiagonal	Q with equal opposite diagonal segment sums	$a'+c'=b'+d'$
Semidiagonal	Q with equal consecutive diagonal segments sums	$a'+b'=c'+d'$

Table 1. Definitions of the six level 1 tetragons

Tangential, extangential and cyclic are usually defined in association with a circle: tangential with sides tangent to the same circle, extangential with the extensions of the sides tangent to the same circle, cyclic with vertices belonging to the same circle. (See Figure 3) The respective characterizations of these three Qs, which we assume as definitions, are well known. Trapezium is usually defined as a quadrilateral with at least one pair of parallel sides. The definition of trapezoid ($A+B=C+D$) that we assume here first appears in [1], as far as we know. Equidiagonal ($a'+c'=b'+d'$) is defined in a new way using the diagonal segments and not as a Q with equal diagonals ($AC=BD$) as usual. The choice to define equidiagonal in this new way is because it has already proven fruitful, as can be seen in [2]. Semidiagonal ($a'+b'=c'+d'$) is Q for which we have no reference.

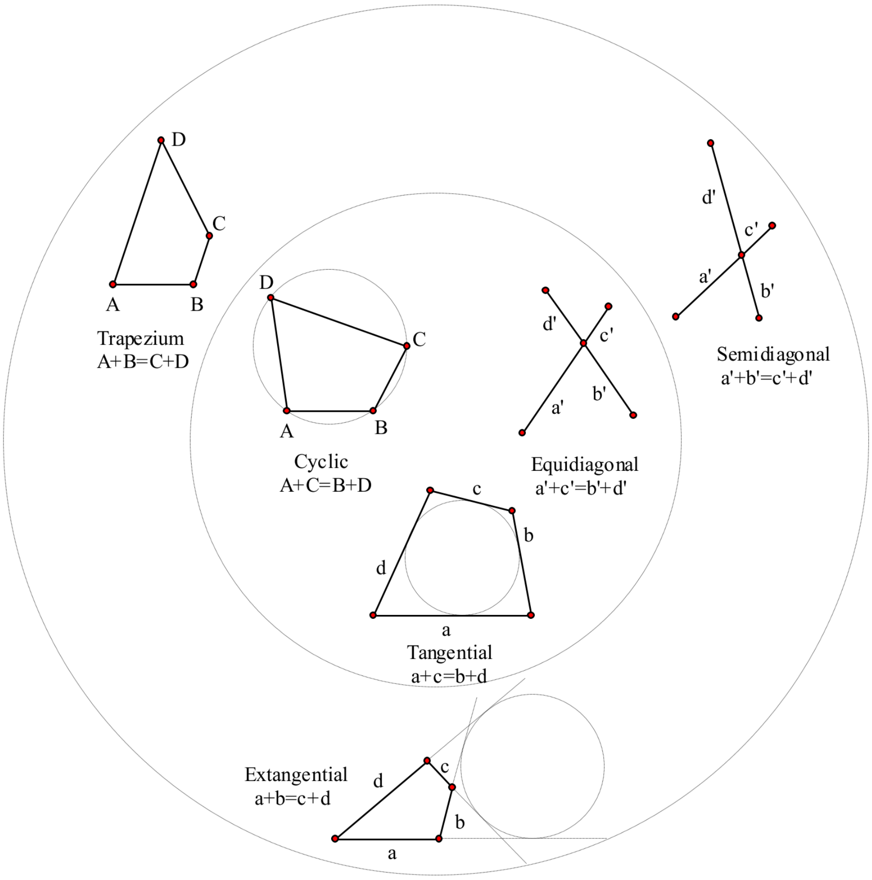

Figure 3. The six level 1 tetragons

Each of these six Qs has its own story. Tangential was defined by Henri Pitot in 1725 and the first characterization given by Jacob Steiner in 1846. In [3], [4], [5], [6], more than one hundred tangential characterizations are given. Jacob Steiner, in 1848, too defined and gave the first characterization of extangential. In [7], [8], [9], [10] characterizations of extangential are included. Cyclic was defined in Euclid's *Elements* and characterizations can be seen in [11], [12], [13]. Trapezium was defined by Posidonius (1st century B.C.) and characterizations can be found in [14], [15]. Equidiagonal characterizations are given in [16]. Generate characterizations to semidiagonal is a pending task.

In the cases of consecutive sides there are two options: $a + b = c + d$ or $d + a = b + c$. Both can be integrated into the expression $|a - c| = |b - d|$. Without losing generality, in what follows we will use only $a + b = c + d$. It will be done in a similar way with the angles and diagonal segments.

Why place the six tetragons of level 1 in a circular diagram (see Figure 3) and the distinction therein between sum of opposites and sum of consecutives will be understood in the next sections.

3. DUALITY

The backbone of this article is duality. De Villiers defines it precisely as: "In mathematics, two theorems or configurations are called *dual* if one can be obtained from the other by replacing each concept and operator by its dual concept or operator." [17] Two different Qs can have dual definitions: tangential ($a + c = b + d$) definition and cyclic ($A + C = B + D$) definition are dual sides/angles. Swapping sides for angles and angles for sides in the definitions of tangential and cyclic yields the definitions of cyclic and tangential respectively. Definitions of tangential ($a + c = b + d$) and equidiagonal ($a' + c' = b' + d'$) are dual sides/diagonal segment. Semidiagonal ($a' + b' = c' + d'$) was defined to be dual sides/diagonal segments of extangential ($a + b = c + d$). Two definitions of the same Q too can be dual: the definitions of parallelogram as Q that verifies $a = c$ and $b = d$ or as Q that verifies $A = C$ and $B = D$ are dual sides/angles. And also the definition of a Q can be self-dual: the definition of square as Q that verifies $a = b = c = d$ and $A = B = C = D$ is self dual sides/angles. Duality is a concept that has been present in some works on the definition and classification of Qs, as can be seen in [1], [2], [18], [19], [20], [21], [22], [23].

A second objective is to analyze which Qs are defined -and in what way- when 2, 3 or 4 of the mentioned conditions of level 1 are considered. A third objective is to define the Qs of levels 2, 3, 4 as particularizations of two Qs of lower levels. A fourth objective is to explain three dualities between the definitions of the Qs defined in the article.

In the following Table 2 we include definitions of Qs that will be defined in another uniform way in the article.

Tetragon	Definition	Definition
Kite	Q with two pairs of equal consecutive sides	$d = a$ and $b = c$ (1) or $a = b$ and $c = d$ (2)
Isosceles trapezium	Q with two pairs of equal consecutive angles	$D = A$ and $B = C$ (1) or $A = B$ and $C = D$ (2)
	Q with two pairs of equal consecutive diagonal segments	$d' = a'$ and $b' = c'$ (1) or $a' = b'$ and $c' = d'$ (2)
Parallelogram	Q with two pairs of equal opposite sides	$a = c$ and $b = d$
	Q with two pairs of equal opposite angles	$A = C$ and $B = D$
	Q with two pairs of equal opposite diagonal segments	$a' = c'$ and $b' = d'$
Right kite	Kite with at least two opposite angles measuring $\frac{1}{4}(A + B + C + D)$	$d = a, b = c$ and $B = D$ $= \frac{1}{4}(A + B + C + D)$ (1) or $a = b, c = d$ and $A = C$ $= \frac{1}{4}(A + B + C + D)$ (2)
Kite $p'/4$	Kite with at least two opposite diagonal segments measuring $\frac{1}{4}(a' + b' + c' + d')$	$d = a, b = c$ and $b' = d'$ $= \frac{1}{4}(a' + b' + c' + d')$ (1) or $a = b, c = d$ and $a' = c'$ $= \frac{1}{4}(a' + b' + c' + d')$ (2)

Isosceles trapezium $p/4$	Isosceles trapezium at least two opposite sides measuring $\frac{1}{4}(a+b+c+d)$	$D=A$, $B=C$ and $a=c$ $=\frac{1}{4}(a+b+c+d)$ (1) or $A=B$, $C=D$ and $b=d$ $=\frac{1}{4}(a+b+c+d)$ (2)
		$d'=a'$, $b'=c'$ and $a=c$ $=\frac{1}{4}(a+b+c+d)$ (1) or $a'=b'$, $c'=d'$ and $b=d$ $=\frac{1}{4}(a+b+c+d)$ (2)
Rhombus	Q with equal sides	$a=b=c=d$
Rectangle	Q with equal angles	$A=B=C=D$
	Q with equal diagonal segments	$a'=b'=c'=d'$
Square	Q with equal sides and equal angles	$a=b=c=d$ and $A=B=C=D$
	Q with equal sides and equal diagonal segments	$a=b=c=d$ and $a'=b'=c'=d'$

Table 2. Convex tetragon definitions

The three definitions of parallelogram, considered in pairs, are dual sides / angles, sides / diagonal segments, angles / diagonal segments. These same dualities are present between the definitions of kite and isosceles trapezium, between the definitions of right kite, kite $p'/4$ and isosceles trapezium $p/4$, between the definitions of rhombus and rectangle. One definition of square is self-dual sides/angles, the other is self-dual sides / diagonal segments, and between them are dual angles / diagonal segments. These three dualities will also be present in the uniform definitions that will be generated in the next sections. The meaning of distinguishing between (1) and (2) in some definitions in Table 2 will be highlighted in section 7.

Another way to appreciate the dualities between the definitions in Table 2 is through the diagram in the next Figure 4.

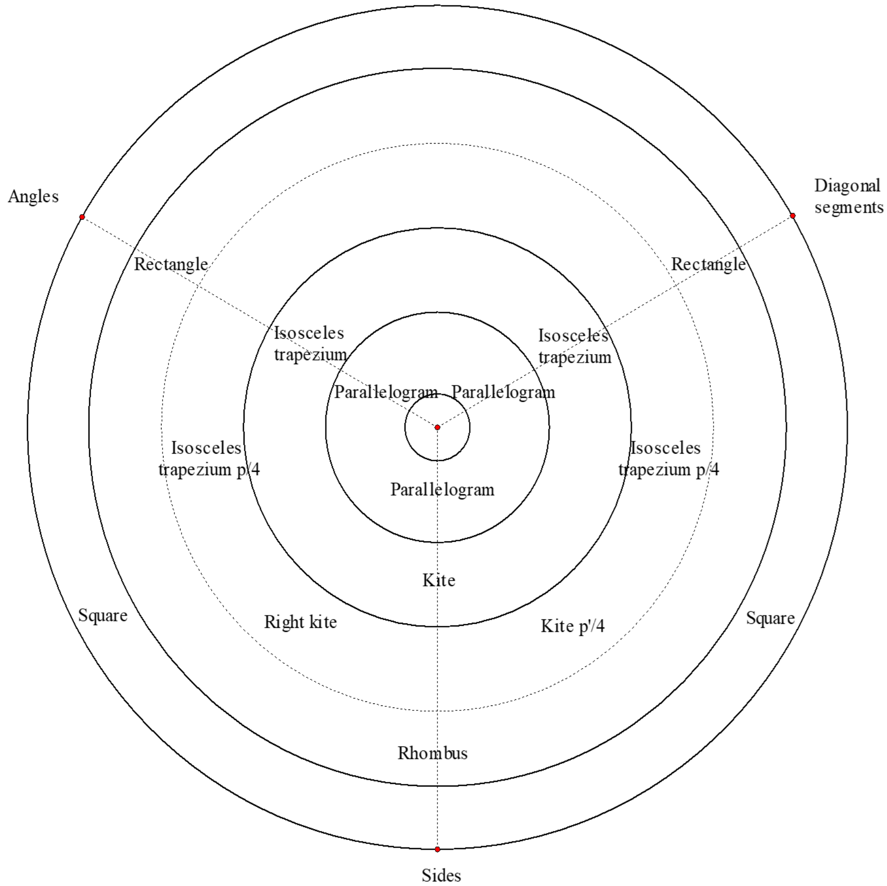

Figure 4. Dualities in convex tetragons definitions

4. SIDES/ANGLES DUALITY

To achieve our objectives we will expand on an idea by Roger Wheeler in the two-page article [1] from 1958. The diagram in [1] is not exhaustive. What Qs are defined in the cases not considered by Wheeler? How are these Qs defined? At level 1 we will place tangential ($a + c = b + d$), extangential ($a + b = c + d$), cyclic ($A + C = B + D$) and trapezium ($A + B = C + D$). In levels 2, 3, 4 we will consider all the Qs defined by two, three, and four conditions of level 1, respectively.
In level 2 we will consider the six Qs defined by two conditions of level 1. (See Table 3)

Proposition 4.1. *Q is tangential and cyclic if and only if it is bicentric.*

This definition of bicentric ($a + c = b + d$, $A + C = B + D$) is self-dual sides/angles. Characterizations of bicentric can be found in [24] and [25]. It is through some of these characterizations that we can make Euclidean constructions for bicentric.

Proposition 4.2. *Q is tangential and trapezium if and only if it is tangential trapezium.*

We don't know if there are characterizations for tangential trapezium besides the one proved in [26].

Proposition 4.3. *Q is extangential and cyclic if and only if it is exbicentric.*

Characterizations of exbicentric can be found in [27].

The definitions of tangential trapezium ($A+B=C+D$, $a+c=b+d$) and exbicentric ($A+C=B+D$, $a+b=c+d$) are dual sides/angles.
In the proofs that follows we will only demonstrate the sufficient conditions, the necessary conditions are simple to verify.

Proposition 4.4. *Q is extangential and trapezium if and only if it is parallelogram.*

Q verifies $A+B=C+D$. If $A=90°$, then $B=90°$. If $C=90°$, $ABCD$ rectangle. If $C>90°$ (see Figure 5, left), exist E in AD such that $ABCD$ is rectangle. According to $a+b=c+d$, so $a+b=c+(b+ED)$, then $a=c+DE$. This is a contradiction with $a<c+DE$. If $A<90°$, then $B>90°$. If $b<d$ (see Figure 5, right), exist E in AD such that $ABCD$ is parallelogram. According to $a+b=c+d$, so $a+b=c+(b+ED)$, then $a=c+DE$. This is a contradiction with $a<c+DE$. An analogous contradiction holds if $b>d$.

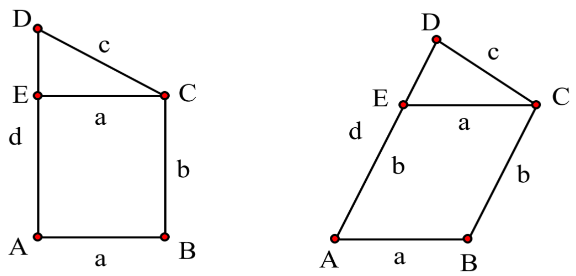

Figure 5. Extangential ($a+b=c+d$) and trapezium ($A+B=C+D$) $\Leftrightarrow$ $ABCD$ parallelogram

This definition of parallelogram ($A+B=C+D$, $a+b=c+d$) is self-dual sides/angles. Other characterizations of parallelogram can be found in [28], [29], [30].

Proposition 4.5. *Q is tangential and extangential if and only if it is kite.*

If $a+c=b+d$ and $a+b=c+d$, then $c-b=b-c$, so $b=c$ and $d=a$. $abcd$ is kite.

Proposition 4.6. *Q is cyclic and trapezium if and only if it is isosceles trapezium.*

If $A+C=B+D$ and $A+B=C+D$, then $C-B=B-C$, so $B=C$ and $D=A$. $ABCD$ is isosceles trapezium.
The definitions of kite ($a+c=b+d$, $a+b=c+d$) and isosceles trapezium ($A+$

$C = B + D$, $A + B = C + D$) are dual sides/angles. The proofs of the two previous characterizations too are dual sides/angles.

In level 3 we will consider the four Qs defined by three conditions of level 1. (See Table 3)

Proposition 4.7. *Q is tangential, extangential and cyclic if and only if it is right kite.*

If Q verifies $a + c = b + d$ and $a + b = c + d$, then $d = a$ and $b = c$, so $abcd$ is kite, then $B = D$. If the condition $A + C = B + D$ is added it's verify $B = D = \frac{1}{4}(A + B + C + D)$ and $abcd$ is right kite.

Proposition 4.8. *Q is tangential, cyclic and trapezium if and only if it is isosceles trapezium $p/4$.*

If Q verifies $A + C = B + D$ and $A + B = C + D$, then $D = A$ and $B = C$, so $ABCD$ is isosceles trapezium, then $a = c$. If the condition $a + c = b + d$ is added it's verify $AB = CD = \frac{1}{4}(a + b + c + d)$ and $ABCD$ is isosceles trapezium $p/4$.
The definitions of right kite ($a + c = b + d$, $a + b = c + d$, $A = D = \frac{1}{4}(A + B + C + D)$) and isosceles trapezium $p/4$ ($A + C = B + D$, $A + B = C + D$, $a = c = \frac{1}{4}(a + b + c + d)$) are dual sides/angles. The two previous proofs too are dual sides/angles.

Proposition 4.9. *Q is tangential, extangential and trapezium if and only if it is rhombus.*

If Q verifies $a + c = b + d$ and $a + b = c + d$, then $d = a$ and $b = c$, so $abcd$ is kite, then $B = D$. If condition $A + B = C + D$ is verified this implies $A + B = C + D = \frac{1}{2}(A + B + C + D) = 180°$, then $B = 180° - A$ and $D = B$, so $C = A$. $ABCD$ is parallelogram with $d = a$, so $ABCD$ is rhombus.

Proposition 4.10. *Q is extangential, cyclic and trapezium if and only if it is rectangle.*

If Q verifies $A + C = B + D$ and $A + B = C + D$, then $D = A$ and $B = C$, so $ABCD$ is isosceles trapezium, then $a = c$. If the condition $a + b = c + d$ is added it's verify $b = d$ and $ABCD$ so parallelogram, so $A = C$. $A + C = B + D = \frac{1}{2}(A + B + C + D) = 180°$, then $A = C = 90°$ and $B = D = 90°$, so $ABCD$ is rectangle.
The definitions of rhombus ($A + B = C + D$, $a + c = b + d$, $a + b = c + d$) and rectangle ($a + b = c + d$, $A + C = B + D$, $A + B = C + D$) are dual sides/angles. The two previous proofs too are dual sides/angles.

In level 4 we will consider the Qs defined by four conditions of level 1. (See Table 3)

Proposition 4.11. *Q is tangential, extangential, cyclic and trapezium if and only if it is square.*

Q verifies $a + c = b + d$, $A + B = C + D$ and $a + b = c + d$ if and only if it is rhombus, so $a = b = c = d$. Q verifies $A + C = B + D$, $A + B = C + D$ and $a + b = c + d$ if and only if it is rectangle, so $A = B = C = D$. If Q verifies $a = b = c = d$ and $A = B = C = D$ is a

square.
Characterizations of square can be found in [31].

In the following Table 3, with (*) we indicate definitions with self-duality sides/angles. In the right column we distinguish the characterizations according to whether they are sums of opposites or consecutives. At level 2 we distinguish between definitions in which two equalities are sums of opposites (*OO*), one sum of opposites and the other of consecutives (*OC*), or both are sums of consecutives (*CC*). At level 3 we distinguish the definitions according to whether they consist of two sums of opposites and one of consecutives (*OOC*) or one of opposites and two of consecutives (*OCC*).
In Table 3 the numbers associated with each quadrilateral have no other function that to facilitate the reading of what follows.

Level	Angles	Angles-Angles-Sides	Angles-Sides	Sides-Sides-Angles	Sides	Opposite (O)/ Consecutive (C)
1	3 Cyclic $A+C=B+D$				1 Tangential $a+c=b+d$	O
1	4 Trapezium $A+B=C+D$				2 Extangential $a+b=c+d$	C
2			13(*) Bicentric $A+C=B+D$, $a+c=b+d$			OO
2	34 Isosceles trapezium $A+C=B+D$, $A+B=C+D$		14 Tangential trapezium $A+B=C+D$, $a+c=b+d$		12 Kite $a+c=b+d$, $a+b=c+d$	OC
2	34 Isosceles trapezium $A+C=B+D$, $A+B=C+D$		23 Exbicentric $A+C=B+D$, $a+b=c+d$,		12 Kite $a+c=b+d$, $a+b=c+d$	OC
2			24(*) Parallelogram $A+B=C+D$ $a+b=c+d$			CC
3		134 Isosceles trapezium $p/4$ $A+C=B+D$, $A+B=C+D$, $a+c=b+d$		123 Right kite $a+c=b+d$, $a+b=c+d$, $A+C=B+D$		OOC
3		234 Rectangle $A+C=B+D$, $A+B=C+D$, $a+b=c+d$		124 Rhombus $a+c=b+d$, $a+b=c+d$, $A+B=C+D$		OOC
4			1234(*) Square $A+C=B+D$, $A+B=C+D$, $a+c=b+d$, $a+b=c+d$			$OOCC$

Table 3. Sides/angles duality

According to the right column of Table 3, at level 2 it is observed that the definition with two equal sums of consecutives defines parallelogram and therefore that $a = c$, $b = d$ and $A = C$, $B = D$. The definition with two equal sums of opposites defines bicentric but these two conditions do not allow inferring the equality of sides or angles as in the definition of parallelogram. The definition with equalities of sums of consecutives is more restrictive than the definition with equalities of sums of opposites. Something similar happens at level 3. The definitions that consist of two sums of opposites and one of consecutives defines right kite or isosceles trapezium $p/4$, so $d = a$, $b = c$, $B = D = \frac{1}{4}(A+B+C+D)$ or $D = A$, $B = C$ and $a = c = \frac{1}{4}(a+b+c+d)$. The

definitions that consist of one sum of opposites and two sums of consecutives defines rhombus or rectangle, so $a = b = c = d$ or $A = B = C = D$.
In Table 3 five Qs are added that Wheeler in [1] did not consider. Bicentric, tangential trapezium, exbicentric appear in level 2 along with kite, isosceles trapezium and parallelogram which are well-known. At level 3 right kite and isosceles trapezium $p/4$ appear along with rhombus and rectangle which are very well-known Qs.
The definitions of all Qs that appear in Table 2 are dual or self-dual sides/angles. They are dual sides/angles the definitions of kite and isosceles trapezium, tangential trapezium and exbicentric, right kite and isosceles trapezium $p/4$, rhombus and rectangle. They are self-dual bicentric, parallelogram, square.
Due to the way Table 3 was structured, Qs of level 2 can be defined as a particularization of two Qs of level 1. Qs of level 3 can be defined as a particularization of a Q of level 2 and a Q of level 1, as a particularization of two Qs of level 2 or as a particularization of three Qs of level 1. Qs of level 4 can be defined as a particularization of two Qs of level 3, as a particularization of one Q of level 3 and one Q of level 1, as two Qs of level 2, as four Qs of level 1. (See Table 4)

<table>
<tr><td colspan="3">Cyclic</td><td colspan="3">Trapezium</td><td colspan="3">Extangential</td><td colspan="3">Tangential</td></tr>
<tr><td colspan="12"></td></tr>
<tr><td colspan="12">Bicentric</td></tr>
<tr><td colspan="12">Tangential and cyclic</td></tr>
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<tr><td colspan="4"></td><td colspan="4">Tangential and trapezium</td><td colspan="4"></td></tr>
<tr><td colspan="4">Isosceles trapezium</td><td colspan="4"></td><td colspan="4">Kite</td></tr>
<tr><td colspan="4">Cyclic and trapezium</td><td colspan="4"></td><td colspan="4">Tangential and extangential</td></tr>
<tr><td colspan="4"></td><td colspan="4">Exbicentric</td><td colspan="4"></td></tr>
<tr><td colspan="4"></td><td colspan="4">Extangential and cyclic</td><td colspan="4"></td></tr>
<tr><td colspan="12"></td></tr>
<tr><td colspan="12">Parallelogram</td></tr>
<tr><td colspan="12">Extangential and trapezium</td></tr>
<tr><td colspan="12"></td></tr>
</table>

Isosceles trapezium $p/4$	*Right kite*
Isosceles trapezium and tangential	Kite and cyclic
Cyclic and tangential trapezium	Tangential and exbicentric
Tangential trapezium and bicentric	Extangential and bicentric
Isosceles trapezium and bicentric	Kite and bicentric
Isosceles trapezium and tangential trapezium	Kite and exbicentric
Bicentric and tangential trapezium	Bicentric and exbicentric
Cyclic, trapezium and tangential	Tangential, extangential and cyclic
Rectangle	*Rhombus*
Parallelogram and cyclic	Parallelogram and tangential
Isosceles trapezium and extangential	Kite and trapezium
Exbicentric and trapezium	Tangential trapezium and extangential
Parallelogram and isosceles trapezium	Parallelogram and kite
Parallelogram and exbicentric	Parallelogram and tangential trapezium
Isosceles trapezium and exbicentric	Kite and tangential trapezium
Cyclic, trapezium and extangential	Tangential, extangential and trapezium
Square	
Rhombus and rectangle	
Rhombus and isosceles trapezium $p/4$	
Rectangle and right kite	
Right kite and isosceles trapezium $p/4$	
Rhombus and cyclic	
Rectangle and tangential	
Right kite and trapezium	
Isosceles trapezium $p/4$ and extangential	
Kite and isosceles trapezium	
Bicentric and parallelogram	
Exbicentric and tangential trapezium	
Cyclic, trapezium, extangential and tangential	

Table 4. Definitions as particularizations of other tetragons

Square is frequently defined as Q that is rhombus and rectangle, that is, as a particularization of two Qs of level 3. But we have no reference for the definition of square as Q that is kite and isosceles trapezium, as a particularization of two Qs of level 2. And neither for any of the other ten definitions. Rhombus, rectangle and right kite, isosceles trapezium $p/4$ are each defined in seven different forms dual sides/angles. Right kite and isosceles trapezium $p/4$ appear with the same importance as rhombus and rectangle, although usually the latter are much more familiar.

5. SIDES/DIAGONAL SEGMENTS DUALITY

In this section we will proceed analogously to the previous section. At level 1 we will place tangential ($a+c=b+d$), extangential ($a+b=c+d$), equidiagonal ($a'+c'=b'+d'$) and semidiagonal ($a'+b'=c'+d'$). In level 2 we will consider the four Qs defined by two conditions of level 1 (See Table 5)

Proposition 5.1. *Q is tangential and equidiagonal if and only if it is tangential equidiagonal.*

Tangential equidiagonal can be defined as Q that verifies $a+c=b+d$, $a'+c'=b'+d'$, a definition self-dual sides/diagonal segments. Tangential equidiagonal is a Q for which we have no reference. It is a pending task to characterize it as well as to provide Euclidean constructions for it.

Proposition 5.2. *Q is tangential and semidiagonal if and only if it is tangential semidiagonal.*

Tangential semidiagonal can be defined as Q that verifies $a+c=b+d$, $a'+b'=c'+d'$. Tangential semidiagonal is a Q for which we have no reference. It is a pending task to characterize it as well as to provide Euclidean constructions for it.

Proposition 5.3. *Q is extangential and equidiagonal if and only if it is extangential equidiagonal.*

Extangential equidiagonal can be defined as Q that verifies $a+b=c+d$, $a'+c'=b'+d'$. Extangential equidiagonal is a Q for which we have no reference. It is a pending task to characterize it as well as to provide Euclidean constructions for it.

Proposition 5.4. *Q is extangential and semidiagonal if and only if it is parallelogram.*

$ABCD$ is extangential $\Leftrightarrow a+b=c+d$. $A(-c,0)$ and $C(c,0)$ are the focus of an ellipse of equation $b^2x^2+a^2y^2-a^2b^2=0$ whose major axis measures $2a$ and whose minor axis measures $2b$. If points B and D belong to the ellipse, it follows that $AB+BC=CD+DA$. (See Figure 6) Let us consider a line of equation $y=m(x-k)$ passing through $K(k,0)$ ($k>0$) that cuts the ellipse at B and D. We will show that the only possibility for $AK+KB=CK+KD$ is that $k=0$.

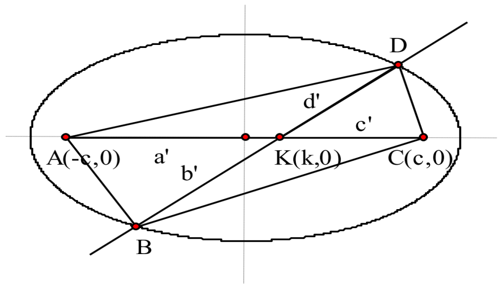

Figure 6. Extangential ($a+b=c+d$) and semidiagonal ($a'+b'=c'+d'$) $\Leftrightarrow$ $ABCD$ parallelogram

The coordinates of points B and D are

$$B\left(\frac{a^2m^2k - ab\sqrt{a^2m^2+b^2-m^2k^2}}{a^2m^2+b^2}, \frac{m\left(-b^2k - ab\sqrt{a^2m^2+b^2-m^2k^2}\right)}{a^2m^2+b^2}\right)$$

$$D\left(\frac{a^2m^2k + ab\sqrt{a^2m^2+b^2-m^2k^2}}{a^2m^2+b^2}, \frac{m\left(-b^2k + ab\sqrt{a^2m^2+b^2-m^2k^2}\right)}{a^2m^2+b^2}\right)$$

$$AK+KB = c+k+\frac{b\left(bk+a\sqrt{a^2m^2+b^2-m^2k^2}\right)\sqrt{1+m^2}}{a^2m^2+b^2}$$

$$CK+KD = c-k+\frac{b\left(-bk+a\sqrt{a^2m^2+b^2-m^2k^2}\right)\sqrt{1+m^2}}{a^2m^2+b^2}$$

$$AK+KB = CK+KD \Leftrightarrow k\left(\frac{2b^2\sqrt{1+m^2}}{a^2m^2+b^2}+2\right)=0 \Leftrightarrow k=0$$

Parallelogram can be defined as Q that verifies $a+b=c+d$, $a'+b'=c'+d'$.

Proposition 5.5. *Q is tangential and extangential if and only if it is kite.*

If $a+c=b+d$ and $a+b=c+d$, then $c-b=b-c$, so $b=c$ and $d=a$. $abcd$ is kite.

Proposition 5.6. *Q is equidiagonal and semidiagonal if and only if it is isosceles trapezium.*

If $a'+c'=b'+d'$ and $a'+b'=c'+d'$, then $c'-b'=b'-c'$, so $b'=c'$ and $d'=a'$. $a'b'c'd'$ is isosceles trapezium.
Definitions of kite and isosceles trapezium are dual sides/diagonal segments. The previous proofs too.

In level 3 we will consider the four Qs defined by three conditions of level 1.

Proposition 5.7. *Q is tangential, extangential and equidiagonal if and only if it is kite $p'/4$.*

If Q verifies $a+c=b+d$ and $a+b=c+d$, then $d=a$ and $b=c$, so $abcd$ is kite, then $b'=d'$. If the condition $a'+c'=b'+d'$ is added it's verify $b'=d'=\frac{1}{4}(a'+b'+c'+d')$ and $abcd$ is kite $p'/4$.

Proposition 5.8. *Q is tangential, equidiagonal and semidiagonal if and only if it is isosceles trapezium $p/4$.*

If Q verifies $a'+c'=b'+d'$ and $a'+b'=c'+d'$, then $d'=a'$ and $b'=c'$, so $a'b'c'd'$ is isosceles trapezium, then $a=c$. If the condition $a+c=b+d$ is added it's verify $a=c=\frac{1}{4}(a+b+c+d)$ and $a'b'c'd'$ is isosceles trapezium $p/4$.

Proposition 5.9. *Q is tangential, extangential and semidiagonal if and only if it is rhombus.*

If Q verifies $a + c = b + d$ and $a + b = c + d$, then $d = a$ and $b = c$, so $abcd$ is kite, then $b' = d'$. If condition $a' + b' = c' + d'$ is verified this implies $a' + b' = c' + d' = \frac{1}{2}(a' + b' + c' + d') = p'/2$, then $b' = p'/2 - a'$ and $d' = b'$, so $c' = a'$. $a'b'c'd'$ is parallelogram with $d = a$, so $a'b'c'd'$ is rhombus.

Proposition 5.10. *Q is extangential, equidiagonal and semidiagonal if and only if it is rectangle.*

If Q verifies $a' + c' = b' + d'$ and $a' + b' = c' + d'$, then $d' = a'$ and $b' = c'$, so $a'b'c'd'$ is isosceles trapezium, then $a = c$. If the condition $a + b = c + d$ is added it's verify $b = d$ and $ABCD$ is parallelogram, so $a' = c'$. $a' + c' = b' + d' = \frac{1}{2}(a' + b' + c' + d') = p'/2$, then $a' = c' = p'/4$ and $b' = d' = p'/4$, so $a'b'c'd'$ is rectangle.

In level 4 we will consider the Q defined by four conditions of level 1.

Proposition 5.11. *Q is tangential, extangential, equidiagonal and semidiagonal if and only if it is square.*

If Q verifies $a + c = b + d$, $a + b = c + d$ and $a' + b' = c' + d'$, then it is rhombus, so $a = b = c = d$. If Q verifies $a' + c' = b' + d'$, $a' + b' = c' + d'$ and $a + b = c + d$, then it is rectangle, so $a' = b' = c' = d'$. If Q verifies $a = b = c = d$ and $a' = b' = c' = d'$ is a square.

In the following table, with (+) we indicate definitions with self-duality sides/diagonal segments.

Level	Sides	Sides-Sides-Diagonal segments	Sides-Diagonal segments	Diagonal segments-Diagonal segments-Sides	Diagonal segments	Opp. (O)/ Cons. (C)
1	1 Tangential $a+c=b+d$				5 Equidiagonal $a'+c'=b'+d'$	O
	2 Extangential $a+b=c+d$				6 Semidiagonal $a'+b'=c'+d'$	C
2			15(+) Tangential equidiagonal $a'+c'=b'+d'$, $a+c=b+d$			OO
	25 Kite $a+c=b+d$, $a+b=c+d$		16 Tangential semidiagonal $a'+b'=c'+d'$, $a+c=b+d$ 25 Extangential equidiagonal $a'+c'=b'+d'$, $a+b=c+d$		16 Isosceles trapezium $a'+c'=b'+d'$, $a'+b'=c'+d'$	OC
			26(+) Parallelogram $a'+b'=c'+d'$, $a+b=c+d$			CC
3		125 Kite $p'/4$ $a+c=b+d$, $a+b=c+d$, $a'+c'=b'+d'$		156 Isosceles trapezium $p/4$ $a'+c'=b'+d'$, $a'+b'=c'+d'$, $a+c=b+d$		OOC
		126 Rhombus $a+c=b+d$, $a+b=c+d$, $a'+b'=c'+d'$		256 Rectangle $a'+c'=b'+d'$, $a'+b'=c'+d'$, $a+b=c+d$		OCC
4			1256(+) Square $a'+c'=b'+d'$, $a'+b'=c'+d'$, $a+c=b+d$, $a+b=c+d$			$OOCC$

Table 5. Sides/diagonal segments duality

In Table 5, four new Qs are defined that had not been defined in the previous sections: tangential equidiagonal, tangential semidiagonal, extangential equidiagonal, kite $p'/4$. The remaining nine Qs are listed in Table 3, but with the exception of tangential, extangential, and kite, the remaining six are defined differently than they were defined in the previous section.

In the next Table 6 the definitions of Qs are presented as a particularization of other Qs.

<table>
<tr><td colspan="3">Tangential</td><td colspan="3">Extangential</td><td colspan="3">Semidiagonal</td><td colspan="3">Equidiagonal</td></tr>
<tr><td colspan="12"></td></tr>
<tr><td colspan="12">Tangential equidiagonal</td></tr>
<tr><td colspan="12">Tangential and equidiagonal</td></tr>
<tr><td colspan="12"></td></tr>
<tr><td colspan="4"></td><td colspan="4">Tangential semidiagonal</td><td colspan="4"></td></tr>
<tr><td colspan="4"></td><td colspan="4">Tangential and semidiagonal</td><td colspan="4"></td></tr>
<tr><td colspan="4">Kite</td><td colspan="4"></td><td colspan="4">Isosceles trapezium</td></tr>
<tr><td colspan="4">Tangential and extangential</td><td colspan="4"></td><td colspan="4">Equidiagonal and semidiagonal</td></tr>
<tr><td colspan="4"></td><td colspan="4">Extangential equidiagonal</td><td colspan="4"></td></tr>
<tr><td colspan="4"></td><td colspan="4">Extangential and equidiagonal</td><td colspan="4"></td></tr>
<tr><td colspan="12"></td></tr>
<tr><td colspan="12">Parallelogram</td></tr>
<tr><td colspan="12">Extangential and semidiagonal</td></tr>
<tr><td colspan="12"></td></tr>
<tr><td colspan="6">Kite $p'/4$</td><td colspan="6">Isosceles trapezium $p/4$</td></tr>
<tr><td colspan="6">Kite and equidiagonal</td><td colspan="6">Isosceles trapezium and tangential</td></tr>
<tr><td colspan="6">Tangential and extangential equidiagonal</td><td colspan="6">Equidiagonal and tangential semidiagonal</td></tr>
<tr><td colspan="6">Extangential and tangential equidiagonal</td><td colspan="6">Tangential semidiagonal and tangential equidiagonal</td></tr>
<tr><td colspan="6">Kite and tangential equidiagonal</td><td colspan="6">Isosceles trapezium and tangential equidiagonal</td></tr>
<tr><td colspan="6">Kite and extangential equidiagonal</td><td colspan="6">Isosceles trapezium and tangential semidiagonal</td></tr>
<tr><td colspan="6">Tangential equidiagonal and extangential equidiagonal</td><td colspan="6">Tangential equidiagonal and tangential semidiagonal</td></tr>
<tr><td colspan="6">Tangential, extangential and equidiagonal</td><td colspan="6">Equidiagonal, semidiagonal and tangential</td></tr>
<tr><td colspan="12"></td></tr>
<tr><td colspan="6">Rhombus</td><td colspan="6">Rectangle</td></tr>
<tr><td colspan="6">Parallelogram and tangential</td><td colspan="6">Parallelogram and equidiagonal</td></tr>
<tr><td colspan="6">Kite and semidiagonal</td><td colspan="6">Isosceles trapezium and extangential</td></tr>
<tr><td colspan="6">Tangential semidiagonal and extangential</td><td colspan="6">Extangential equidiagonal and semidiagonal</td></tr>
<tr><td colspan="6">Parallelogram and kite</td><td colspan="6">Parallelogram and isosceles trapezium</td></tr>
<tr><td colspan="6">Parallelogram and tangential semidiagonal</td><td colspan="6">Parallelogram and extangential equidiagonal</td></tr>
<tr><td colspan="6">Kite and tangential semidiagonal</td><td colspan="6">Isosceles trapezium and extangential equidiagonal</td></tr>
<tr><td colspan="6">Tangential, extangential and semidiagonal</td><td colspan="6">Equidiagonal, semidiagonal and extangential</td></tr>
<tr><td colspan="12"></td></tr>
<tr><td colspan="12">Square</td></tr>
<tr><td colspan="12">Rhombus and rectangle</td></tr>
<tr><td colspan="12">Rhombus and isosceles trapezium $p/4$</td></tr>
<tr><td colspan="12">Rectangle and kite $p'/4$</td></tr>
<tr><td colspan="12">Right $p'/4$ and isosceles trapezium $p/4$</td></tr>
<tr><td colspan="12">Rhombus and equidiagonal</td></tr>
<tr><td colspan="12">Rectangle and tangential</td></tr>
<tr><td colspan="12">Kite $p'/4$ and trapezium</td></tr>
<tr><td colspan="12">Isosceles trapezium $p/4$ and extangential</td></tr>
<tr><td colspan="12">Kite and isosceles trapezium</td></tr>
<tr><td colspan="12">Tangential equidiagonal and parallelogram</td></tr>
<tr><td colspan="12">Extangential equidiagonal and tangential semidiagonal</td></tr>
<tr><td colspan="12">Equidiagonal, semidiagonal, extangential and tangential</td></tr>
</table>

Table 6. Definitions as particularizations of other quadrilaterals

6. ANGLES/DIAGONAL SEGMENTS DUALITY

In section 4, all definitions of Qs only involve sides and angles conditions, as can be seen in Table 3. In section 5, all definitions of Qs only involve sides and diagonal segments conditions, as can be seen in Table 5. The following Table 7 includes Table 3 (without its sides column) on the left and Table 5 on the right (without its sides column).

L	Angles	Angles-Angles-Sides	Angles-Sides	Angles-Sides-Sides	Diagonal segments-Sides-Sides	Diagonal segments-Sides	Diagonal segments-Diagonal segments-Sides	Diagonal segments	O/C
1	3 Cyclic $A + C = B + D$							5 Equidiagonal $a' + c' = b' + d'$	O
	4 Trapezium $A + B = C + D$							6 Semidiagonal $a' + b' = c' + d'$	C
2			13 Bicentric $A + C = B + D$, $a + c = b + d$			15 Tangential equidiagonal $a' + c' = b' + d'$, $a + c = b + d$			OO
	34 Isosceles trapezium $A + C = B + D$, $A + B = C + D$		14 Tangential trapezium $A + B = C + D$, $a + c = b + d$ 25 Exbicentric $A + C = B + D$, $a + b = c + d$			16 Tangential semidiagonal $a' + b' = c' + d'$, $a + c = b + d$ 25 Extangential equidiagonal $a' + c' = b' + d'$, $a + b = c + d$		56 Isosceles trapezium $a' + c' = b' + d'$, $a' + b' = c' + d'$	OC
			24 Parallelogram $A + B = C + D$, $a + b = c + d$			26 Parallelogram $a' + b' = c' + d'$, $a + b = c + d$			CC
3		134 Isosceles trapezium $p/4$ $A + C = B + D$, $A + B = C + D$, $a + c = b + d$		123 Right kite $a + c = b + d$, $a + b = c + d$, $A + C = B + D$	125 Kite $p'/4$ $a + c = b + d$, $a + b = c + d$, $a' + c' = b' + d'$		156 Isosceles trapezium $p/4$ $a' + c' = b' + d'$, $a' + b' = c' + d'$, $a + c = b + d$		OOC
		234 Rectangle $A + C = B + D$, $A + B = C + D$, $a + b = c + d$		124 Rhombus $a + c = b + d$, $a + b = c + d$, $A + B = C + D$	126 Rhombus $a + c = b + d$, $a + b = c + d$, $a' + b' = c' + d'$		256 Rectangle $a' + c' = b' + d'$, $a' + b' = c' + d'$, $a + b = c + d$		OCC
4			1234 Square $A + C = B + D$, $A + B = C + D$, $a + c = b + d$, $a + b = c + d$			1256 Square $a' + c' = b' + d'$, $a' + b' = c' + d'$, $a + c = b + d$, $a + b = c + d$			$OOCC$

Table 7. Angles/diagonal segments duality

So, all Qs from Tables 3 and 5 appear in Table 7 with the exception of tangential ($a + c = b + d$), extangential ($a + b = c + d$) and kite ($a + c = b + d$, $a + b = c + d$), defined exclusively as equalities referring to sides.
In the new duality of this section the definitions include conditions referring to sides-angles and sides-diagonal segments, as can be seen in levels 2, 3 and 4 of Table 7.
In the definition of kite, no angles or diagonal segments are involved, but this properties of kite are dual angles/diagonal segments: Q with at least one pair of equal opposite angles and Q with at least one pair of equal opposite diagonal segments.
The following section includes 4 new definitions to this duality.

7. ANOTHER ANGLES/DIAGONAL SEGMENTS DUALITY

In this section we will proceed analogously to the previous sections 4 and 5. At level 1 we will place cyclic ($A + C = B + D$), trapezium ($A + B = C + D$), equidiagonal ($a' + c' = b' + d'$) and semidiagonal ($a' + b' = c' + d'$).
In level 2 we will consider the six Qs defined by two conditions of level 1. (See Table 8)

Proposition 7.1. *Q is cyclic and equidiagonal if and only if it is isosceles trapezium (1) or isosceles trapezium (2).*

If Q is cyclic ($A + C = B + D$), using intersecting chord theorem implies $a'c' = b'd'$. Q is equidiagonal ($a' + c' = b' + d'$), so $a'^2 + c'^2 + 2a'c' = b'^2 + d'^2 + 2b'd'$. If Q is cyclic and equidiagonal it verifies $a'^2 + c'^2 = b'^2 + d'^2$ which is equivalent to $a'^2 - b'^2 = d'^2 - c'^2$ and to $(a' - b')(a' + b')$=$(d' - c')(d' + c')$. If $a' = b'$, then $b' = d'$, so $a'b'c'd'$ is isosceles trapezium (2). If $a' \neq b'$, then $a' = d'$ and $b' = c'$, so $a'b'c'd'$ is isosceles trapezium (1).

Proposition 7.2. *Q is trapezium and equidiagonal if and only if it is isosceles trapezium.*

If a Q is trapezium ($A + B = C + D$) using Tales theorem implies $a'/c' = d'/b'$, so $c' = a'b'/d'$. Substituting c' in the equidiagonal definition condition ($a' + c' = b' + d'$) we have $a'(b' + d') = d'(b' + d')$, they are $a'/d' = (b' + d')/(b' + d') = 1$ from where $a' = d'$. This implies $b' = c'$. $a'b'c'd'$ is isosceles trapezium.

Proposition 7.3. *Q is cyclic and semidiagonal if and only if it is isosceles trapezium.*

If a Q is cyclic ($A + C = B + D$), using intersecting chord theorem implies $a'c' = b'd'$, so $c' = b'd'/a'$. Substituting c' in the semidiagonal definition condition ($a' + b' = c' + d'$) we have $a'(a' + b') = d'(a' + b')$, they are $a'/d' = (a' + b')/(a' + b') = 1$ from where $a' = d'$. This implies $b' = c'$. $a'b'c'd'$ is isosceles trapezium.

Proposition 7.4. *Q is trapezium and semidiagonal if and only if it is isosceles trapezium (1) or parallelogram.*

If Q is trapezium ($A + B = C + D$), using Tales theorem implies $a'b' = c'd'$. If Q is semidiagonal ($a' + b' = c' + d'$), so $a'^2 + b'^2 + 2a'b' = c'^2 + d'^2 + 2c'd'$. If Q is trapezium semidiagonal it verifies $a'^2 + b'^2 = c'^2 + d'^2$ which is equivalent to $a'^2 - c'^2 = d'^2 - b'^2$ and to $(a' - c')(a' + b')$=$(d' - b')(d' + b')$. If $a' = c'$, then $b' = d'$, so $a'b'c'd'$ is parallelogram. If $a' \neq c'$, then $a' = d'$ and $b' = c'$, so $a'b'c'd'$ is isosceles trapezium (1).

Definitions of cyclic semidiagonal ($A + C = B + D$, $a' + b' = c' + d'$) and trapezium equidiagonal ($A + B = C + D$, $a' + c' = b' + d'$) are dual angles/diagonal segments. The proofs (Propositions 7.2 and 7.3) are not dual but they are very similar. Definitions of Q that is cyclic and equidiagonal ($A + C = B + D$, $a' + c' = b' + d'$) and of Q that is trapezium and semidiagonal ($A + B = C + D$, $a' + b' = c' + d'$) are self-dual angles/diagonal segments. Both involve equalities of sums of angles and diagonal segments, in the first case of opposites, in the second case of consecutives. The novelty of these two Qs is that they are bi-frontal like the Roman god Janus. In the case of Q that is cyclic and equidiagonal defines both isosceles trapezium (1) and isosceles trapezium (2). In the case of Q that is trapezium and semidiagonal defines both an isosceles trapezium (1) and a parallelogram. (See Figure 11) The proofs (Propositions 7.1 and 7.4) are not dual but they are very similar.

Proposition 7.5. *Q is cyclic and trapezium if and only if it is isosceles trapezium.*

If Q is cyclic ($A + C = B + D$) and trapezium ($A + B = C + D$), then $C - B = B - C$, so $B = C$ and $D = A$. $ABCD$ is isosceles trapezium.

Proposition 7.6. *Q is equidiagonal and semidiagonal if and only if it is isosceles trapezium.*

If Q is equidiagonal ($a' + c' = b' + d'$) and semidiagonal ($a' + b' = c' + d'$), then $c' - b' = b' - c'$, so $b' = c'$ and $d' = a'$. $a'b'c'd'$ is isosceles trapezium.
Both definitions of isosceles trapezium are dual angles/diagonal segments. The previous proofs too.

In level 3 we will consider the six Qs defined by three conditions of level 1. (See Table 8)

Proposition 7.7. *Q is cyclic, trapezium and equidiagonal if and only if it is isosceles trapezium.*

Q is cyclic and equidiagonal if and only if it is isosceles trapezium (1) or isosceles trapezium (2). Then $D = A$, $B = C$ or $A = B$, $C = D$. Q is trapezium if and only if $A + B = C + D$. So $D = A$, $B = C$ and is isosceles trapezium or $A = B = C = D$ and is rectangle. Rectangle is a particular case of isosceles trapezium, so Q is isosceles trapezium.

Proposition 7.8. *Q is cyclic, equidiagonal and semidiagonal if and only if it is isosceles trapezium.*

Q is cyclic and equidiagonal if and only if it is isosceles trapezium (1) or isosceles trapezium (2). Then $d' = a'$, $b' = c'$ or $a' = b'$, $c' = d'$. Q is trapezium if and only if $a' + b' = c' + d'$. So $d' = a'$, $b' = c'$ and is isosceles trapezium or $a' = b' = c' = d'$ and is rectangle. Rectangle is a particular case of isosceles trapezium, so Q is isosceles trapezium.

Proposition 7.9. *Q is cyclic, trapezium and semidiagonal if and only if it is isosceles trapezium.*

Q is trapezium and semidiagonal if and only if it is isosceles trapezium (1) or parallelogram. Then $D = A$, $B = C$ or $A = C$, $B = D$. Q is cyclic if and only if $A + C = B + D$. So $D = A$, $B = C$ and is isosceles trapezium or $A = B = C = D$ and is rectangle. Rectangle is a particular case of isosceles trapezium, so Q is isosceles trapezium.

Proposition 7.10. *Q is trapezium, equidiagonal and semidiagonal if and only if it is isosceles trapezium.*

Q is trapezium and semidiagonal if and only if it is isosceles trapezium (1) or parallelogram. Then $d' = a'$, $b' = c'$ or $a' = b'$, $c' = d'$. Q is equidiagonal if and only if $a' + c' = b' + d'$. So $d' = a'$, $b' = c'$ and is isosceles trapezium or $a' = b' = c' = d'$ and is rectangle. Rectangle is a particular case of isosceles trapezium, so Q is isosceles trapezium.

The four previous definitions of isosceles trapezium are non-economic. Definitions in geometry are non-economic if one part of the definition can be deduced from another part, and are economical if they do not contain superfluous information. For example, Q is cyclic ($A + C = B + D$) and trapezium ($A + B = C + D$) if and only if it is isosceles trapezium. Q is cyclic ($A + C = B + D$), trapezium ($A + B = C + D$) and equidiagonal ($a' + b' = c' + d'$) if and only if it is isosceles trapezium. If we remove the equidiagonal condition from this second definition, isosceles trapezium is still defined, therefore this second definition is non-economic. The non-economic definitions are indicated with (ne) in the Table 8.

In level 4 we will consider the quadrilateral defined by four conditions of level 1.

Proposition 7.11. *Q is cyclic, trapezium, equidiagonal and semidiagonal if and only if it is isosceles trapezium.*

Q is cyclic and trapezium if and only if is isosceles trapezium. Q is equidiagonal and semidiagonal if and only if is isosceles trapezium. So Q is cyclic, trapezium, equidiagonal and semidiagonal if and only if it is isosceles trapezium. This definition of isosceles trapezium too is non-economic.

In the following table, with (Δ) we indicate definitions with self-duality angles/diagonal segments.

L	Angles	Angles-Angles-Diagonal segments	Angles- Diagonal segments	Diagonal segments-Diagonal segments-Angles	Diagonal segments	O/C
1	3 Cyclic $A + C = B + D$				1 Equidiagonal $a' + c' = b' + d'$	O
	4 Trapezium $A + B = C + D$				6 Semidiagonal $a' + b' = c' + d'$	C
2			13(Δ) Isosceles trapezium (1) or isosceles trapezium (2) $A + C = B + D$, $a' + c' = b' + d'$			OO
	34 Isosceles trapezium $A + C = B + D$, $A + B = C + D$		14 Isosceles trapezium $A + B = C + D$, $a' + c' = b' + d'$ 36 Isosceles trapezium $A + C = B + D$, $a' + b' = c' + d'$		16 Isosceles trapezium $a' + c' = b' + d'$, $a' + b' = c' + d'$	OC
			46(Δ) Isosceles trapezium (1) or parallelogram $A + B = C + D$, $a' + b' = c' + d'$			CC
3		134 Isosceles trapezium (ne) $A + C = B + D$, $A + B = C + D$, $a' + c' = b' + d'$		136 Isosceles trapezium (ne) $a' + c' = b' + d'$, $a' + b' = c' + d'$, $A + C = B + D$		OOC
		346 Isosceles trapezium (ne) $A + C = B + D$, $A + B = C + D$, $a' + b' = c' + d'$		146 Isosceles trapezium (ne) $a' + c' = b' + d'$, $a' + b' = c' + d'$, $A + B = C + D$		COO
4			1346(Δ) Isosceles trapezium (ne) $A + C = B + D$, $A + B = C + D$, $a' + c' = b' + d'$, $a' + b' = c' + d'$			$OOCC$

Table 8. Another angles/diagonal segments duality

The two minimal definitions of isosceles trapezium at level 2 of Table 8 that only include angle conditions (34) or only include conditions for diagonal segments (16) are already listed in Table 7. Two other definitions of isosceles trapezium (14 and 36) in Table 8 are dual angles/diagonal segments. The definition of isosceles trapezium (1) or isosceles trapezium (2) and the definition of isosceles trapezium (1) or parallelogram are self-dual angles/diagonal segments. These four dual definitions refer to angles and diagonal segments, but not to side-angles and side-diagonal segments as in section 6.

In the next Table 9 the economic new definitions of the *Qs* are presented as a particularization of other *Qs*.

<table>
<tr><th>Cyclic</th><th>Trapezium</th><th>Semidiagonal</th><th>Equidiagonal</th></tr>
<tr><td colspan="4"></td></tr>
<tr><td colspan="4">Isosceles trapezium (1) or isosceles trapezium (2)</td></tr>
<tr><td colspan="4">Cyclic and equidiagonal</td></tr>
<tr><td colspan="4"></td></tr>
<tr><td></td><td colspan="2">Isosceles trapezium</td><td></td></tr>
<tr><td></td><td colspan="2">Cyclic and semidiagonal</td><td></td></tr>
<tr><td></td><td colspan="2"></td><td></td></tr>
<tr><td></td><td colspan="2"></td><td></td></tr>
<tr><td></td><td colspan="2">Isosceles trapezium</td><td></td></tr>
<tr><td></td><td colspan="2">Trapezium and equidiagonal</td><td></td></tr>
<tr><td colspan="4"></td></tr>
<tr><td colspan="4">Isosceles trapezium (1) or parallelogram</td></tr>
<tr><td colspan="4">Trapezium and semidiagonal</td></tr>
<tr><td colspan="4"></td></tr>
</table>

Table 9. Definitions as particularizations of other quadrilaterals

8. CONCLUSIONS

The first (standardize the definitions of a large part of Qs based on six equalities of sums of sides, sums of angles, sums of diagonal segments and whether they are opposite or consecutive) and second (analyze which Qs are defined -and in what way- when 2, 3 or 4 of the mentioned conditions of level 1 are considered) objectives are reflected in the following Table 10. The third (define the Qs of levels 2, 3, 4 as particularizations of Qs of lower levels) in Tables 3, 5 and 7 and the fourth (explain three dualities between the definitions of the Qs defined in the article) in Tables 2, 4, 6, 7, 8.

Tetragon	Level			
	1	*2*	*3*	*4*
Tangential	1 $a + c = b + d$			
Extangential	2 $a + b = c + d$			
Cyclic	3 $A + C = B + D$			
Trapezium	4 $A + B = C + D$			
Equidiagonal	5 $a' + c' = b' + d'$			
Semidiagonal	6 $a' + b' = c' + d'$			
Bicentric		13 $A + C = B + D,$ $a + c = b + d$		
Tangential trapezium		14 $A + B = C + D,$ $a + c = b + d$		
Exbicentric		23 $A + C = B + D,$ $a + b = c + d$		
Tangential equidiagonal		15 $a' + c' = b' + d',$ $a + c = b + d$		
Tangential semidiagonal		16 $a' + b' = c' + d',$ $a + c = b + d$		
Extangential equidiagonal		25 $a' + c' = b' + d',$ $a + b = c + d$		
Parallelogram		24 $A + B = C + D,$ $a + b = c + d$ 26 $a' + b' = c' + d',$ $a + b = c + d$		
Kite		12 $a + c = b + d,$ $a + b = c + d$		

Isosceles trapezium		34 $A + C = B + D$, $A + B = C + D$ 56 $a' + c' = b' + d'$, $a' + b' = c' + d'$ 36 $A + C = B + D$, $a' + b' = c' + d'$ 45 $A + B = C + D$, $a' + c' = b' + d'$		
Isosceles trapezium (1) or isosceles trapezium (2)		35 $A + C = B + D$, $a' + c' = b' + d'$		
Isosceles trapezium (1) or parallelogram		46 $A + B = C + D$, $a' + b' = c' + d'$		
Right kite			123 $a + c = b + d$, $a + b = c + d$, $A + C = B + D$	
Kite $p'/4$			125 $a + c = b + d$, $a + b = c + d$, $a' + c' = b' + d'$	
Isosceles trapezium $p/4$			134 $A + C = B + D$, $A + B = C + D$ $a + c = b + d$ 156 $a' + c' = b' + d'$, $a' + b' = c' + d'$, $a + c = b + d$	
Rectangle			234 $A + C = B + D$, $A + B = C + D$, $a + b = c + d$ 256 $a' + c' = b' + d'$, $a' + b' = c' + d'$, $a + b = c + d$	
Rhombus			124 $a + c = b + d$, $a + b = c + d$, $A + B = C + D$ 126 $a + c = b + d$, $a + b = c + d$, $a' + b' = c' + d'$	
Square				1234 $A + C = B + D$, $A + B = C + D$, $a + c = b + d$, $a + b = c + d$ 1256 $a' + c' = b' + d'$, $a' + b' = c' + d'$, $a + c = b + d$, $a + b = c + d$

Table 10. Uniform definitions of quadrilaterals

The standardization of definitions reaches 23 *Q*s. Some *Q*s are defined in more than one way: parallelogram of 2, isosceles trapezium of 4, isosceles trapezium $p/4$ of 2, rectangle of 2, rhombus of 2, square of 2. 31 different definitions were generated.

Another way to appreciate which *Q*s are defined in the article -and in what way-, as well as the three dualities sides/angles, sides/diagonal segments and angles/diagonal segments is through the diagrams in Figures 7 and 8. Only economic definitions are included. For this reason the upper part of Figure 8 does not include definitions.

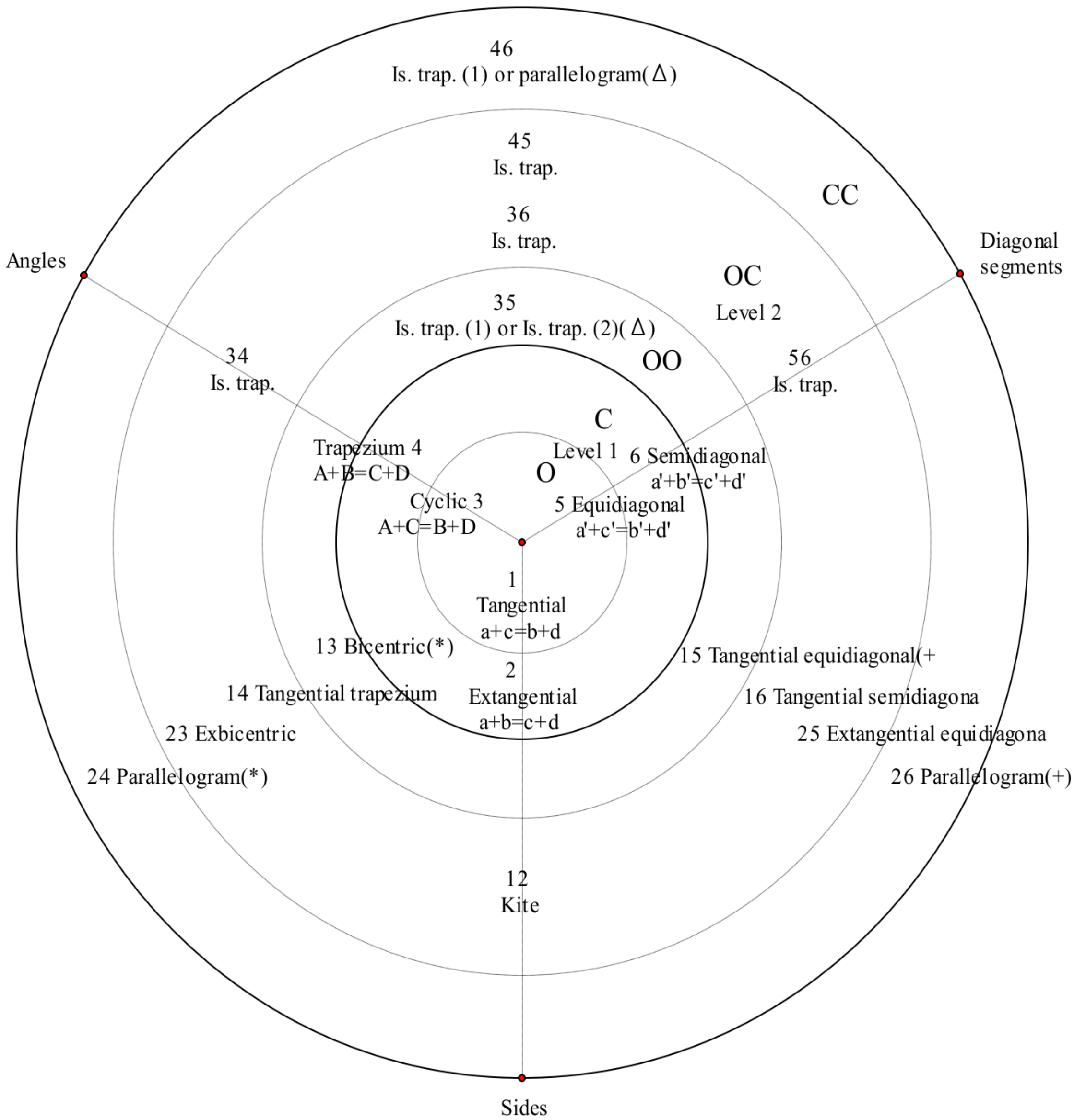

Figure 7. Dualities in levels 1 and 2

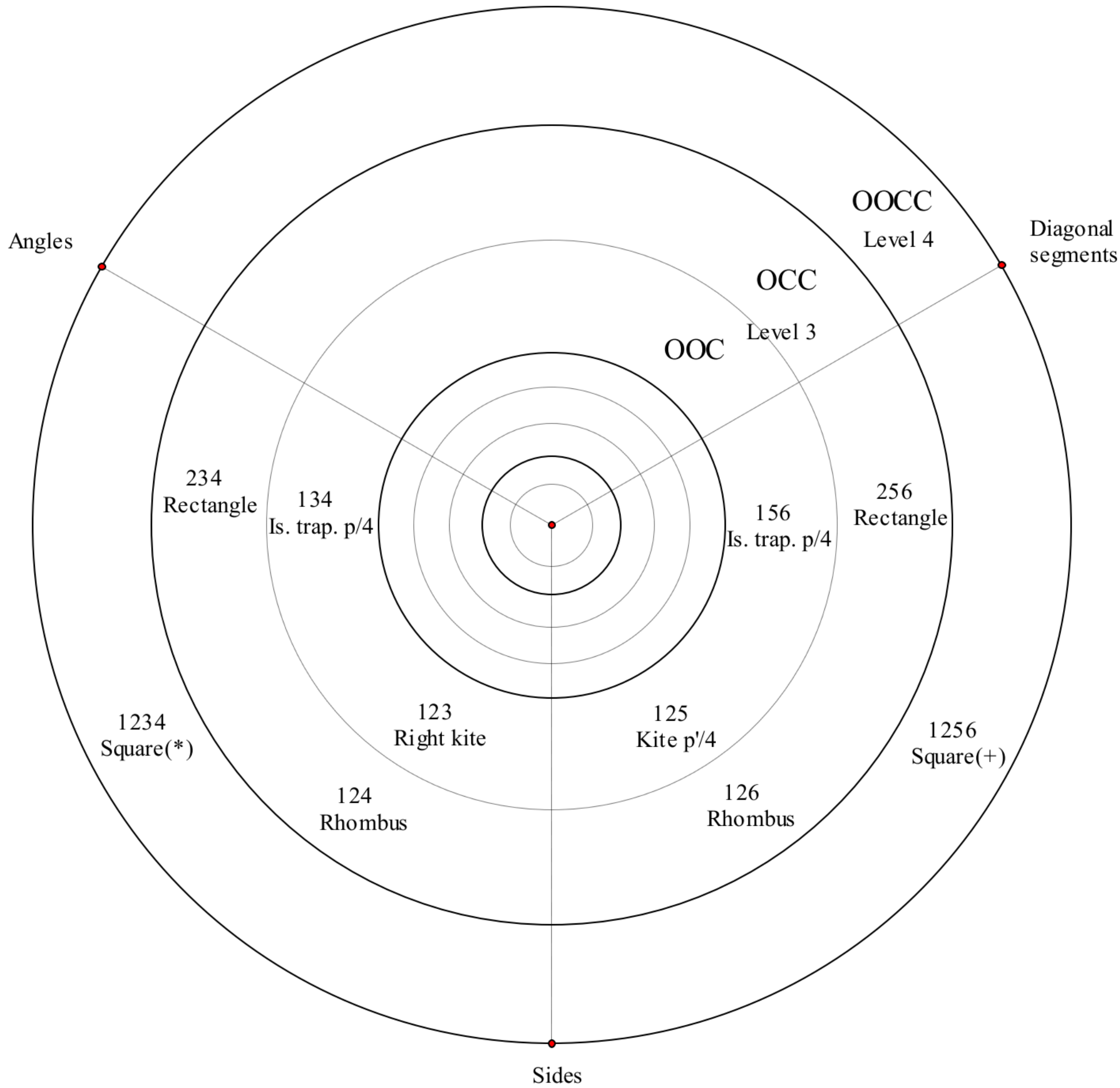

Figure 8. Dualities in levels 3 and 4

9. THE SEMIDIAGONAL TETRAGON AND OTHER NEW TETRAGONS

New Qs were defined. Semidiagonal ($a' + b' = c' + d'$), at level 1, was defined to be dual sides/diagonal segments of extangential ($a + b = c + d$) and dual angles/diagonal segments of trapezium ($A + B = C + D$). Introducing the semidiagonal quadrilateral defined in this way is the main idea of this article since it makes the three dualities of the previous sections possible. This new Q is defined at level 1 along with tangential, extangential, cyclic, trapezium and equidiagonal, which are long-known and studied Qs. This definition proved to be fruitful and leaves the pending task of generating characterizations and properties for semidiagonal.
The first characterization of semidiagonal was already generated by Martin Josefsson.

Proposition 9.1. *ABCD is a convex tetragon whose diagonals intersect at O. h_1 and h_4 heights of triangle AOD with ends at A and D respectively, h_2 and h_3 heights of triangle BOC with ends at B and C respectively. It is a necessary and sufficient condition for ABCD to be semidiagonal that $h_1 + h_2 = h_3 + h_4$.*

If the measure of the angle $BOC = \alpha$, it is true that $h_1 = a' \sin(\alpha)$, $h_2 = b' \sin(\alpha)$, $h_3 = c' \sin(\alpha)$, $h_4 = d' \sin(\alpha)$. (See Figure 9) So $a' + b' = c' + d' \Leftrightarrow (a' + b') \sin(\alpha) = (c' + d') \sin(\alpha) \Leftrightarrow h_1 + h_2 = h_3 + h_4$.

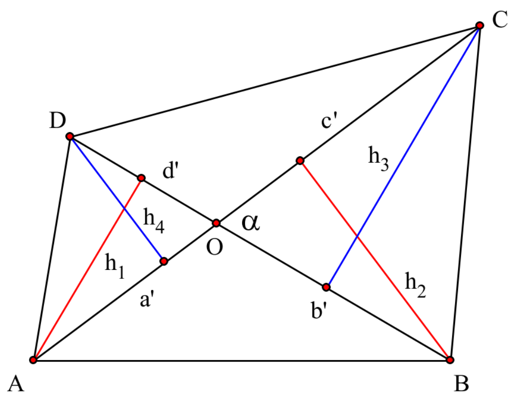

Figure 9. $a'b'c'd'$ is semidiagonal $\Leftrightarrow h_1 + h_2 = h_3 + h_4$

Of the six Qs considered at level 1, there are five that coincide with those of [23, p. 81]. The difference is that orthodiagonal was replaced here by semidiagonal. At level 2 the new Qs tangential equidiagonal, tangential semidiagonal, extangential equidiagonal are defined. Generating characterizations and Euclidean constructions for these Qs is a pending task.

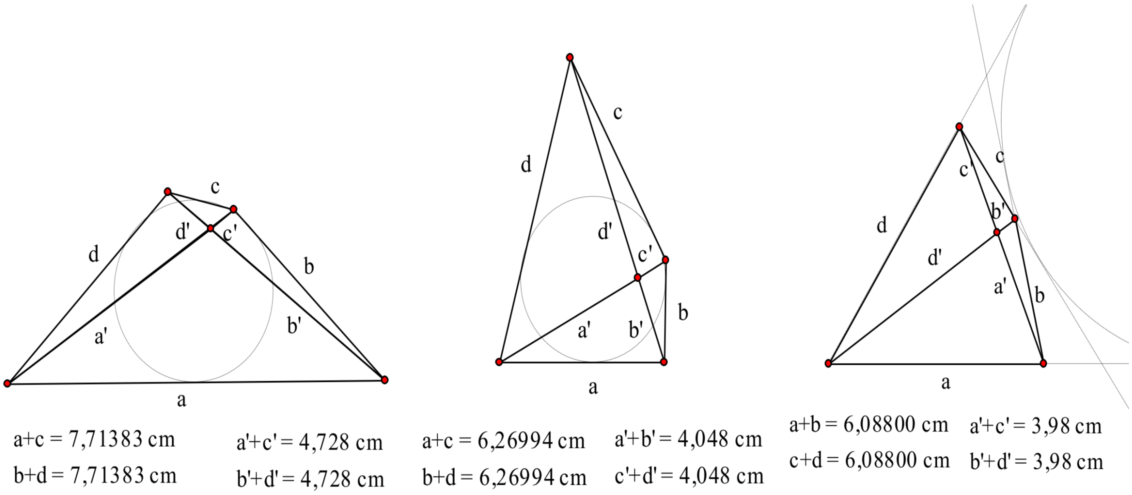

Figure 10. Approximate constructions of abcd tangential equidiagonal (left), tangential semidiagonal (center) and extangential equidiagonal (right)

In level 2 too isosceles trapezium (1) or isosceles trapezium (2) and isosceles trapezium (1) or parallelogram are defined. (See Figure 11) The novelty of each of these definitions is that each *Q* admits two possible faces.

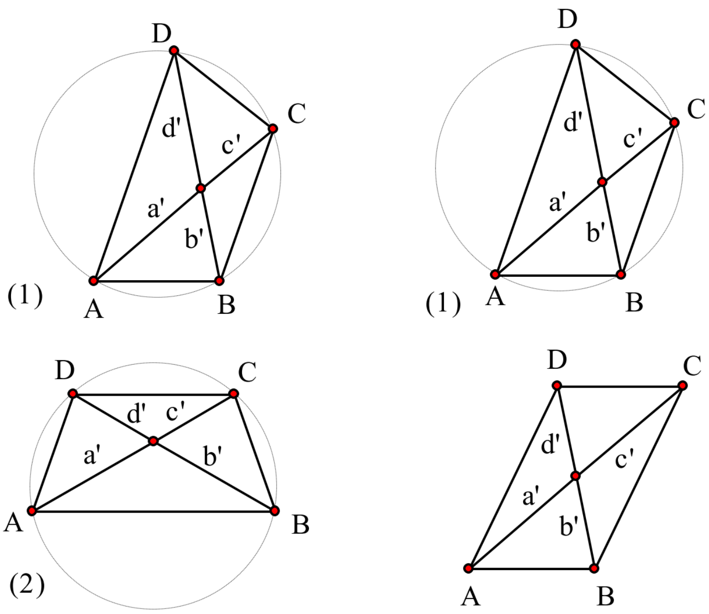

Figure 11. Bifrontal tetragons: isosceles trapezium (1) or isosceles trapezium (2) (left) and isosceles trapezium (1) or parallelogram (right)

At level 3 the new Q is kite $p'/4$ which appears in close relationship with right kite and isosceles trapezium $p/4$ which is defined in two different ways. (See Figure 12) These three unfamiliar Qs appear on the same level as rhombus and rectangle. Isosceles trapezium $p/4$, in [23] is called bicentric trapezium.

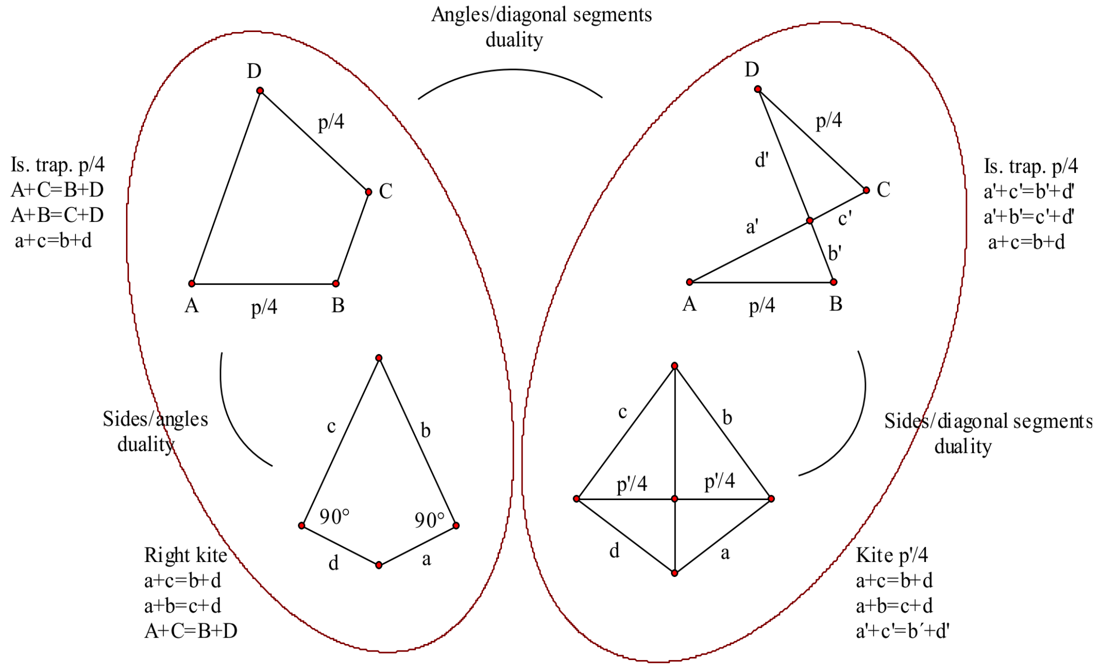

Figure 12. Dualities between definitions of right kite, kite $p'/4$ and isosceles trapezium $p/4$

10. FINAL REFLECTIONS

What was done in section 4 is an expansion of the brief and beautiful [1] and [17] on the sides/angles duality. What was done in sections 5, 6, 7 is an extension of what was done in [2] and [18] on sides/diagonal segments and angles/diagonal segments dualities. From this work it turns out that 15 definitions are uniform and dual or self-dual sides/angles, 15 definitions are uniform and dual or self-dual sides/diagonal segments, 31 definitions are uniform and dual or self-dual angles/diagonal segments.
The notion of duality has its origin in Euclid's *Elements*. Euclid's classification of Qs included only five Qs. But four of them were defined based on the equilateral/right-angled duality. There was a concern that these four Qs were defined in close relation to one another. To this day it is difficult to know how many Qs have been defined throughout history. But more important than the number of Qs defined is that we can establish dualities between their definitions. In this work the duality between definitions was extended, in most cases, to dualities between proofs. Perhaps duality is a way to explore

new properties and that from a property of a certain Q a dual property can be generated for the dual Q.

In [32] the definitions of Qs from around 100 texts are analyzed. The conditions used to define each Q are varied: parallel sides, equality of sides, equality of angles, angle measures, reference to another quadrilateral to which some condition is added, union of two or more of the above conditions. Each Q is defined in isolation and without explicit links to the definitions of other Qs. The definitions of Qs leave the impression of a rather chaotic area. In this article each definition is linked to at least one other definition through one of the three dualities explained. The standardization of definitions reaches 23 Qs and 31 definitions.

The presence of parallelism in the definitions of Qs is frequent. Here we return to Euclid's path of not using parallelism to define Qs. None of the 31 definitions generated mention parallelism.
This classification considers all the possibilities of Qs that fulfill one, two, three or four of the starting conditions. The definitions that emerged at each level have the same hierarchy. At each level, known Qs are defined along with new or little-studied ones. The difference between known, little known and new is only due to the historical non lineal development of geometry. The new Qs that were defined in this article leave a possible area of development for Euclidean geometry.

This article has its origin and a close relationship with the beautiful [23] where five of the six Qs considered at level 1 coincide. In [23] orthodiagonal is considered as a dual of equidiagonal and here semidiagonal was used as a dual of equidiagonal. A pending task is to investigate which Qs are defined when adding the condition of perpendicular diagonals and analyze how dualities behave in this situation.

An interesting aspect of the classification of Qs is that different and varied classifications are possible and provide complementary knowledge about the different Qs and the relationships between them.

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