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TRIANGLES WITH THE 2ND NAPOLEON POINT AT INFINITY

TEMISTOCLE BÎRSAN

ABSTRACT. Given a triangle ABC, consider the set $\mathcal{N}$ of triangles that have the 2nd Napoleon point N_- at infinity. The properties of the set $\mathcal{N}$ are studied. Properties related to the basic elements of the given triangle, the Lester circle, the Brocard angle and the Neuberg circles are highlighted.

1. INTRODUCTION

Consider a triangle ABC with sidelenghts a, b, c and adopt the standard notations for its remarkable points that will occur in this note, namely: G – centroid, O – circumcenter, H – orthocenter, O_9 – nine-point center, M – midpoint of HG, K – symmedian point, $F_\pm$ – 1st and 2nd Fermat points, $J_\pm$ – 1st and 2nd isodynamic points, $N_\pm$ – 1st and 2nd Napoleon points (Fig. 1). As is customary, the points in the plane of triangle ABC will also be called *finite points*. On the other hand, it is agreed that the lines of a family of parallel lines have in common a point, called the *point at infinity* corresponding to this family (or direction).
It is known or easy to show that the points mentioned above are finite points, except for the point N_-. The purpose of this note is to establish the conditions under which N_- *is a point at infinity* and to study triangles with this property.
For the general properties of Fermat points, isodynamics and Napoleon the reader is referred to the treatises [1], [5], [6], [9]. I will also need some results established in the papers [3], [4].

2. NOTATIONS AND PRELIMINARIES

Let A_+ and A_- be the vertices of the equilateral triangles built on the BC outside and inside the triangle ABC, respectively; similar for B_+, B_- and C_+, C_-. It is well known that the lines AA_+, BB_+, CC_+ meet in F_+, the lines AA_-, BB_-, CC_- in F_-, and we have $AA_+ = BB_+ = CC_+$ and $AA_- = BB_- = CC_-$. Let us denote by l_+ the common length of the segments AA_+, BB_+, CC_+ and by l_- that of the segments AA_-, BB_-, CC_-.

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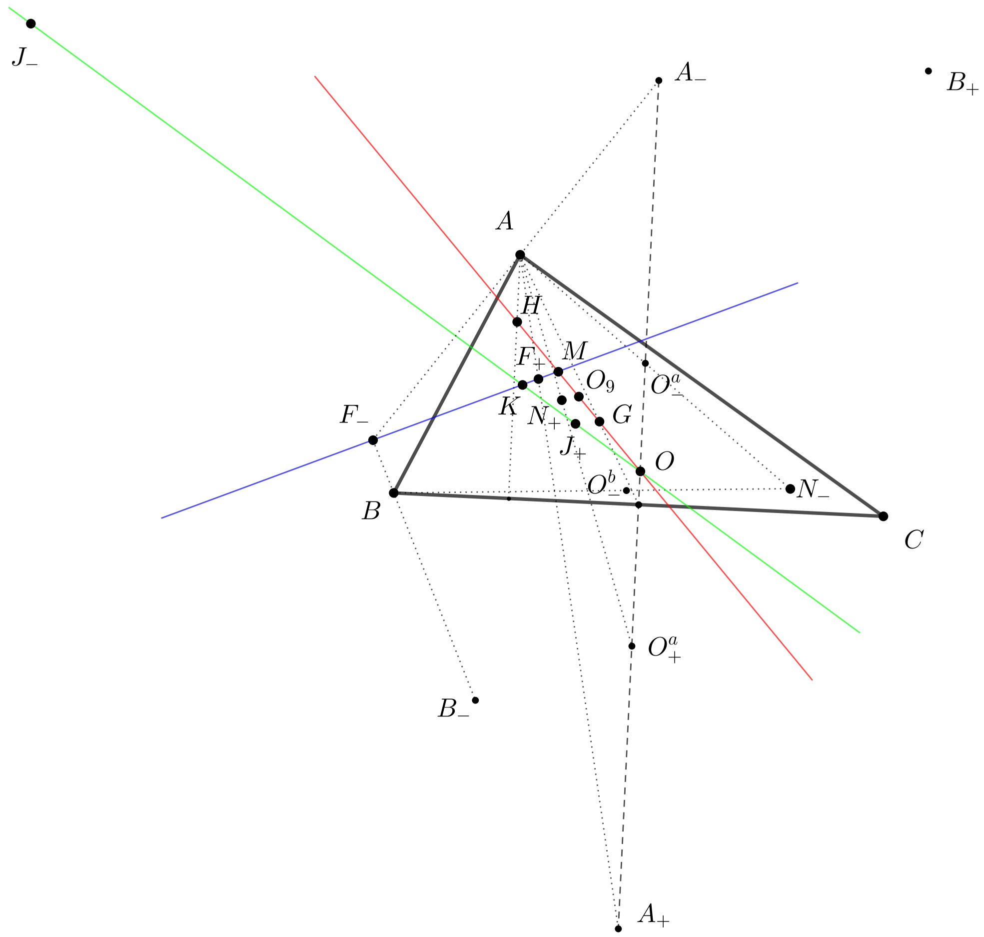

Fig. 1 Points $G, O, H, O_9, M, K, F_\pm, J_\pm, N_\pm$

It is known that we have

$$l_+^2 = \frac{1}{2}\left(a^2 + b^2 + c^2 + 4\sqrt{3}\Delta\right), \quad l_-^2 = \frac{1}{2}\left(a^2 + b^2 + c^2 - 4\sqrt{3}\Delta\right), \tag{2.1}$$

([6], #354 d) where Δ denotes the area of triangle ABC. The reader can easily verify this statement using the law of cosines for triangles ABA_+ and ABA_- (Fig. 1). Adding and subtracting these equalities, we obtain

$$a^2 + b^2 + c^2 = l_+^2 + l_-^2, \quad 4\sqrt{3}\Delta = l_+^2 - l_-^2, \tag{2.2}$$

which will be very useful in the calculations below.

Let us denote by O^a_-, O^b_- and O^c_- the centers of the equilateral triangles BCA_-, CAB_- and ABC_-. We also denote P and Q the midpoints of the segments AB and $O^a_-O^b_-$, respectively (Fig. 2).

Now, let us express the lengths of some segments related to the quadrilateral $ABO^b_-O^a_-$ by a, b, c, and l_+, l_-. First, for the circumradii AO^b_-, CO^b_- and BO^a_-, CO^a_- of the equilateral triangles ACB_- and respectively BCA_-, we have

$$AO^b_- = CO^b_- = \frac{b\sqrt{3}}{3}, \quad BO^a_- = CO^a_- = \frac{a\sqrt{3}}{3}. \tag{2.3}$$

Without loss of generality we can label the angles of triangle ABC so that $C < B < A$ (as in Fig. 2). Then $C < \frac{\pi}{3}$, and applying the cosine formula to triangle $CO^a_-O^b_-$ we have

$$(O^a_-O^b_-)^2 = \frac{a^2}{3} + \frac{b^2}{3} - 2\frac{ab}{3}\cos(\frac{\pi}{3} - C).$$

With the formulas $\cos C = (a^2 + b^2 - c^2)/(2ab)$ and $\sin C = 2\Delta/(ab)$ we immediately obtain

$$(O^a_-O^b_-)^2 = \frac{1}{6}(a^2 + b^2 + c^2 - 4\sqrt{3}\Delta)$$

or, by (2.1),

$$(O^a_-O^b_-)^2 = \frac{l^2_-}{3}. \tag{2.4}$$

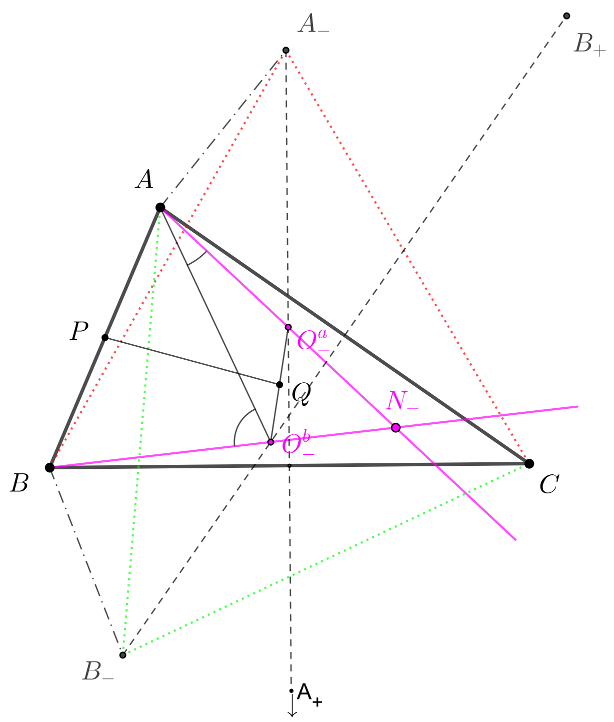

Fig. 2 Quadrilateral $ABO^b_-O^a_-$ and the bimedian PQ

Applying Leibnitz formula relative to point A and triangle BCA_- and then to point B and triangle CAB_-, we obtain

$$c^2+b^2+l_-^2-3(AO_-^a)^2=3\left(\frac{a\sqrt{3}}{3}\right)^2 \quad \text{and} \quad c^2+a^2+l_-^2-3(BO_-^b)^2=3\left(\frac{b\sqrt{3}}{3}\right)^2,$$

and taking into account the first formula in (2.2), we have

$$(AO_-^a)^2=\frac{1}{3}\left(l_+^2+2l_-^2-2a^2\right), \quad (BO_-^b)^2=\frac{1}{3}\left(l_+^2+2l_-^2-2b^2\right). \tag{2.5}$$

We calculate the bimedian PQ of the quadrilateral $ABO_-^bO_-^a$ using a formula from [10], #11.1:

$$4PQ^2+AB^2+(O_-^aO_-^b)^2=\left(AO_-^b\right)^2+(BO_-^a)^2+(AO_-^a)^2+\left(BO_-^b\right)^2.$$

The reader can obtain this equality by applying the median theorem to triangles $O_-^aPO_-^b$, ABO_-^a, ABO_-^b. By (2.3), (2.4), (2.5) and making routine calculations, we will obtain

$$4PQ^2=\frac{1}{3}\left(l_+^2+2l_-^2-2c^2\right). \tag{2.6}$$

So, the bimedian PQ is found.

3. A general condition for N_- to be a point at infinity

Recall that N_- is the point where the lines AO_-^a, BO_-^b, CO_-^c meet. This point can be a finite point or a point at infinity depending on the given triangle. In this section, the triangles with the property that N_- is a point at infinity will be characterized by a simple relationship between l_+ and l_-.

Proposition 3.1. *N_- is a point at infinity if and only if the triangle satisfies the condition*

$$l_+=2l_-. \tag{3.1}$$

Proof. N_- is a point at infinity if and only if the given triangle satisfies the condition $AO_-^a\|BO_-^b\|CO_-^c$, or, equivalently, the condition $AO_-^a\|BO_-^b$ (Fig. 2). So, let's show that

$$AO_-^a\|BO_-^b \iff l_+=2l_-. \tag{3.2}$$

From elementary geometry we know that the quadrilateral $ABO_-^bO_-^a$ is a trapezoid with $AO_-^a\|BO_-^b$ if and only if the relation $2PQ = AO_-^a + BO_-^b$ holds. Therefore, (3.2) is rewritten in the form

$$2PQ=AO_-^a+BO_-^b \iff l_+=2l_-. \tag{3.3}$$

We start with the left-hand side of the equivalence (3.3). Taking into account formulas (2.5) and (2.6), it is written in the form

$$\sqrt{l_+^2+2l_-^2-2c^2}=\sqrt{l_+^2+2l_-^2-2a^2}+\sqrt{l_+^2+2l_-^2-2b^2}.$$

Squaring and taking into account (2.2), we obtain:

$$l_+^2-4c^2=2\sqrt{l_+^2+2l_-^2-2a^2}\cdot\sqrt{l_+^2+2l_-^2-2b^2}$$

or

$$l_+^2-4c^2=2\sqrt{l_-^2-a^2+b^2+c^2}\cdot\sqrt{l_-^2+a^2-b^2+c^2}$$

Squaring again, we have successively:

$$\begin{aligned}
l_+^4 - 8c^2 l_+^2 + 16c^4 &= 4l_-^4 + 4l_-^2 \cdot 2c^2 + 4\left[c^4 - \left(a^2 - b^2\right)^2\right] \Longleftrightarrow \\
l_+^4 - 4l_-^4 &= 8c^2\left(l_+^2 + l_-^2\right) - 16c^4 + 4\left[c^4 - \left(a^2 - b^2\right)^2\right] \Longleftrightarrow \\
l_+^4 - 4l_-^4 &= 4\left(2a^2b^2 + 2a^2c^2 + 2b^2c^2 - a^4 - b^4 - c^4\right) \Longleftrightarrow \\
l_+^4 - 4l_-^4 &= 4 \cdot 16\Delta^2 \Longleftrightarrow l_+^4 - 4l_-^4 = \frac{4}{3}\left(4\sqrt{3}\Delta\right)^2 \Longleftrightarrow \\
l_+^4 - 4l_-^4 &= \frac{4}{3}\left(l_+^2 - l_-^2\right)^2 \Longleftrightarrow \left(l_+^2 - 4l_-^2\right)^2 = 0 \Longleftrightarrow \\
l_+ &= 2l_-.
\end{aligned}$$

Thus, the equivalence (3.3) is proven. □

Remark 3.1. *Proposition 3.1 can also be proven using the equivalence* $AO_-^a \| BO_-^b \Longleftrightarrow \cos \widehat{O_-^a A O_-^b} = \cos \widehat{AO_-^b B}$, *but the calculations are not simpler. An analytical proof of this proposition using barycentric coordinates is present in the proof of Lemma 1.2 in* [4].

Remark 3.2. *There is a concordance between the formulas in* [4] *that give the distances of* N_- *to the points* $O, H, G, O_9, K, F_\pm, J_\pm, N_+$ *and the previous proposition. Indeed, in the denominator of these formulas appears the expression* $4l_-^2 - l_+^2$ *which is null if and only if* $l_+ = 2l_-$, *that is, if and only if* N_- *is a point at infinity.*

4. The set $\mathcal{N}$ of triangles with N_- at infinity

We denote by $\mathcal{N}$ the set of triangles in the plane with the particularity that the 2nd Napoleon point corresponding to them is at infinity. Naturally, this particularity of point N_- will influence the relative position of points O, H, O_9, K, $F_\pm$, $J_\pm$, N_-, and of the lines, triangles and other figures determined by these points.
First, we will highlight a number of characteristic properties of triangles in $\mathcal{N}$, formulated through the basic elements of the given triangle: sides, angles, area, etc.

Proposition 4.1. *The following statements regarding a triangle ABC are equivalent:*
(i) point N_- *of the triangle is at infinity,*
(ii) $7\left(a^4 + b^4 + c^4\right) = 11\left(a^2b^2 + a^2c^2 + b^2c^2\right)$,
(iii) $\Delta = \frac{\sqrt{3}}{20}\left(a^2 + b^2 + c^2\right)$,
(iv) $R = \frac{5\sqrt{3}}{3}\frac{abc}{a^2 + b^2 + c^2}$ *(R - circumradius of the triangle ABC),*
(v) $\tan A = \frac{\sqrt{3}}{5}\frac{a^2 + b^2 + c^2}{-a^2 + b^2 + c^2}$ *and its analogues.*

Proof. (i) $\Longleftrightarrow$ (iii) Indeed, taking into account Proposition 3.1 and formulas (2.1), we have:

$$\begin{aligned}
(i) &\Longleftrightarrow l_+ = 2l_- \Longleftrightarrow l_+^2 = 4l_-^2 \\
&\Longleftrightarrow a^2 + b^2 + c^2 + 4\sqrt{3}\Delta = 4\left(a^2 + b^2 + c^2 - 4\sqrt{3}\Delta\right)
\end{aligned}$$

$$\begin{aligned} &\iff 20\sqrt{3}\Delta = 3\left(a^2+b^2+c^2\right) \\ &\iff (iii). \end{aligned}$$

(iii) $\Longleftrightarrow$ (iv) The formula $4R\Delta = abc$ is used.
(iii) $\Longleftrightarrow$ (ii) We have

$$\begin{aligned} (iii) &\iff 25\cdot 16\Delta^2 = 3\left(a^2+b^2+c^2\right)^2 \\ &\iff 25\cdot\left(2\cdot\Sigma a^2b^2 - \Sigma a^4\right) = 3\left(\Sigma a^4 + 2\Sigma a^2b^2\right) \\ &\iff 7\left(a^4+b^4+c^4\right) = 11\left(a^2b^2+a^2c^2+b^2c^2\right) \\ &\iff (ii). \end{aligned}$$

(iii) $\Longleftrightarrow$ (v) The formula $\tan A = \dfrac{4\Delta}{-a^2+b^2+c^2}$ and its analogues are taken into account.

□

Using condition (ii) of Proposition 4.1, it is easy to see whether or not certain particular triangles belong to the set $\mathcal{N}$. In this regard, we have:
1) Equilateral triangles do not belong to $\mathcal{N}$, because for them $N_- = O$, and therefore N_- is a finite point.
2) An isosceles triangle belongs to $\mathcal{N}$ only if it has a particular shape. Indeed, if the triangle has sides (a, l, l) $(0 < a < 2l)$, then condition (ii) mentioned above is written:

$$7\left(a^4+2l^4\right) = 11\left(2a^2l^2+l^4\right) \Longleftrightarrow 7\left(\frac{a^2}{l^2}\right)^2 - 22\left(\frac{a^2}{l^2}\right) + 3 = 0.$$

Solving this equation, we obtain: $a = \sqrt{3}l$ and $a = \sqrt{7}l$. Then, only the triangles $\left(\sqrt{3}k, k, k\right)$ and $\left(\sqrt{7}k, k, k\right)$, $k > 0$ are in $\mathcal{N}$.
3) In the case of right triangles (a, b, c) with $a^2 = b^2 + c^2$, the condition (ii) gives

$$7a^4 + 7\left[\left(b^2+c^2\right)^2 - 2b^2c^2\right] = 11a^4 + 11b^2c^2 \Longleftrightarrow \sqrt{3}a^2 = 5bc.$$

If we choose $a = 1$ and assume $b \geq c$, we obtain the system

$$b^2 + c^2 = 1, \quad 5bc = \sqrt{3},$$

with the solution $b = \dfrac{1}{2}\left(\sqrt{1+\frac{2\sqrt{3}}{5}} + \sqrt{1-\frac{2\sqrt{3}}{5}}\right)$, $c = \dfrac{1}{2}\left(\sqrt{1+\frac{2\sqrt{3}}{5}} - \sqrt{1-\frac{2\sqrt{3}}{5}}\right)$ or yet $b = \sqrt{\frac{1}{2}+\frac{\sqrt{13}}{10}}$, $c = \sqrt{\frac{1}{2}-\frac{\sqrt{13}}{10}}$. Then, only the right triangles of the shape $\left(k, k\sqrt{\frac{1}{2}+\frac{\sqrt{13}}{10}}, k\sqrt{\frac{1}{2}-\frac{\sqrt{13}}{10}}\right)$, $k > 0$ are in $\mathcal{N}$.

A new characterization is related to Lester circle, i.e. the circle determined by the points F_+, F_-, O and O_9 ([7], #8.16).

Proposition 4.2. *The following statements are equivalent:*
(i) N_- is at infinity,
(ii) the chords of the Lester circle determined by the Fermat axis F_+F_- and the Euler line OH are equal, i.e. $F_-F_+ = OO_9$,
(iii) the cyclic quadrilateral $F_-F_+O_9O$ is an isosceles trapezoid with bases F_-O and F_+O_9 (Fig. 3).

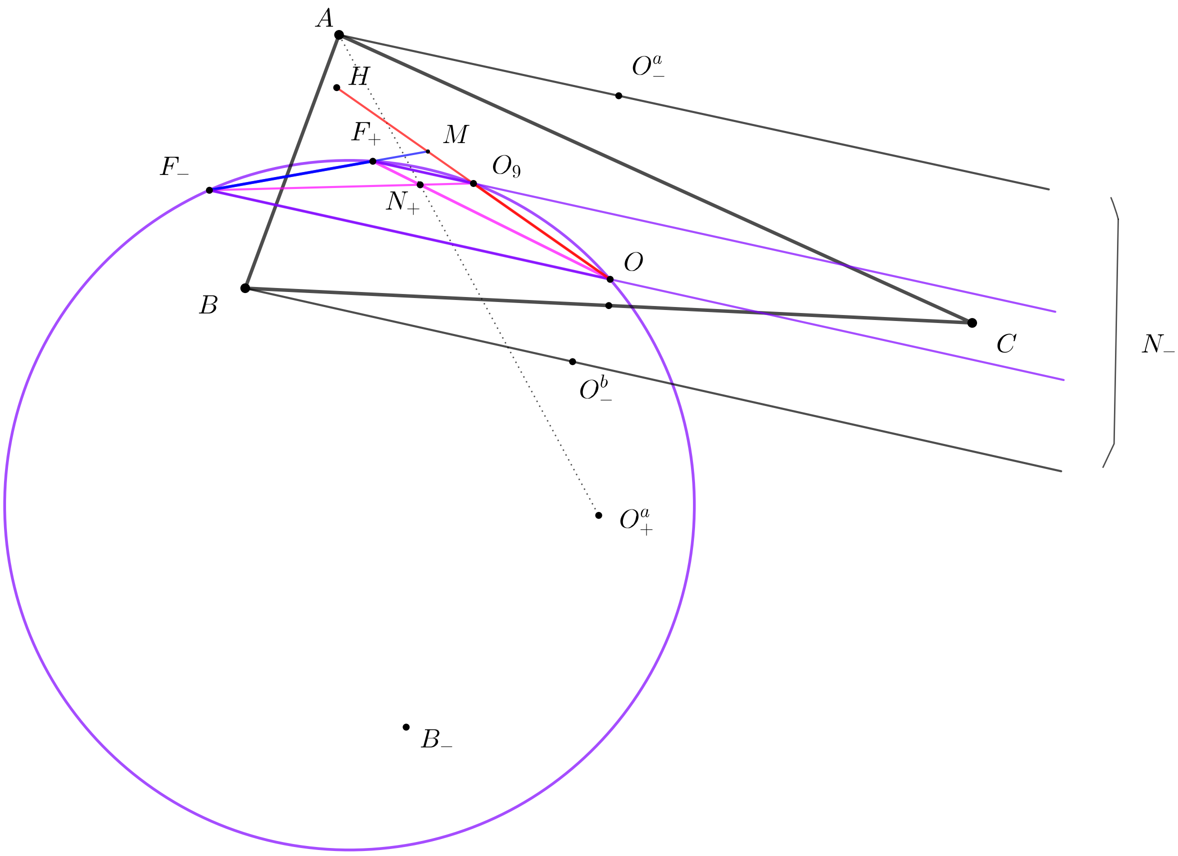

Fig. 3 Lester circle and the quadrilateral $F_-F_+O_9O$

Proof. (i) $\Longleftrightarrow$ (ii) According to Proposition 3.1, (i) is equivalent to $l_+ = 2l_-$. Then, it remains to show that

$$l_+ = 2l_- \Longleftrightarrow F_-F_+ = OO_9. \tag{4.1}$$

For this purpose, we use the formulas:

$$F_-F_+ = \frac{1}{\sqrt{3}} \cdot \frac{\sqrt{f}}{l_+l_-}([3], 4.1),$$

$$HO = 2OO_9 = \frac{\sqrt{f}}{4\Delta} \ ([3], 2.10).$$

We have

$$\begin{aligned} F_-F_+ &= OO_9 \Longleftrightarrow \frac{1}{\sqrt{3}} \cdot \frac{\sqrt{f}}{l_+l_-} = \frac{1}{2} \cdot \frac{\sqrt{f}}{4\Delta} \Longleftrightarrow 8\sqrt{3}\Delta = 3l_+l_- \\ \overset{(2.2)}{\Longleftrightarrow} 2\left(l_+^2 - l_-^2\right) &= 3l_+l_- \Longleftrightarrow (l_+ - 2l_-)\,(2l_+ + l_-) = 0 \Longleftrightarrow l_+ = 2l_-, \end{aligned}$$

so (4.1) is true.
(ii) $\Longleftrightarrow$ (iii) Since the quadrilateral $F_-F_+O_9O$ is cyclic, it follows that $F_-F_+ = OO_9 \Longleftrightarrow F_-O \| FO_9$.
The proof is complete. $\square$

Corollary 4.1. *For any triangle in $\mathcal{N}$ we have:*

$$F_+F_- = \frac{1}{2}OH, \quad J_+J_- = \frac{3}{2}R. \tag{4.2}$$

Proof. The first formula in (4.2) follows from (4.1). The second follows immediately from the equality

$$3R \cdot F_+F_- = OH \cdot J_+J_-. \quad ([2], (2.15))$$

□

Lemma 4.1. *We have:*
(i) the points F_+, N_+, O are collinear in the order $F_+ - N_+ - O$;
(ii) the points F_-, N_+, O_9 are collinear in the order $F_- - N_+ - O_9$ (Fig. 3).

Proof. The collinearities in (i) and (ii) appear in [7], Table 5.1. The claimed orders is indicated in [4], Lemma 1.1. □

Proposition 4.3. *The following statements are equivalent:*
(i) N_- is at infinity,
(ii) $N_+F_- = N_+O$, and
(iii) $N_+F_+ = N_+O_9$.

Proof. (i) $\Longleftrightarrow$ (ii) According to Lemma 4.1, the diagonals of the quadrilateral $F_-F_+O_9O$ meet in N_+ (Fig. 3). Obviously, the cyclic quadrilateral $F_-F_+O_9O$ is an isosceles trapezoid with $F_-F_+ = OO_9$ if and only if F_-N_+O is an isosceles triangle with $N_+F_- = N_+O$. It remains to take into account Proposition 4.2 to conclude the proof.
(i) $\Longleftrightarrow$ (iii) Similarly.
The proposition is proven. □

Corollary 4.2. *The position of point N_+ on the diagonals of the isosceles trapezoid $F_-F_+O_9O$ is given by*

$$\frac{\overline{N_+F_-}}{\overline{N_+O_9}} = \frac{\overline{N_+O}}{\overline{N_+F_+}} = -4. \tag{4.3}$$

Proof. Using convenient similar triangles (Fig. 3), we can write:

$$\frac{N_+O}{N_+F_+} = \frac{OF_-}{O_9F_+} = \frac{OM}{O_9M} = \left(\frac{2}{3}OH\right) / \left(\frac{1}{6}OH\right) = 4,$$

from where do we deduce (4.3). □

In [7], Table 5.1, the following collinearities regarding point N_- are indicated: 1) O, F_-, N_-, 2) O_9, F_+, N_-, 3) H, J_-, N_-, 4) K, N_+, N_-. Consequently, the following proposition is immediate.

Proposition 4.4. *The following statements are equivalent:*
(i) N_- is at infinity,
(ii) any two lines chosen arbitrarily from the set $\{OF_-, O_9F_+, HJ_-, KN_+\}$ are parallel (Fig. 4).

Proof. Obviously, we have $N_- = OF_- \cap O_9F_+ \cap HJ_- \cap KN_+$ etc. □

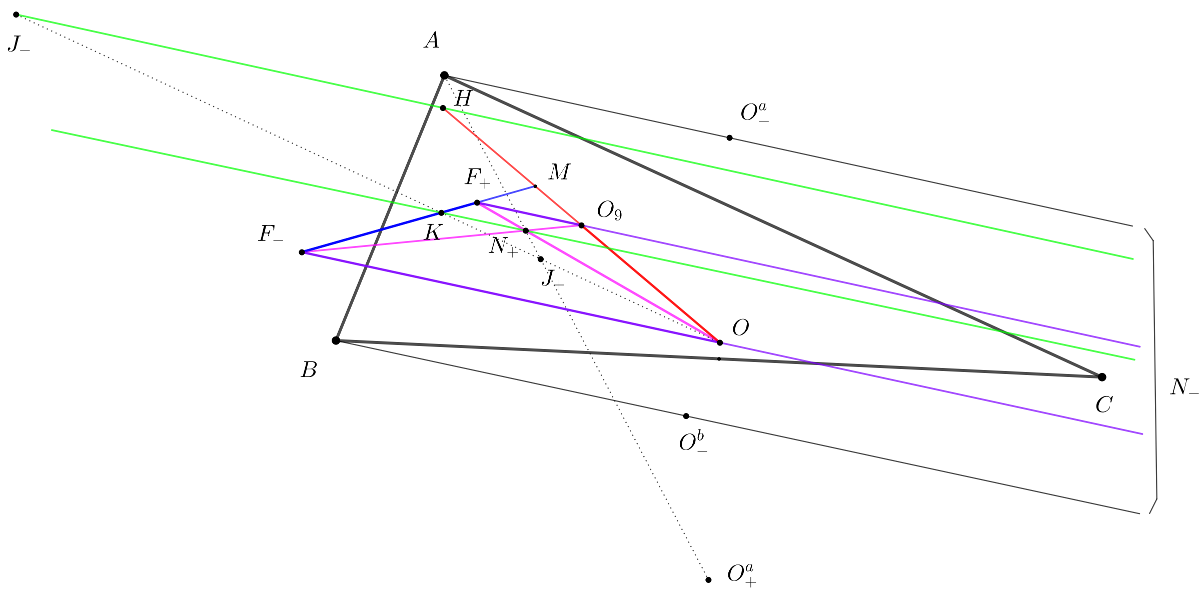

Fig. 4 N_- is at infinity if $OF_-\|O_9F_+\|HJ_-\|KN_+$

Proposition 4.5. *The triangles in $\mathcal{N}$ have the following properties:*
(i) the position of point K on the Fermat axis F_+F_- is given by

$$\frac{\overline{KF_-}}{\overline{KF_+}} = -4; \tag{4.4}$$

(ii) the positions of points J_+, J_- on the Brocard axis OK are given by

$$\frac{\overline{J_+O}}{\overline{J_+K}} = -\frac{5}{3}, \quad \frac{\overline{J_-O}}{\overline{J_-K}} = \frac{5}{3}. \tag{4.5}$$

Proof. (i) Indeed, according to the formulas (2.13) and (2.14) in [3], we have:

$$\frac{KF_-}{KF_+} = \left(\frac{l_+}{l_-}\right) / \left(\frac{l_-}{l_+}\right) = 4.$$

(ii) Taking into account formulas (3.3) and (3.5) from [3], we obtain:

$$\begin{aligned}\frac{J_+O}{J_+K} &= \left(\frac{abc}{4\Delta}\frac{l_-}{l_+}\right) / \left(\frac{\sqrt{3}abc}{a^2+b^2+c^2}\frac{l_-}{l_+}\right) = \frac{a^2+b^2+c^2}{4\sqrt{3}\Delta} = \\ &= \frac{l_+^2+l_-^2}{l_+^2-l_-^2} = \frac{5l_-^2}{3l_-^2} = \frac{5}{3}.\end{aligned}$$

Similarly, using formulas (3.4) and (3.6) from [3], we obtain that $\dfrac{J_-O}{J_-K} = \dfrac{5}{3}$. □

Proposition 4.6. *For the triangles in $\mathcal{N}$ we have:*

$$F_+J_+ = \frac{1}{4}OH, \quad F_-J_- = OH, \quad F_+J_- = 2F_-J_+. \tag{4.6}$$

The quadrilateral F_-J_-HO is a parallelogram (Fig. 4).

Proof. These formulas can be obtained using the fact that $l_+ = 2l_-$ and the formulas (4.3), (4.4), (4.5), (4.6), (2.10) in [3] for the distances F_+J_+, F_-J_-, F_+J_-, F_-J_+, OH, respectively. Since $F_-J_- = OH$ and the line F_-J_- is parallel to the Euler line OH ([7], Table 5.3), the quadrilateral F_-J_-HO is a parallelogram. □

Remark 4.1. *It is well known that on the Euler line the following constant distance-ratios occur: $H\dot{M} : HO_9 : HG : HO = 4 : 6 : 8 : 12$. Table 5.5 of* [7] *lists other constant distance-ratios that occur on various lines of any triangle. Naturally, if the triangle is particular, there may be surprises in this regard. Corollary 4.2 and Propositions 4.5 show that the triangles in $\mathcal{N}$ are such an example.*

Remark 4.2. *Relative to the triangles in $\mathcal{N}$, the distances between some notable points are simply expressed by their basic elements (see the formulas (4.2)). So, we have: $J_+O = \frac{1}{2}R$, $J_-O = 2R$, $OK = \frac{4}{5}R$, $F_+K = \frac{1}{10}OH$, $F_-K = \frac{2}{5}OH$, $KM = \frac{4}{15}OH$, $J_+K = \frac{3}{10}R$, $J_-K = \frac{6}{5}R$ (all obtained from the formulas in* [3] *and* [4] *for $l_+ = 2l_-$) etc.*

5. The set $\mathcal{N}$ versus Neuberg circles

Triangles in $\mathcal{N}$ can also be characterized by the Brocard angle ω.

Proposition 5.1. *N_- is a point at infinity if and only if the equality*

$$\cot\omega = \frac{5\sqrt{3}}{3} \tag{5.1}$$

holds.

Proof. Relative to the angle ω, the formula

$$\cot\omega = \frac{a^2+b^2+c^2}{4\Delta}$$

is well known ([6],#435; [9],#17.2). Combining this with formula (iii) for area Δ given in Proposition 4.1, we obtain the required formula. □

Corollary 5.1. *The triangles in $\mathcal{N}$ are equibrocardian.*

Remark 5.1. *According to the formula (5.1), an approximate value of the Brocard angle of the triangles in $\mathcal{N}$ is $\omega \approx 19°7'$.*

Now, let ABC be a fixed triangle belonging to $\mathcal{N}$. We consider the family of triangles in $\mathcal{N}$ that have B, C as vertices and denote it $\mathcal{N}_a$. We define the families $\mathcal{N}_b$ and $\mathcal{N}_c$ similarly.

Proposition 5.2. *The locus of the third vertex of the triangles in $\mathcal{N}_a$ is composed of the Neuberg circle of the triangle ABC passing through vertex A and its reflection about BC. Similar for the families $\mathcal{N}_a$ and $\mathcal{N}_b$.*

Proof. For all triangles XBC in $\mathcal{N}_a$ the points A_+ and A_- are fixed (these triangles have the same equilateral triangles BCA_+ and BCA_-) (Fig. 5). According to Proposition 3.1, we have

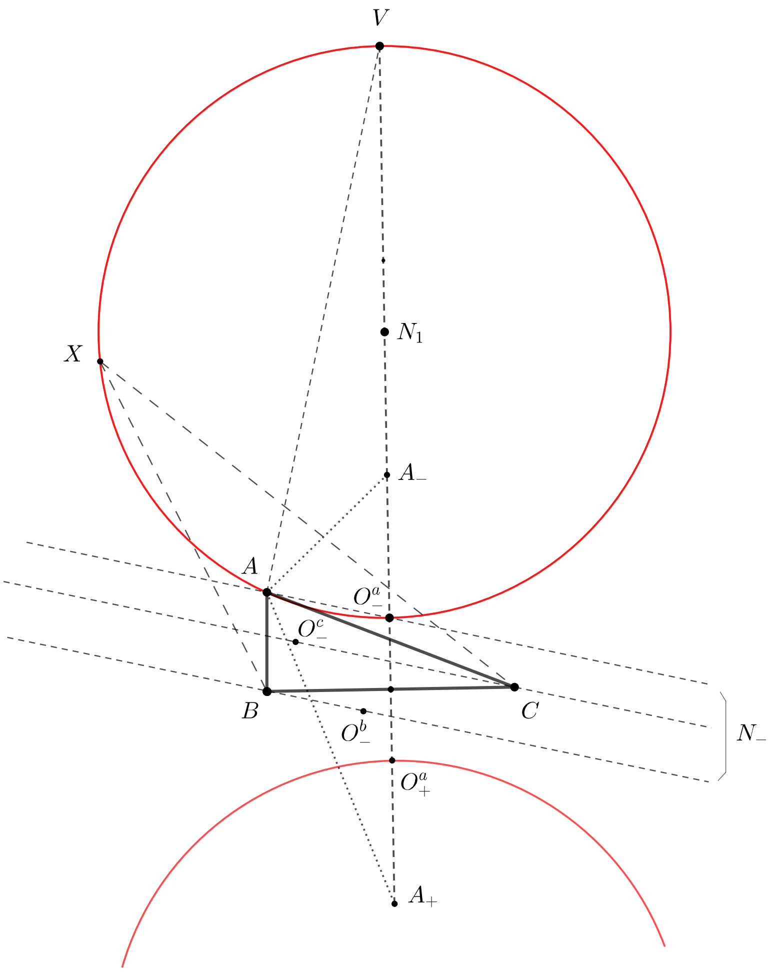

Fig. 5 Neuberg circle of triangle ABC through A and O^a_-

$$\frac{XA_+}{XA_-} = 2. \tag{5.2}$$

Thus, the required locus returns to the locus of the point X that satisfies the condition (5.2). This is a well-known locus (for ex., [6], #58), and X moves on the circle of diameter UV, where $U, V \in A_+A_-$, $U \equiv O^a_-$ (i.e. U is the center of the equilateral triangle BCA_-) and V such that $VA_+ = 2VA_-$ and the reflection of this circle about BC (Fig. 5).

It remains to show that the circle of diameter UV is the Neuberg circle passing through A. Indeed, through an elementary calculation, we obtain that its radius is equal to $\frac{2\sqrt{3}}{3}a$, and $\cot \widehat{BN_1A_+} = \frac{5\sqrt{3}}{3} (= \cot\omega)$, where N_1 is its center. Finally, we use the Theorem from [6], #480.

The proof is complete. □

Proposition 5.3. *The 2nd Napoleon point N_- of triangle ABC is at infinity if and only if the Neuberg circle passing through vertex A also passes through the center of equilateral triangle BCA_-.*

Proof. The statement follows immediately from the proof of the previous proposition. □

Remark 5.2. *The results in this section can be viewed as a particular case of the theory of echibrocardian triangles (see, for example,* [6], *ch.XVII;* [9], *##8.23–8.29) and, therefore, can be obtained in this way.*

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Gh. Asachi Technical University of Iaşi
Carol I Blv., no 11a, 700506 Iaşi
Iaşi, România
Email address: tbirsan111@gmail.com