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RECTANGULAR HYPERBOLA, A QUARTIC AND THE TRIANGLE CONSTRUCTION $\{a, \widehat{B}, b_A\}$

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ABSTRACT. In this article we study a rectangular hyperbola resulting from a triangle construction problem and prove a related to the problem property. The hyperbola is the image of a circle related to the triangle under a harmonic homology. This projectivity leaves also invariant a quartic related to the construction problem. All lines, through a certain point, intercept on that quartic two segments of constant length. These structures are used to prove the existence of solution of the triangle construction problem.

1. INTRODUCTION

The present study is the result of an effort to construct a triangle ABC from its elements $\{a, \widehat{B}, b_A\}$, i.e. the side $a = |BC|$, the angle $\widehat{B}$ and the internal/external bisector of $\widehat{A}$. It is known ([1]) that this construction is not possible with straightedge and compass. Thus, the task was to search for two simple curves, whose intersections would determine a crucial point for the construction. Figure 1 shows the configuration related to the problem: $b_A = |AD|$ is the bisector, and the vertex of the given angle $\widehat{B}$ varies on a known circle κ having the segment AD as a chord. The other vertex C of the triangle coincides with the intersection $C = BD \cap AB'$, where B' is the reflection of B in AD. The construction problem is solved if we can locate $B \in \kappa$ such that $|BC| = a$.
In section 2 we prove that, as B varies on κ, point C describes a rectangular hyperbola κ'. Subsequently we discuss some geometric properties of this hyperbola. In section 3 we consider the last requirement for the side BC, to have a given length $|BC| = a$. This leads to a circular quartic, the solution of the construction problem reducing to the determination of the intersections of the hyperbola with that quartic. The existence of solution to this problem is discussed in section 4.

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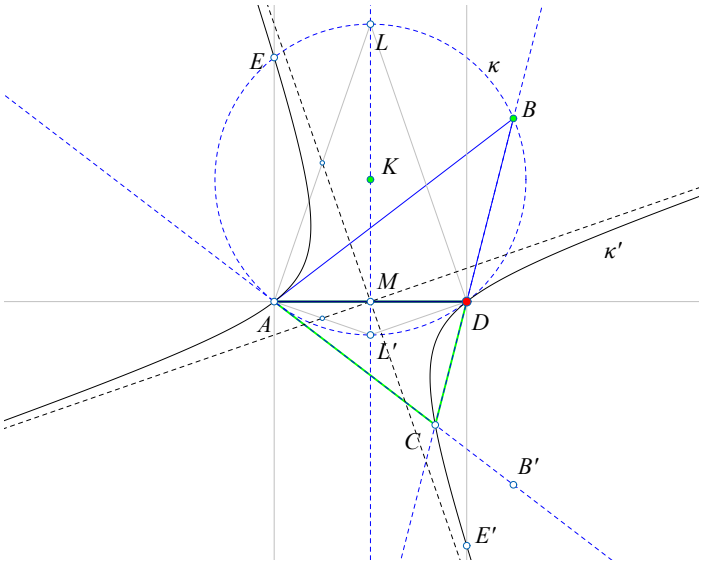


Figure 1. Rectangular hyperbola related to construction $\{a, \widehat{B}, b_A\}$

2. THE RECTANGULAR HYPERBOLA

Referring to figure 1, we adopt the Cartesian coordinate system with origin at A and the segment $\{AD,\ D = (d, 0),\ d > 0\}$, realizing the bisector, along the x-axis. The vertex B of the given angle $\widehat{B}$ moves on a known circle $\kappa(K, r)$. For each position B on κ we consider the reflection B' in AD and the intersection of lines $C = BD \cap AB'$. Triangle ABC fulfills only two of the requirements: it has the given bisector length $b_A = |AD| = d$ and the given angle $\widehat{B}$.

Theorem 2.1. *Under the preceding assumptions, as B varies on the circle κ, point C describes a rectangular hyperbola κ' with the following properties.*

(1) It passes through $\{A, D\}$ and through the second intersection E of the circle κ with the y-axis, as well as through its symmetric E' in the middle M of AD.
(2) AD is a diameter and its middle M is the center of the hyperbola.
(3) The circle and the hyperbola are tangent at D.
(4) The asymptotes are inclined to the coordinate axes at the angle $\widehat{B}/2$.

Proof. Setting $\{c = cos(\phi)\ ,\ s = sin(\phi)\}$, we have the expression of B :

$$K = (d/2, k)\ \ ,\ \ r^2 = k^2 + \frac{d^2}{4}\ \ ,\ \ B = \left(\frac{d}{2} + rc, k + rs\right)\ . \tag{2.1}$$

It then follows that the two lines and their intersection are

$$\begin{aligned} \text{line}\ \ BD\ \ &:\ \ -2(rs+k)x + (2cr-d)y + 2d(rs+k) = 0\,, \\ \text{line}\ \ AB'\ \ &:\ \ (k+rs)x + (d/2+rc)y = 0\,, \\ \text{and}\ \ C = BD \cap AB'\ \ &=\ \ \frac{d}{4cr}(2cr+d\ ,\ -2(rs+k))\,. \end{aligned} \tag{2.2}$$

Eliminating ϕ from the coordinates (x,y) of C we find the equation:

$$d(y^2-x^2)+4kxy+d^2x-2dky \;=\; 0\,. \tag{2.3}$$

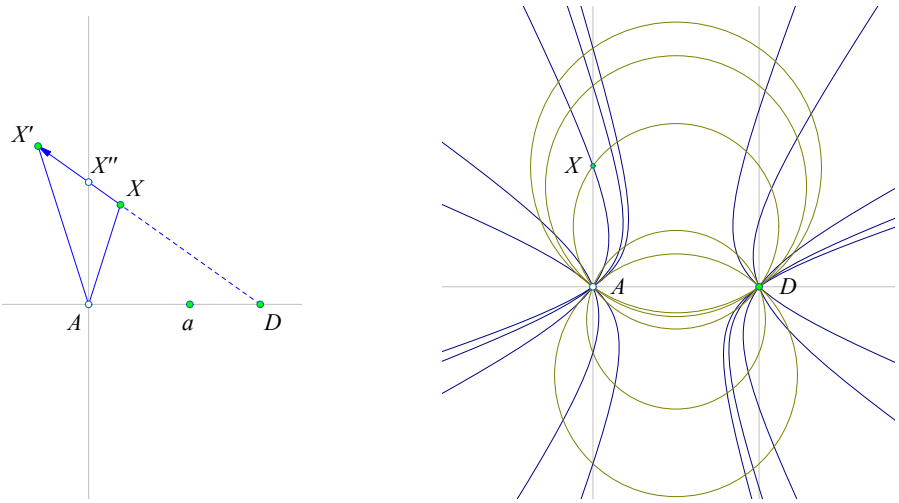
Using elementary facts about quadratic equations ([2]), we see that this describes a rectangular hyperbola.
Nr-1 : It obviously passes through the origin $A(0,0)$, through the point $D(d,0)$ and the through points $\{E(0,2k), E'(d,-2k)\}$. *Nr-2* follows from *nr-1*. Alternatively, it can be easily seen that the tangents at A and D are parallel. *Nr-3* follows from the fact, that as B approaches D on κ, the corresponding point C on κ' approaches D too. *Nr-4* follows by considering the limiting position of C when B tends to the diametral points $\{L,L'\}$ of the diameter of κ along KM. At these points namely, it is easily checked, that the lines $\{BD, AC\}$ become parallel to the claimed directions. □

Theorem 2.2. *With the notation and definitions of this section, the map $B\mapsto C$ is the restriction on κ of a harmonic homology f of the plane, transforming κ to κ'. The center of the homology is D and its axis coincides with the y-axis.*

Proof. In fact, define the projective transformation of the plane f through:

$$f(x,y) \;=\; (x',y') = \frac{d}{2x-d}\,(x,-y)\;. \tag{2.4}$$

From (2.1) and (2.2) follows that $f(B)=C$ for all $B\in\kappa$. Besides, it is easily checked that f fixes $D(d,0)$ and also fixes every point of the y-axis. Finally, for every point X of the plane and its image $X'=f(X)$ is on the line DX, which intersects the y-axis at a point X''. The cross ratio $(XX';DX'')=-1$ since $\{D,X''\}$ are the traces on XX' of the bisectors at A of the triangle XAX' (see Figure 2-(a)). □



(a) Homology $f:\; X\mapsto X'$ **(b)** f-corresponding pencils

Figure 2. Harmonic homology f and f-corresponding pencils

Notice, that f being a harmonic homology ([3, p.248] , [4, p.55], [5, I,p.223]), it is also involutive, being identical with its inverse: $f^{-1}=f$. This implies, that conversely f maps κ' onto κ and proves the following properties of the rectangular hyperbola.

Corollary 2.1. *For a diameter AD and an arbitrary point C of a rectangular hyperbola, extend CD to CB so that AD becomes a bisector of the angle $\widehat{A} = \widehat{CAB}$ (see Figure 1). Then the angle $\widehat{B} = \widehat{ABD}$ is constant and equal to the angle formed by the diameter AD and the tangent to the hyperbola at D. Consequently, the point B describes a circle having chord the segment AD, as the point C varies on the hyperbola.*

Corollary 2.2. *Given a right angle $\widehat{DAX}$, the harmonic homology f with center at D and axis the line AX maps the pencil of circles through $\{A, D\}$ to the pencil of rectangular hyperbolas through the same points having their center at the middle of AD, and vice versa (see Figure 2-(b)). The corresponding members of the two pencils are tangent at D.*

3. THE QUARTIC

With the notation and conventions of the preceding sections, the second curve κ'' related to the construction problem $\{a, \widehat{B}, b_A\}$ is defined through the harmonic homology f as the set of points $X(x, y)$ of the plane satisfying

$$|f(X) - X|^2 \;=\; a^2\,, \tag{3.1}$$

with a a positive constant. Next theorem describes the main properties of this curve.

Theorem 3.1. *Using the expression of f in formula* (2.4)*, the curve is of degree 4 (a quartic) :*

$$\kappa'' \;:\; g(x,y) \;=\; 4x^2((x-d)^2+y^2) - a^2(2x-d)^2 \;=\; 0\,, \tag{3.2}$$

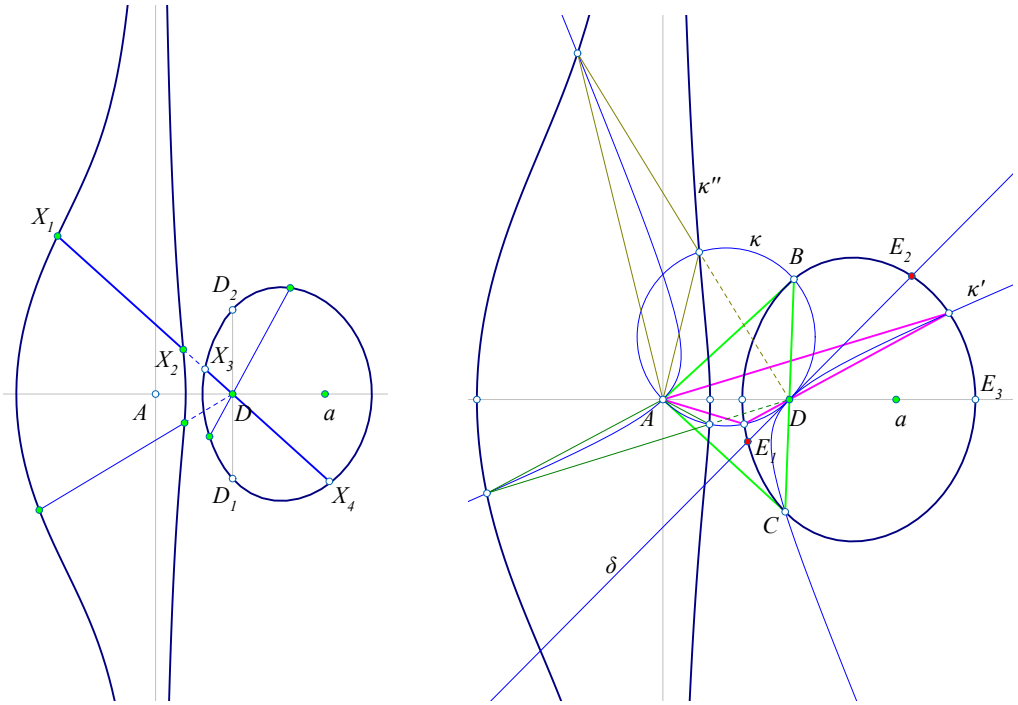
and has the following properties (see Figure 3-(a)):

1. *The curve is non-singular and is invariant under f : $f(\kappa'') = \kappa''$.*
2. *Every line through $D(d, 0)$ intercepts on the curve two disjoint segments $\{X_1X_2, X_3X_4\}$ of length a.*
3. *The curve has three connected components. Two of them extending to infinity and one forming a simple closed curve.*
4. *The transformation f interchanges the two infinite components and maps the closed component onto itself.*

Proof. A short calculation leads from equation (3.1) to (3.2) and shows that the gradient $(\nabla g)(x, y) \neq (0, 0)$ for points $X(x, y)$ on κ'', implying ([6, p.41]) that the curve κ'' is infinite differentiable. From its definition, f being a harmonic homology with center D and axis the y-axis, maps every point $X \in \kappa''$ to X' on the line DX, such that $(XX'; DX'') = -1$, where X'' is the intersection of DX with the y-axis. *Nr-1* follows directly from this definition and the implied involutive property $f^{-1} = f$, since

$$|f(X) - X| = a \quad \Rightarrow \quad |f(f(X)) - f(X)| = |X - f(X)| = a\,.$$

By Bezout's theorem ([7, p.59]), every line through D will meet the curve at four points and if one of them, X say, is real, then, by *nr-1*, its image $X' = f(X)$ will be also a real point on the same line and at distance a from X. This implies, that if we have four distinct real points on a line through D, then these define two segments on it, of length a. To further investigate the location of these segments we use a parameterization of the



(a) Segments of length a **(b)** Triangles with given $\{a, \widehat{B}, b_A\}$

Figure 3. The quartic κ''

line through D and compute the intersection points of the line with the quartic:

$$X(t) = (d + atc\,,\ ats) \quad \text{with} \quad c = \cos(\phi), s = \sin(\phi)\,,$$
$$a^2(2act^2 + 2dt - 2act - d)(2act^2 + 2dt + 2act + d) = 0 \quad \Rightarrow$$
$$t_1 = -\frac{\sqrt{d^2 + a^2c^2} + d + ac}{2ac}\,,\ t_2 = -\frac{\sqrt{d^2 + a^2c^2} + d - ac}{2ac},$$
$$t_3 = \frac{\sqrt{d^2 + a^2c^2} - d - ac}{2ac}\,,\ t_4 = \frac{\sqrt{d^2 + a^2c^2} - d + ac}{2ac}\,.$$

This shows that every line through D meets the curve in four real points. Denoting the corresponding points by $\{X_i = X(t_i) = (x_i = d + act_i, y_i = ast_i), i = 1..4\}$, we notice that: $t_4 - t_3 = t_2 - t_1 = 1$, in accordance with *nr-2*. For $-\frac{\pi}{2} \leq \phi \leq \frac{\pi}{2}$, equivalently for $c > 0$: $\{d + a > x_4 > x_3 > x_2 > 0 > x_1\}$. This shows that the segments on the line through D : $\{X_1X_2, X_3X_4\}$, are disjoint and completes the proof of *nr-2*.

From the non-singularity of the curve, we know that its connected components (branches) are infinite differentiable curves. The closed branch is described by X_4 and its polar radius at_4 from D and the corresponding radius $at_4 - a = at_3$ of X_3, as ϕ varies in the interval $(-\pi/2, \pi/2)$. In fact, as ϕ varies in this interval, X_4 describes a continuous curve approaching $D_1(d, -a/2)$ for $\phi \to -\pi/2$ and also approaching D_2 for $\phi \to \pi/2$ (see Figure 3-(b)). Analogously X_3 describes a continuous curve approaching D_2 for $\phi \to -\pi/2$ and D_1 for $\phi \to \pi/2$. Thus, the union of their points defines a continuous closed curve which is invariant under f.

An analogous analysis for X_1 and its polar radius at_1 and the corresponding radius at_2 fo X_2 shows that X_1 describes a curve lying in the domain $x < 0, X_2$ describes a curve lying in the domain $x > 0$ and both extend to $\pm\infty$ for $\phi \to \mp\pi/2$. □

4. Existence

By Bezout's theorem, and depending on the given data$\{a, \widehat{B}, b_A = d\}$, the construction problem can have, in principle, up to $2 \cdot 4 = 8$ real solutions, eventually some of them realizing AD as an external bisector. In general, the determination of necessary and sufficient conditions for the existence and uniqueness of solutions is quite more complicated for non-constructible triangles than for constructible ones. Some well known examples in this direction, to cite a few, are discussed in the articles by Wu [8] on the *"fundamental inequality for triangles"* (see also [9]) relating the radii of the in/circum-circle and the perimenter, the article by Mironescu and Panaitopol [10] (see also [11]) on the existence and uniqueness of a triangle from its three bisectors, and the article by Oxman [12] on the uniqueness and existence of a triangle from a side and the adjacent bisectors.

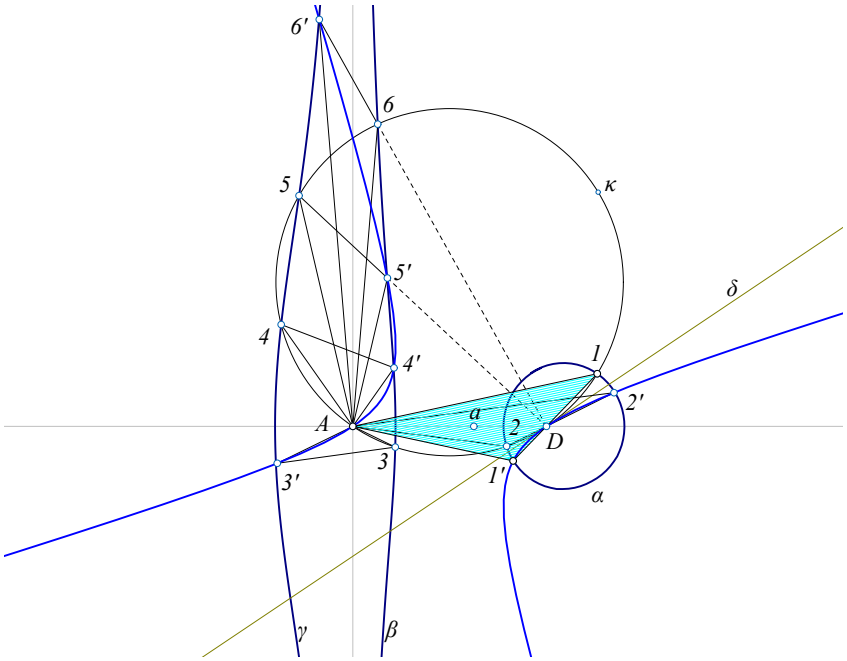


Figure 4. Six triangles satisfying/almost-satisfying the requirements

In our case we'll discuss here the existence, leaving the question of determination of quantitative conditions for uniqueness or/and the number of solutions of the triangle construction problem, to a future instance.
To begin with, we notice first, that our quartic is a *circular* one, i.e. its projective extension or homogenization, resulting from equation (3.2) by replacing $\{x, y\}$ with $\{x/z, y/z\}$, is satisfied by the *cyclic points at infinity* $\{(1, \pm i, 0)\}$. These points satisfy also the homogenized equation of any circle of the plane. Thus, this quartic and the circle κ, having these two points in common, may have at most six additional real intersection points. Consequently also the rectangular hyperbola $\kappa' = f(\kappa)$ has at most six intersection points with

the quartic, each such point corresponding via f to an intersection point of the quartic with the circle. Figure 4 shows a case in which the circle κ has actually the maximal possible real intersections with the quartic. These points are labeled with $\{1,2,3,4,5,6\}$ and their f-images lying on the rectangular hyperbola are labeled with $\{1',2',3',4',5',6'\}$. Together with point A, set at the origin of coordinates, these points define six triangles $\{AII', I = 1..6\}$, which satisfy the requirements of the construction problem for an angle equal to $\widehat{B}$ or $\pi - \widehat{B}$ and b_A internal or external bisector.
In fact, the triangle satisfying the requirements of the construction problem $(a, \widehat{B}, b_A)$ in a strict sense, i.e. having the angle $\widehat{B}$ and b_A for internal bisector, is $ABC = A11'$, having its vertices $\{B, C\}$ on the bounded component α of the quartic. Triangle $ABC = A22'$ has at $B = 2$ the supplementary angle $\pi - \widehat{B}$ of the given one and represents also a solution of the problem $(a, \pi - \widehat{B}, b_A)$. Since the circle/hyperbola passes through point D, which is inside the simple closed component α, the circle/hyperbola intersects α at least in two points, seen easily to be on either side of the x-axis. Thus, the preceding two triangles show that there is always a solution of the construction problem $(a, \widehat{B}, b_A)$ for all possible values of the given data, whether $\widehat{B}$ is an acute or obtuse angle. In particular, when $\widehat{B}$ is a right angle, then the circle κ has AD for diameter and the two preceding triangles are right angled and symmetric w.r.t. the x-axis.
As we noticed in § 3, the β-branch of the curve runs entirely in the $x > 0$ half plane and the circle κ, passing through the origin A, intersects it in at least two points $\{3, 6\}$ lying on either side of the x-axis and defining the triangles $\{A33', A66'\}$. The first triangle $ABC = A33'$ has at B the given angle, but AD is now an external bisector. Triangle $ABC = A66'$ has at B the supplementary $\pi - \widehat{B}$ of the given angle and AD is again external bisector of that angle. Thus, the construction problem $(a, \widehat{B}, b_A)$ has also always a solution, whether $\widehat{B}$ is acute or obtuse and b_A is an external bisector.
From the preceding discussion, and since the circle κ intersects the branches $\{\alpha, \beta\}$ of the quartic at least at four points, it cannot intersect the branch γ in more than two points. Figure 4 shows the case with the maximal possible intersections of the circle κ with the quartic, two of them lying on the branch γ. These are the points $\{4, 5\}$ producing the triangles $\{ABC = A44', ABC = A55'\}$. Both have the given angle $\widehat{B}$ at B but AD is again external bisector of $\widehat{A}$, as in the case $ABC = A33'$. Thus in this case we get three solutions with the angle $\widehat{B}$ and b_A external bisector. If the circle κ is *tangent* to the branch γ, equivalently the hyperbola is tangent to the branch β, then we have two solutions for the given data and b_A external bisector, and if κ does not intersect γ, equivalently the hyperbola does not intersect the branch β, then we get one only solution with these data. Figure 3-(b) illustrates this case. Concluding, we should notice that the shape of the quartic and its decomposition in three branches, of the kind we determined in § 3, does not depend on the relative measures of the parameters $\{a > 0, d > 0\}$ appearing in its equation. Figures 3 and 4 illustrate the generality of this behavior. In the first we have $d < a$ and in the second $d > a$. Next theorem summarizes our discussion.

Theorem 4.1. *The triangle construction problem $(a, \widehat{B}, b_A)$ has a solution for b_A internal or external bisector of $\widehat{A}$ for any possible values of the given data.*

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