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THE MOVING SOFA PROBLEM IN ELEMENTARY LIGHT: CAN WE IMPROVE THE SOFA CONSTANT?

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Abstract. Moving sofa problem asks for finding the rigid planar object of any shape, the largest constituting the sofa constant, to maneuver through an L-shaped planar corridor. This article first searches the largest object for convex polygons, then generalize allowing bent edges.

1. Introduction

The moving sofa problem or sofa problem is a two-dimensional idealisation of real-life furniture-moving problems and asks for the rigid two-dimensional shape of largest area $'A'$ that can be maneuvered through an L-shaped planar region with legs of unit width. The area $'A'$ thus obtained is referred to as the sofa constant.[1][2] Joseph Gerver found a sofa described by 18 curve sections each taking a smooth analytic form. This further increased the lower bound for the sofa constant to approximately 2.2195. [2][3]

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2. The Moving Sofa Problem for Convex Polygon

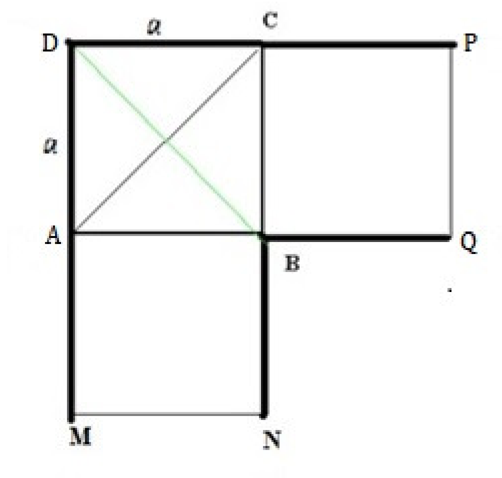

Figure 1. Dimensions of the corridor

Let $MNPQ$ an L-shaped corridor (Fig1) with the width $= a = AB = BC = CD = AD$, $\square ABCD$ =corner square common to both arms of the corridors, $AC = BD$ =Diagonal of the square= greatest dimension of the corridor for the passage of an object with significant width. The best fit with the least waste of areas will be the objects with straight sides, as much parallel and close to the wall of the corridor as possible, a rectangle or square in this case. The square $\square ABCD = a^2$ would need no maneuver as it would pass the corridor by any of its sides completely filling its passage.

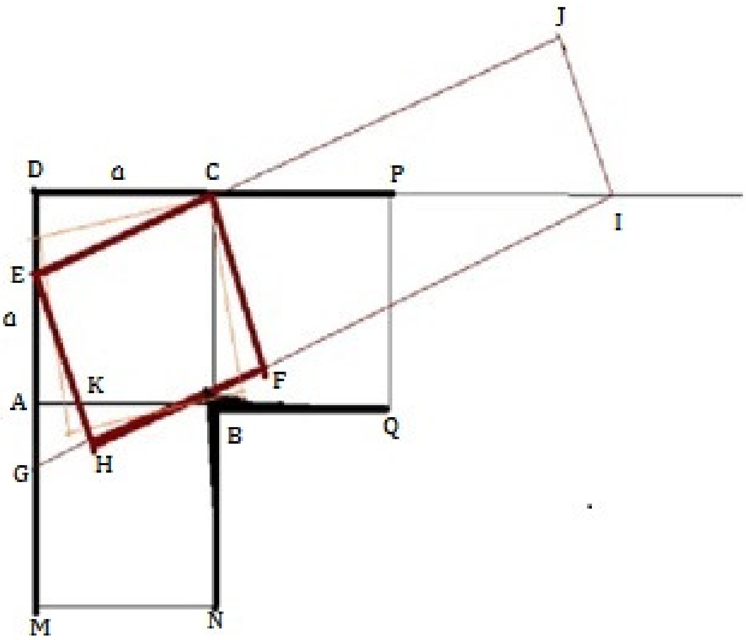

Figure 2. Area of the possible tilted rectangle

In Fig2, Let $\square EHFC =$ A rectangle, being of its width, passing through the corridor
$= \square EKBC + \triangle BFC + \triangle BHK$
$= \square EKBC + (\triangle BFC - \square AGHK) + (\triangle BHK + \square AGHK)$
$= \square EKBC + (\triangle EGH - \square AGHK) + (\triangle BHK + \square AGHK)$ [$\triangle BFC = \triangle EGH$, right triangles with equal same sides]
$= \square EKBC + \triangle EKA + \triangle AGB$
$= \square EKBC + \triangle EKA + \triangle EDC$ [$\triangle AGB = \triangle EDC$ right triangles with equal same sides][4]
$= \square ABCD = a^2$
So, the tilted rectangle, $\square EHFC$ would have an area equal to $\square ABCD$ equaling a^2 , and could be maneuvered through the corridor, and can have a triangular appendage $\square FCI$ when $CI <$ diagonal of $\square ABCD$ for the tilted object's turn over the corner. In that case, $\triangle FCI < \frac{1}{2}(\sqrt{2} \cdot a \cdot a) = \frac{\sqrt{2} \cdot a^2}{2}$. The maneuverable area would be $< a^2 + \frac{\sqrt{2} \cdot a^2}{2} = (\frac{2+\sqrt{2}}{2})a^2 = 1.707a^2$.

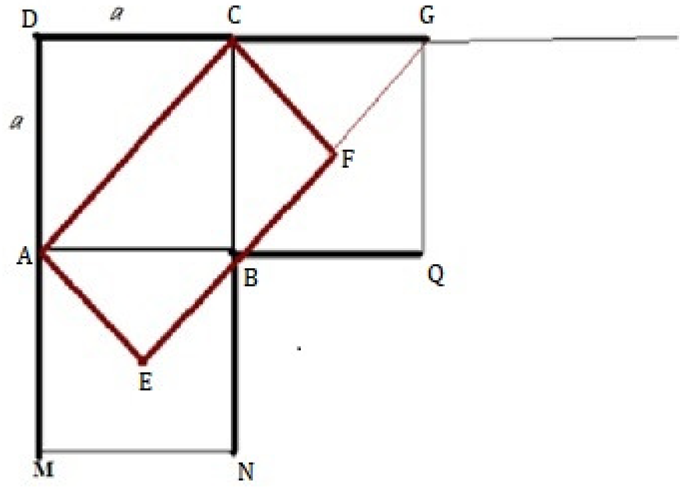

Figure 3. A special case

In Fig3, $\square AEFC$ has its length AC falling on the diagonal of $\square ABCD$, and extension of its arm $EF \parallel AC$ cutting $CG = AB = CD$.
Then, $\triangle AEB = \triangle BFC = \triangle FCG = \frac{1}{2}\triangle BCG = \frac{1}{2}\triangle ABC = \frac{1}{2}\triangle ADC = \frac{1}{4}\square ABCD$
So, $\square AEFC = \triangle ABC + \triangle AEB + \triangle BFC = \triangle ABC + \frac{1}{2}\triangle ADC + \frac{1}{2}\triangle ADC = \triangle ABC + \triangle ADC = \square ABCD = a^2$
And, $\square AEGC = \square AEFC + \triangle FCG = \square ABCD + \frac{1}{4}\square ABCD = 1\frac{1}{4}\square ABCD = 1\frac{1}{4}a^2$
Or simply $\square AEGC = \square AEFC + \triangle FCG = \sqrt{2}a \cdot \frac{\sqrt{2}}{2}a + \frac{1}{4}a^2 = a^2 + \frac{1}{4}a^2 = 1\frac{1}{4}a^2 = 1.25a^2$

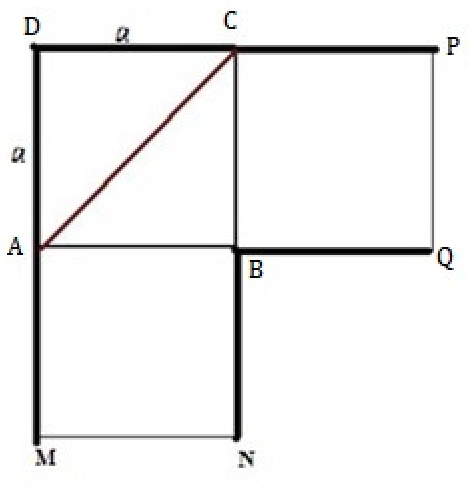

Figure 4. A linear object equaling the diagonal of the corner square has an easy passage

In Fig4, we see that a line without width, a linear object AC (comparable to a stick) can easily pass through the corridor without any special maneuver.

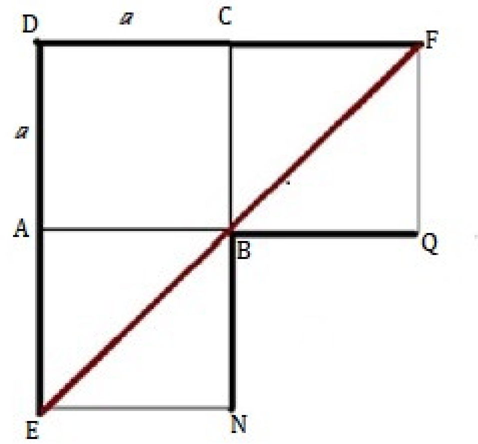

Figure 5. A linear object twice the diagonal of the corner square requires maneuver to pass

In Fig5, we see that linear object EF twice the diagonal of the corner square, touching the inner angle of the corner and the outer wall requires maneuver to pass through the corner like a car on a narrow street turning a sharp bend, with forward and backward movements, advancing forward, while moving up, a little bit when the point F of the object gets free from the outer wall.

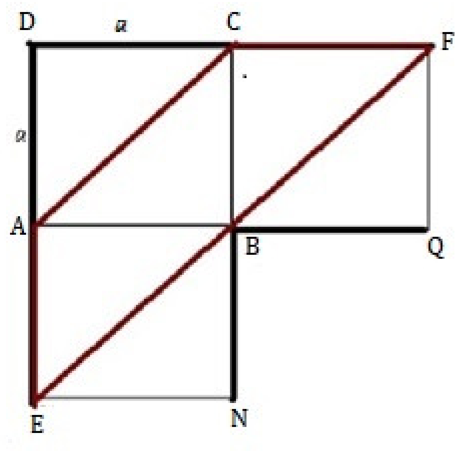

Figure 6. An object bounded by two lines previously shown, along its length

In Fig6, $\square AEFC$ is an object bounded by two lines previously shown (Fig4, Fig5) along its length. With a little plasticity of only any one of the outer wall, or only any of AE or CF end of the object or a just a fraction of a unit yield to frictional pressure of only any

of the points C or A of the object will allow a small movement forward which will give freedom for further forward movement.
$\square AEFC = \triangle AEB + \triangle ABC + \triangle BFC = \frac{1}{2}\square ABCD + \frac{1}{2}\square ABCD + \frac{1}{2}\square ABCD = 1\frac{1}{2}\square ABCD = 1\frac{1}{2}a^2 = 1.5a^2$

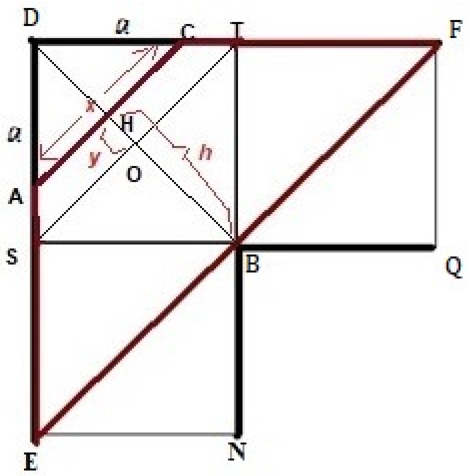

Figure 7. An object with its height equaling the width of the corridor and leading edge equals the diagonal of the corner square

The largest passable convex polygon should have its parallel sides conforming with and almost touching the sides of the corridor, spanning a height equaling the width of the corridor, the leading edge conforming with and as close as possible in length to the diagonal of the corner square. In Fig7, $\square AEFC$ is a trapezium shaped object with its height equaling the width of the corridor, its leading edge as close as possible in length to the diagonal of the corner square to be passable as the largest object.
Height $BH = h = a$, Base $EF = 2\sqrt{2}a$, Roof $AC = x$, Leading edge AE moving down.
$OS = OB = OD = OT = \frac{\sqrt{2}}{2}a.$
$OH = y = a - \frac{\sqrt{2}}{2}a = (1 - \frac{\sqrt{2}}{2})a$
In $\triangle ACD$ and $\triangle STD$,

$$\frac{SD}{AS} = \frac{OD}{OH}$$

Or,

$$\frac{a}{AS} = \frac{\frac{\sqrt{2}}{2}a}{(1 - \frac{\sqrt{2}}{2})a}$$

Or,

$$AS = \frac{(1 - \frac{\sqrt{2}}{2})}{\frac{\sqrt{2}}{2}}a = (\sqrt{2} - 1)a$$

And,

$$\frac{AC}{ST} = \frac{AD}{SD}$$

Or,

$$\frac{AC}{OS+OT} = \frac{SD-AS}{SD}$$

Or,

$$\frac{AC}{\frac{\sqrt{2}}{2}a+\frac{\sqrt{2}}{2}a} = \frac{a-(\sqrt{2}-1)a}{a}$$

Or,

$$AC = (2-\sqrt{2})\sqrt{2}a = (2\sqrt{2}-2)a$$

Also,
$AE = SE + AS = a + (\sqrt{2}-1)a = \sqrt{2}a$ =Diagonal of the corner
And the area of $\square AEFC$,

$$BH \times \frac{AC+EF}{2} = a \times \frac{(2\sqrt{2}-2)a+2\sqrt{2}a}{2} = (2\sqrt{2}-1)a^2 \approx 1.828427a^2$$

When $a = 1$ unit, the largest possible area maneuvered through the L-shaped corridor will be just under and very close to 1.828427 square units. For the convex polygon contours, sofa constant$\leq$ 1.828427 .

3. The Moving Sofa Problem When Curved Edges Allowed

Theorem 3.1. *When curved edges are allowed, sofa constant*$=\sqrt{2}+1 = 2.4142$.

Proof. Now we calculate sofa constant for two-dimensional objects with curved edges allowed, while passing the corner of a unit wide corridor. We see minimum tight fit is unit semicircle which when turn on its centre at corridor corner, none of its dimension exceeds those of traversed passage, the sofa constant is $\frac{\pi}{2} = 1.5707$.

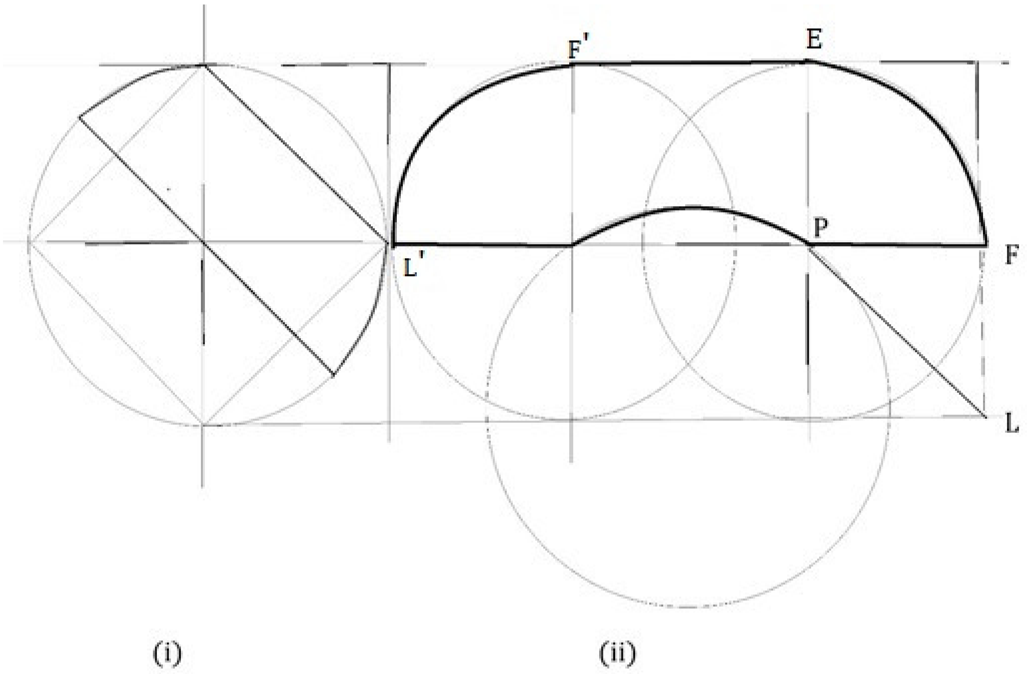

Figure 8. Moving Sofa (i) round ended object (ii) Gerver's Sofa

In Fig8(i) area of the planar convex object, passing the corner of a unit wide corridor, is,

$$\frac{1}{2} + 2 \times \frac{\pi}{8} \approx 1.285$$

In Fig8(ii) in the Gerver's telephone handset sofa, the corner point on which the sofa turns and its inner edge slides, $P = pivotal \quad point$, For the leading end, $F = First \quad contact \quad with \quad the \quad approching \quad wall$, $L = Last \quad contact \quad with \quad the \quad approaching \quad wall$, $Maximum \quad height = 1$, $Passage \quad Corner \quad Diagonal = \sqrt{2}$ which is also the length of the handset handle. The same dimensions will be encountered by the lagging end while traversing the corner. When E comes in place of F, F' comes in place of E, L' comes in place of F'. Outer arcs, $EF = F'L'$ have the same radius of 1, moreover, $PL = straight \quad distance \quad EF = F'E = straight \quad distance \quad F'L' = \sqrt{2}$, inner cut curved edge provides the extra space of maneuvering. The maximum passage dimension at a given transverse cross section along the path is between 1 and $\sqrt{2}$. In case of maximum sized maneuverable object, for the path of P, we have,

$$PF^2 + FL^2 = PL^2$$
$$x^2 + y^2 = PL^2$$

This is a circle, in actual values found here,

$$(0 - (-1))^2 + (0 - (-1))^2 = (\sqrt{2})^2$$

Whose centre is $(-1, -1)$ radius $\sqrt{2}$ if $P(0, 0)$. The cut out segment is a part of this circle which, if subtends the angle θ in the centre, have area of,

$$\frac{1}{2}(\sqrt{2})^2(\theta - sin\theta) = \theta - sin\theta$$

Then the area of the Gerver's sofa,

$$2 \times \frac{\pi}{4} + 1 \times \sqrt{2} - (\theta - sin\theta) = \frac{\pi}{2} + \sqrt{2} - (\theta - sin\theta)$$

While on inner edge, cutting circle radius is $\sqrt{2}$, the centre $(-1,-1)$ is equidistant from axes, the line between centre $(-1,-1)$ and origin $(0,0)$ makes an angle of $\frac{\pi}{4}$ with either of axes, the cut out segment has a base with the distance from its centre 1, the half the segment base is 1, segment base is 2, subtended $\theta = \frac{\pi}{2}$, cut out segment height $= \sqrt{2}-1 = 0.4142$, handset handle is $(1 - 0.4142) = 0.5858$ high (less than half the diagonal $\frac{\sqrt{2}}{2} = 0.7071$ at the passage corner) inner corners of sofa ends are cut by $\frac{1}{2}(segment \quad base - handle \quad length) = \frac{1}{2}(2 - \sqrt{2}) = 0.2928$ on either side, could be calculated by elementary trigonometry and geometry. The optimal area of the Gerver's sofa or the sofa constant,

$$\frac{\pi}{2} + \sqrt{2} - (\frac{\pi}{2} - sin\frac{\pi}{2}) = \sqrt{2} + 1 = 2.4142$$

□

4. Conclusion

This article is meant for dealing with the real life problems of maneuvering furniture or other rigid objects through corridors, stairways with difficult bends, and cars through tricky streets with sharp turns, sometimes rigid endoscope through anatomical passages. In real life we have to deal with factory made wooden and steel objects with less complex shapes which are easier to manufacture in a lot. So, we first specified our solution to the problem for the convex polygons, later generalized. There is a scope for experimentation practically minimizing friction, our article provides a theoretical basis.

Acknowledgment

Those who allowed me hard-found time to do this work.

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