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A VARIANT OF THE EULER LINE

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ABSTRACT. Four points – labeled here as P, Q, S, T – obtained as points of concurrence of certain lines associated with any triangle, are considered: Q and S are endpoints of a diameter of the triangle's circumcircle, while P and T are endpoints of a diameter of the nine-point circle. The line segments PQ and ST always meet the Euler line at the centroid, and the centroid divides the lines in the same 2 : 1 ratio as it does for the Euler line. Q forms an "orthocentric-like" system with the triangle's vertices, just as S does.

1. Introduction and motivations

In [1, 2], Dr Shawyer told how Josh Khler, a student of Steve Sigur, made the following observation: "Through each midpoint of the sides of a triangle, draw a line whose slope is the reciprocal of the slope of the side containing the midpoint. Then the lines concur at a point on the nine-point circle". (See also [3, 4, 5, 6] for related discussions.) Here, we denote this point of concurrence by T. Applying the same construction to the vertices (through each vertex, draw a line whose slope is the reciprocal of the slope of the opposite side) yields lines that concur at a point S on the circumcircle of the triangle. As expected, line ST always meets the Euler line at the centroid of the parent triangle, and the centroid divides ST in a 2 : 1 ratio, with T closer to the centroid than S.

If we replace "reciprocal" with "negative" in the above constructions, we obtain two additional points P (on the nine-point circle) and Q (on the circumcircle) that exhibit similar behaviour. Line PQ meets the Euler line at the centroid, and is divided in a 2 : 1 ratio there. PT is a diameter of the nine-point circle, and QS is a diameter of the circumcircle. S happens to be the reflection of the orthocenter in P, while T is the midpoint of the orthocenter and Q. All four points are related to the circumcenter O, the orthocenter H, and the side-lengths a, b, c via

$$2(PQ^2 + ST^2 - OH^2) = a^2 + b^2 + c^2 \tag{1.1}$$

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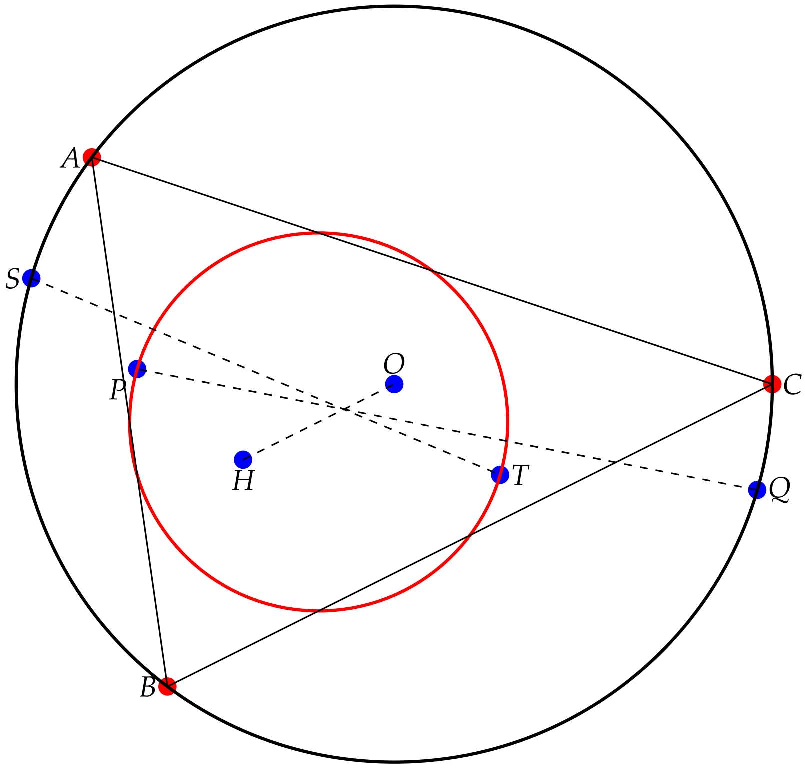

Figure 1. The Euler line HO together with lines PQ and ST

Since S and T can be obtained from P and Q, our focus will be on P and Q. Due to the 2 : 1 ratio in which the centroid divides PQ, a number of easily verified distance-related equations involving P, Q, the vertices, and the side-lengths are obtained:

$$PQ^2 = 3(PA^2 + PB^2 + PC^2) - a^2 - b^2 - c^2 \tag{1.2}$$

$$4PQ^2 = 3(QA^2 + QB^2 + QC^2) - a^2 - b^2 - c^2 \tag{1.3}$$

$$4(PA^2 + PB^2 + PC^2) = QA^2 + QB^2 + QC^2 + a^2 + b^2 + c^2 \tag{1.4}$$

which are, respectively, analogues of the following well-known equations involving the circumcenter, the orthocenter, the vertices, and the side-lengths:

$$OH^2 = 3(OA^2 + OB^2 + OC^2) - a^2 - b^2 - c^2 \tag{1.5}$$

$$4OH^2 = 3(HA^2 + HB^2 + HC^2) - a^2 - b^2 - c^2 \tag{1.6}$$

$$4(OA^2 + OB^2 + OC^2) = HA^2 + HB^2 + HC^2 + a^2 + b^2 + c^2 \tag{1.7}$$

Finally, a less common formula for the length of the Euler line, namely

$$OH^2 = (HM_a^2 - OM_a^2) + (HM_b^2 - OM_b^2) + (HM_c^2 - OM_c^2) \tag{1.8}$$

where M_a, M_b, M_c are the midpoints of BC, CA, AB, will be proved. With P in place of O and Q in place of H, we also prove the following analogue of (1.8):

$$PQ^2 = (QM_a^2 - PM_a^2) + (QM_b^2 - PM_b^2) + (QM_c^2 - PM_c^2) \tag{1.9}$$

2. MAIN RESULTS

Let ABC be a triangle. Unless otherwise stated, we will denote:

- the lengths of sides BC, CA, AB by a, b, c, respectively
- the midpoints of sides BC, CA, AB by M_a, M_b, M_c
- the radius of the circumcircle of ABC by R
- the circumcenter by O, the orthocenter by H, the nine-point center by N
- the interior angles by $\angle A = \angle CAB,\ \angle B = \angle ABC,\ \angle C = \angle BCA$

2.1. An "orthocentric-like" system.

Proposition 2.1. *Given a triangle ABC, define a line from vertex A in such a way that its slope is the negative of the slope of the opposite side BC; similarly, define lines from B and C. Then these lines are concurrent at a point Q on the circumcircle of ABC.*

Proof. To verify concurrence, place the vertices of triangle ABC at $A(x_1, y_1)$,$B(x_2, y_2)$,$C(x_3, y_3)$, where the x_i's, y_i's, $i = 1, 2, 3$, are real numbers. Let the slopes of AB, BC, CA be m_1, m_2, m_3 respectively, then

$$m_1 = \frac{y_1 - y_2}{x_1 - x_2},\ m_2 = \frac{y_2 - y_3}{x_2 - x_3},\ m_3 = \frac{y_1 - y_3}{x_1 - x_3}$$

The equations of the lines so defined are:

$$\begin{aligned}
y - y_1 &= -m_2(x - x_1) &&\implies m_2 x + y - (y_1 + m_2 x_1) = 0\\
y - y_2 &= -m_3(x - x_2) &&\implies m_3 x + y - (y_2 + m_3 x_2) = 0\\
y - y_3 &= -m_1(x - x_3) &&\implies m_1 x + y - (y_3 + m_1 x_3) = 0
\end{aligned}$$

For concurrence, the determinant below should vanish:

$$\begin{vmatrix} m_2 & 1 & -(y_1 + m_2 x_1) \\ m_3 & 1 & -(y_2 + m_3 x_2) \\ m_1 & 1 & -(y_3 + m_1 x_3) \end{vmatrix} = \begin{vmatrix} m_2 & 1 & -(y_1 + m_2 x_1) \\ m_3 - m_2 & 0 & (y_1 - y_2) + m_2 x_1 - m_3 x_2 \\ m_1 - m_2 & 0 & (y_1 - y_3) + m_2 x_1 - m_1 x_3 \end{vmatrix}$$

Expanding along the second column gives:

$$\begin{aligned}
\det &= -\left[(m_3 - m_2)(y_1 - y_3 + m_2 x_1 - m_1 x_3) - (m_1 - m_2)(y_1 - y_2 + m_2 x_1 - m_3 x_2)\right]\\
&= -\left[(m_3 - m_2)(m_3 \times (x_1 - x_3) + m_2 x_1 - m_1 x_3) - (m_1 - m_2)(m_1 \times (x_1 - x_2) + m_2 x_1 - m_3 x_2)\right]\\
&= -\left[(m_3 - m_2) \times ((m_3 + m_2)x_1 - (m_1 + m_3)x_3) - (m_1 - m_2) \times ((m_1 + m_2)x_1 - (m_1 + m_3)x_2)\right]\\
&= -\left[(m_3^2 - m_2^2 + m_2^2 - m_1^2)x_1 + (m_1 - m_2)(m_1 + m_3)x_2 + (m_1 + m_3)(m_2 - m_3)x_3\right]\\
&= -(m_1 + m_3)\left[(m_3 - m_1)x_1 + (m_1 - m_2)x_2 + (m_2 - m_3)x_3\right]\\
&= -(m_1 + m_3) \times 0\\
&= 0
\end{aligned}$$

That $(m_3 - m_1)x_1 + (m_1 - m_2)x_2 + (m_2 - m_3)x_3$ is zero follows from adding the left sides of $y_1 - y_2 = m_1(x_1 - x_2)$, $y_2 - y_3 = m_2(x_2 - x_3)$, $y_3 - y_1 = m_3(x_3 - x_1)$ and re-arranging the right sides. So these lines are indeed concurrent. Let the point of concurrence be Q. To show that Q is on the circumcircle, suppose that sides AB, BC, CA make angles α, β, γ, respectively, with the positive x-axis. For simplicity, let $\beta = 0$. Assume without loss of generality that $\alpha < \gamma$. The interior angles of triangle ABC are: $\angle B = \alpha$, $\angle C = \pi - \gamma$, and $\angle A = \gamma - \alpha$. Through A, draw a line whose slope is the negative of the slope of BC (since the slope of BC is zero, the line drawn will just be parallel to side BC); through B, draw a line whose slope is the negative of the slope of CA (this cevian will then make an angle of $\pi - \gamma$ with the positive x-axis); through C, draw a line whose slope is the negative of the slope of side AB (the cevian so drawn will then make an angle of $\pi - \alpha$ will the positive x-axis).

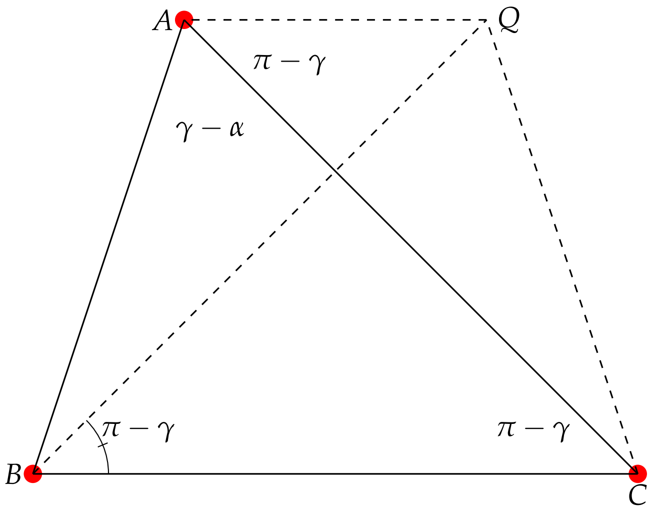

Figure 2. Point Q

Since $\angle CBQ = \angle QAC$ above, it follows from a well-known characterization of cyclic quadrilateral (e.g. see [7]) that A, B, C, Q lie on the same circle. □

Corollary 2.1. *Through the midpoints of each side of a triangle, draw lines in such a way that the slope of each line is the negative of the slope of the side from which the line was drawn. Then the lines are concurrent at a point P on the nine-point circle of the given triangle.*

Proof. We apply the construction in Proposition (2.1) to the medial triangle. The point of concurrence will lie on the circumcircle of the medial triangle, which is the nine-point circle of the parent triangle. □

Proposition 2.2. *Given a triangle ABC, define a line from vertex A in such a way that its slope is the reciprocal of the slope of the opposite side BC; similarly, define lines from B and C. Then these lines are concurrent at a point S on the circumcircle of ABC.*

Corollary 2.2. *Through the midpoints of each side of a triangle, draw lines in such a way that the slope of each line is the reciprocal of the slope of the side from which the line was drawn. Then the lines are concurrent at a point T on the nine-point circle of the given triangle.*

Definition 2.1. *Define an "orthocentric-like" system to mean four points A, B, C, D in which each point is the point of concurrence of three lines, determined using the same rule, from the triangle formed by the other three points.*

In the case of the regular orthocentric system, the "rule" is simply the definition of an altitude. In the case of an "*orthocentric-like*" system, the "rule" will be whatever was applied to the parent triangle to yield the original point of concurrence – such a point of concurrence determined from a parent triangle may not be a triangle center in the mould of orthocenter, circumcenter, etc, so in what follows, we just call such a point of concurrence a "pseudo-center".

Proposition 2.3. *Given any triangle ABC, and the point of concurrence Q defined in Proposition 2.1, the four points A, B, C, Q form an "orthocentric-like" system.*

Proof. Given triangle ABC in which point Q is the point of concurrence of lines defined in the following way: through A draw a line whose slope is the negative of the slope of side BC, etc. Claim: C is the "pseudo-center" of triangle ABQ. Indeed, through A, draw a line whose slope is the negative of the slope of side BQ – but then, by construction, the slope of BQ is the negative of the slope of side AC. So we're drawing, through A, a line whose slope is the slope of side AC, which is side AC itself. Similarly, through B, we draw a line whose slope is the negative of the slope of side AQ – but then, by construction, the slope of AQ is the negative of the slope of BC. This amounts to drawing a line, through B, with same slope as the slope of BC, which is BC itself. The two lines so drawn intersect at C. By Proposition 2.1 we know that the third line through Q with slope the negative of the slope of side AB will also go through C. This shows that C is the "pseudo-center" of triangle ABQ. Repeating for triangles BCQ and CAQ completes the proof. □

Proposition 2.4. *Given any triangle ABC with midpoints M_a, M_b, M_c of sides BC, CA, AB, and the point of concurrence P defined in Corollary 2.1, the four points M_a, M_b, M_c, P form an "orthocentric-like" system.*

Proposition 2.5. *Given any triangle ABC, and the point of concurrence S defined in Proposition 2.2, the four points A, B, C, S form an "orthocentric-like" system.*

Proof. Similar to the proof of Proposition 2.3. Given triangle ABC in which point S is the point of concurrence of lines defined in the following way: through A draw a line whose slope is the reciprocal of the slope of side BC, etc. Claim: C is the "pseudo-center" of triangle ABS. Let m_1, m_2, m_3 be the slopes of sides AB, BC, CA. Through A, draw a line whose slope is the reciprocal of the slope of side BS – but then, by construction, the slope of BS is $\frac{1}{m_3}$. So we're drawing, through A, a line whose slope is $1/\frac{1}{m_3} = m_3$, the slope of side AC, which is side AC itself. Through B, we draw a line whose slope is the reciprocal of the slope of side AS – but then, by construction, the slope of AS is $\frac{1}{m_2}$. This amounts to drawing a line, through B, with same slope as the slope of BC, which is BC itself. The two lines so drawn intersect at C. By Proposition 2.2 we know that the third line through S with slope the reciprocal of the slope of side AB will also go through C. This shows that C is the "pseudo-center" of triangle ABS. Repeating for triangles BCS and CAS completes the proof. □

Proposition 2.6. *Given any triangle ABC with midpoints M_a, M_b, M_c of sides BC, CA, AB, and the point of concurrence T defined in Corollary 2.2, the four points M_a, M_b, M_c, T form an "orthocentric-like" system.*

2.2. **Distance-based comparison with the orthocenter and circumcenter.** In terms of distances, point Q occasionally behaves like the orthocenter, while point P acts like the circumcenter. For example, in any triangle, the distance from a vertex to the orthocenter is twice the distance from the circumcenter to the midpoint of the opposite side. We have an analogue in Theorem 2.1 below; its proof – and that of Theorem 2.2 – easily follows from homothety. However, a separate approach is followed in order to utilize the properties of points P and Q.

Theorem 2.1. *In triangle ABC, let Q and P be as defined in Proposition 2.1 and Corollary 2.1. Then the distance AQ is twice the distance PM_a, where M_a is the midpoint of side BC.*

Proof. Let H, O, N be the orthocenter, circumcenter, and nine-point center, respectively. Let Y be the midpoint of HQ and Z the midpoint of AH. Since Q is on the circumcircle of ABC (Proposition 2.1), then Y is on the nine-point circle, and Z is on the nine-point circle as well. Consider the diagram below:

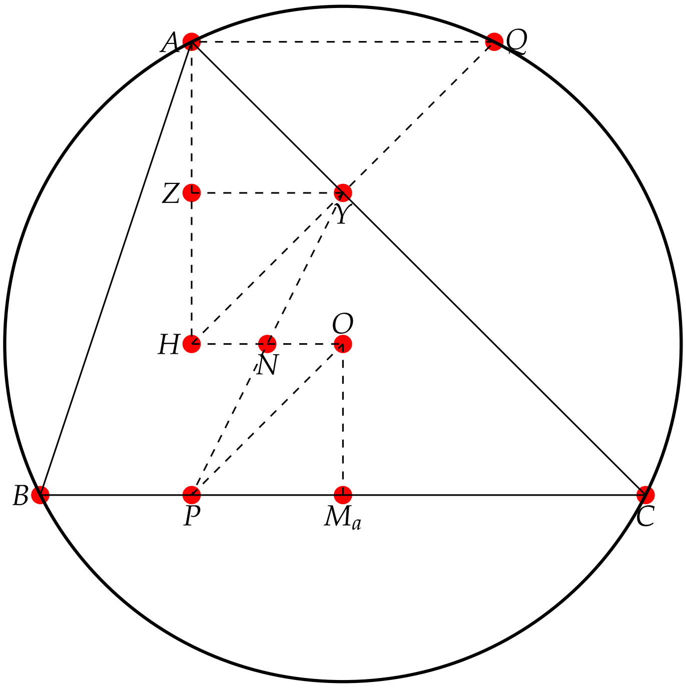

Figure 3. AQ is twice PM_a

By definition, the slope of AQ is the negative of the slope of BC and the slope of PM_a is also the negative of the slope of BC, thus AQ is parallel to PM_a. In triangle AHQ, ZY is parallel to AQ – and equal to half its length – because Z and Y are the midpoints of AH and HQ. This implies that ZY is also parallel to PM_a. It is well-known that AH is parallel to OM_a and $AH = 2 \times OM_a$. This gives: $ZH = OM_a$. Since HO and PY are bisected at N, we have that $HPOY$ is a parallelogram, so HY is parallel to PO, and $\angle HYP = \angle YPO$. In turn, $\angle ZYH = \angle OPM_a$.

According to Corollary 2.1, P is on the nine-point circle, the line segment PY through N is a diameter of the nine-point circle, and so $\angle PZY = 90°$. Since AH is parallel to OM_a and ZY is parallel to PM_a, it follows that $\angle PM_aO = 90°$. Therefore, triangle PM_aO is congruent to triangle ZYH, with $ZH = OM_a$, $ZY = PM_a$, and $HY = PO$. Since triangle ZYH is similar to triangle AQH (similarity ratio of $1/2$), we have that triangle PM_aO is similar to triangle AQH. Thus, $AQ = 2 \times PM_a$. ☐

Theorem 2.2. *Let Q and P be as defined in Proposition 2.1 and Corollary 2.1. The line PQ intersects the Euler line at the centroid of the parent triangle ABC, and the centroid divides PQ in a $2:1$ ratio, with P closer to the centroid than Q.*

Proof. In the diagram below, OP is parallel to QH and $OP = \frac{1}{2}QH$. Let Y be the midpoint of QH, and let Z be the midpoint of PQ.

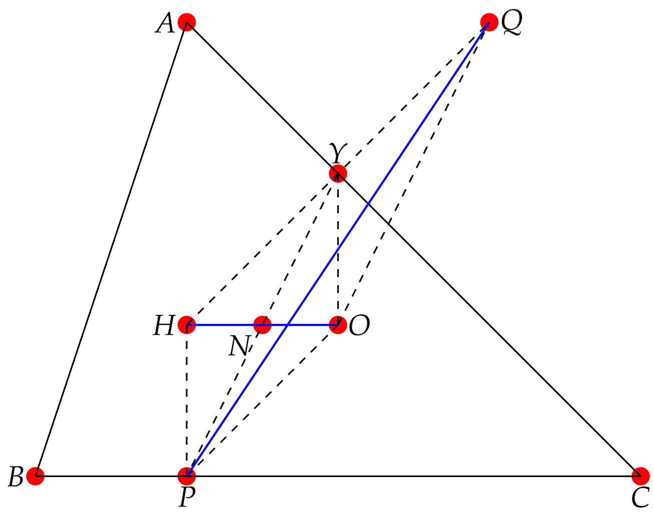

Figure 4. Parallelogram $POYH$

$POYH$ is a parallelogram because PO is parallel to HY and $PO = HY$. Thus, the diagonals PY and HO bisect each other. Since N is the midpoint of HO, then N is equally the midpoint of PY. Similarly, $POQY$ is a parallelogram because PO is parallel to YQ and $PO = YQ$, so the diagonals PQ and OY bisect each other at Z, the midpoint of PQ. In triangle POY, both PZ and ON are medians, so they intersect at the centroid G' of triangle POY. We claim that this is the same centroid as that of the parent triangle ABC. Indeed, since $NG' : G'O = 1 : 2$ we can let $NG' = t$ and get $G'O = 2t$, for some positive t. Then $NO = 3t$ and $HN = 3t$ as well. Thus $HG' : G'O = 4t : 2t = 2 : 1$. It is the centroid G of the parent triangle ABC that divides the Euler line HO in the ratio $2 : 1$, with the centroid closer to O than to H. Therefore, $G' = G$.

Now consider median PZ in triangle POY. We have $PG : GZ = 2 : 1$, and continuing as before leads to $PG : GQ = 1 : 2$. Thus the centroid G divides the line PQ in the same ratio as it divides the Euler line OH, and P is closer to the centroid than Q. □

Proposition 2.7. *If X is the midpoint of PQ, then $OP^2 = HQ \times NX$, where Q, P are as defined in Proposition 2.1 and Corollary 2.1.*

Proof. With X as given and with reference to Figure 4 above, we have that NX is parallel to YQ and $NX = \frac{1}{2}YQ$. As YQ itself and PO both equal $\frac{1}{2}HQ$, we obtain $NX = \frac{1}{4}HQ$, whence

$$PO^2 = \frac{1}{4}HQ^2 = NX \times HQ$$

□

Proposition 2.8. *The equation $PQ^2 - OH^2 = 3(OQ^2 - HP^2)$ holds in any triangle.*

Proof. Let G be the centroid of triangle ABC. Consider the diagram below:

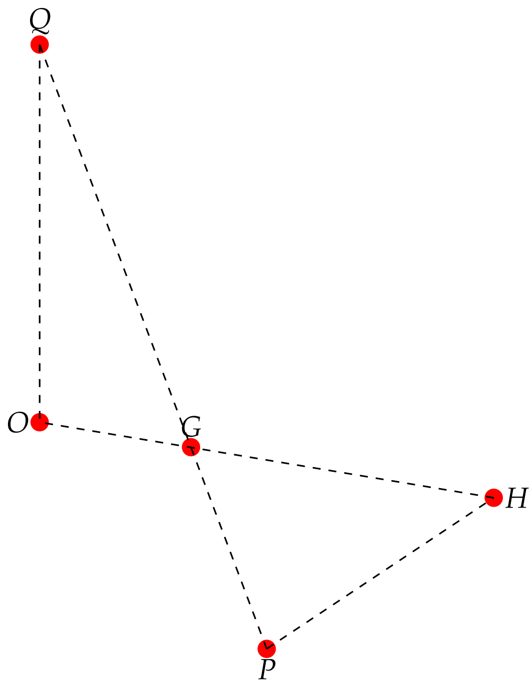

Figure 5. Common point G to triangles QGO and HGP

By Theorem 2.2 we have $QG : GP = 2 : 1$. Let $QG = 2t$ and $GP = t$, for some $t > 0$. Also, $OG : GH = 1 : 2$ means we can set $OG = k$, $GH = 2k$, for some $k > 0$. In triangle QOG, we have:

$$\cos G = \frac{k^2 + (2t)^2 - OQ^2}{2(k)(2t)}$$

In triangle HPG, we have:

$$\cos G = \frac{(2k)^2 + t^2 - HP^2}{2(2k)(t)}$$

Equating these two gives

$$k^2 + 4t^2 - OQ^2 = 4k^2 + t^2 - HP^2$$
$$HP^2 - OQ^2 = 3(k^2 - t^2)$$

Now, $OH = 3k$ implies $k = \frac{OH}{3}$ and $PQ = 3t$ gives $t = \frac{PQ}{3}$. Substituting:

$$HP^2 - OQ^2 = 3\left(\frac{OH^2}{9} - \frac{PQ^2}{9}\right)$$
$$3(HP^2 - OQ^2) = OH^2 - PQ^2$$

□

Similarly, the following holds:

Proposition 2.9. *In any triangle, we have* $2PQ^2 + OH^2 = 6PO^2 + 3QO^2$.

Proposition 2.10. *In any triangle ABC with side-lengths a, b, c we have*

$$PQ^2 = 3(PA^2 + PB^2 + PC^2) - a^2 - b^2 - c^2.$$

Proof. It is well-known that $MA^2 + MB^2 + MC^2 = \frac{1}{3}(a^2 + b^2 + c^2) + 3MG^2$ where G is the centroid and M is any point in the plane of any triangle ABC (e.g., see [8, 9, 10, 11]). Let $M = P$, then $PA^2 + PB^2 + PC^2 = \frac{1}{3}(a^2 + b^2 + c^2) + 3PG^2$. By Theorem 2.2, $PG = \frac{1}{3}PQ$ and so

$$PA^2 + PB^2 + PC^2 = \frac{1}{3}(a^2 + b^2 + c^2) + 3\left(\frac{1}{9}PQ^2\right)$$
$$PQ^2 = 3(PA^2 + PB^2 + PC^2) - a^2 - b^2 - c^2$$

□

If we let $M = Q$ instead and use $GQ = \frac{2}{3}PQ$ we obtain

Proposition 2.11. *In any triangle ABC with side-lengths a, b, c we have*

$$4PQ^2 = 3(QA^2 + QB^2 + QC^2) - a^2 - b^2 - c^2.$$

Proposition 2.12. *In any triangle ABC, let H, O be the orthocenter and circumcenter, and let M_a, M_b, M_c be the midpoints of sides BC, CA, AB respectively. Then:*

$$OH^2 = (HM_a^2 - OM_a^2) + (HM_b^2 - OM_b^2) + (HM_c^2 - OM_c^2)$$

Proof. It is well-known that $OH^2 = 9R^2 - a^2 - b^2 - c^2$ (e.g., page 102 in [8]), where a, b, c are the usual side-lengths. Also $AH^2 = 4R^2 - a^2$; similarly for BH & CH. Consider triangle BHC, where HM_a is a median, and the right-angled triangle COM_a:

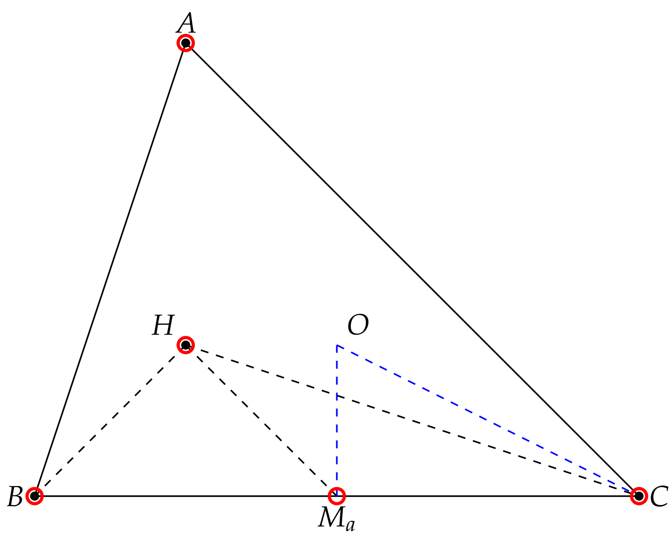

Figure 6. Triangle BHC and right-angled triangle COM_a

We have

$$\begin{aligned} HM_a^2 &= \frac{2BH^2 + 2CH^2 - BC^2}{4} \\ &= \frac{2(4R^2 - b^2) + 2(4R^2 - c^2) - a^2}{4} \\ OM_a^2 &= R^2 - \left(\frac{a}{2}\right)^2 \\ HM_a^2 - OM_a^2 &= \frac{12R^2 - 2b^2 - 2c^2}{4} \end{aligned}$$

Similarly, $HM_b^2 - OM_b^2 = \frac{12R^2-2a^2-2c^2}{4}$ and $HM_c^2 - OM_c^2 = \frac{12R^2-2a^2-2b^2}{4}$. Adding:

$$\begin{aligned} (HM_a^2 - OM_a^2) + (HM_b^2 - OM_b^2) + (HM_c^2 - OM_c^2) &= \frac{36R^2 - 4(a^2 + b^2 + c^2)}{4} \\ &= 9R^2 - a^2 - b^2 - c^2 \\ &= OH^2 \end{aligned}$$

□

Proposition 2.13. *In any triangle ABC with M_a, M_b, M_c the midpoints of BC, CA, AB, we have:*

$$PQ^2 = (QM_a^2 - PM_a^2) + (QM_b^2 - PM_b^2) + (QM_c^2 - PM_c^2)$$

Proof. QM_a is a median in triangle QBC, and so

$$QM_a^2 = \frac{2QB^2 + 2QC^2 - BC^2}{4} = \frac{2QB^2 + 2QC^2 - a^2}{4}$$

Similarly, PM_a is a median in triangle PBC, so $PM_a^2 = \frac{2PB^2+2PC^2-a^2}{4}$. Thus

$$QM_a^2 - PM_a^2 = \frac{2QB^2 + 2QC^2 - 2PB^2 - 2PC^2}{4}$$

In the same way, we get

$$QM_b^2 - PM_b^2 = \frac{2QA^2 + 2QC^2 - 2PA^2 - 2PC^2}{4}, \; QM_c^2 - PM_c^2 = \frac{2QA^2 + 2QB^2 - 2PA^2 - 2PA^2}{4}$$

Adding:

$$(QM_a^2 - PM_a^2) + (QM_b^2 - PM_b^2) + (QM_c^2 - PM_c^2) = QA^2 + QB^2 + QC^2 - PA^2 - PB^2 - PC^2$$

Now, from Propositions 2.10 and 2.11

$$PQ^2 = 3(PA^2 + PB^2 + PC^2) - a^2 - b^2 - c^2, \; 4PQ^2 = 3(QA^2 + QB^2 + QC^2) - a^2 - b^2 - c^2$$

Eliminating $-a^2 - b^2 - c^2$ between these two equations gives

$$PQ^2 = QA^2 + QB^2 + QC^2 - PA^2 - PB^2 - PC^2$$

Thus

$$PQ^2 = (QM_a^2 - PM_a^2) + (QM_b^2 - PM_b^2) + (QM_c^2 - PM_c^2)$$

□

2.3. **Some special cases.**

Proposition 2.14. *If triangle ABC is right-angled at C, then triangle PQC is also right-angled at C.*

Proof. Suppose that $\angle C = 90^\circ$ in triangle ABC. Let O, H, R be the circumcenter, orthocenter, and circumradius, respectively. From Propositions 2.8 and 2.9 we have:

$$PQ^2 - OH^2 = 3(OQ^2 - HP^2)$$
$$2PQ^2 + OH^2 = 6PO^2 + 3QO^2$$

Now, $QO = R$ and $PO = \frac{1}{2}HQ$ in any triangle. Since $\angle C = 90^\circ$ in the present case we have in addition that $OH = R$ and $H = C$. The second equation above becomes

$$2PQ^2 + OH^2 = 6\left(\frac{1}{4}HQ^2\right) + 3QO^2$$
$$4PQ^2 + 2OH^2 = 3HQ^2 + 6QO^2$$
$$4PQ^2 + 2R^2 = 3QC^2 + 6R^2$$
$$4PQ^2 = 3QC^2 + 4R^2$$

Using the same substitutions in $PQ^2 - OH^2 = 3(OQ^2 - HP^2)$ gives

$$PQ^2 = 4R^2 - 3PC^2$$

If we now eliminate $4R^2$ from $4PQ^2 = 3QC^2 + 4R^2$ and $PQ^2 = 4R^2 - 3PC^2$, we obtain

$$PQ^2 = PC^2 + QC^2$$

This shows that triangle PQC is right-angled at C. (Compare with Proposition 2.17.) □

Note that the converse of Proposition 2.14 above does not hold in general. In fact, if side AB of triangle ABC is parallel to the x-axis, then PQC is right-angled at C.

Proposition 2.15. *Triangle HPO is right-angled at P, if and only if the parent triangle is a right triangle.*

Proof. From Propositions 2.8 and 2.9 again, we have:

$$PQ^2 - OH^2 = 3(OQ^2 - HP^2)$$
$$2PQ^2 + OH^2 = 6PO^2 + 3QO^2$$

Using $QO = R$ and eliminating PQ^2 from both equations gives

$$3OH^2 = 6PO^2 + 6HP^2 - 3R^2 \implies OH^2 + R^2 = 2PO^2 + 2HP^2$$

Now if the parent triangle is a right-triangle, then $OH = R$ and so the above equation becomes

$$2OH^2 = 2PO^2 + 2HP^2 \implies OH^2 = PO^2 + HP^2,$$

showing that triangle HPO is right-angled at P. On the other hand, if triangle HPO is right-angled at P, then we have again that $OH^2 = PO^2 + HP^2$; using this in

$$OH^2 + R^2 = 2PO^2 + 2HP^2$$

gives $OH = R$, so the parent triangle is right-angled. □

Proposition 2.16. *In any equilateral triangle, the length of PQ equals the common length of the medians.*

Proof. From Propositions 2.8 and 2.9 again, we have:

$$PQ^2 - OH^2 = 3(OQ^2 - HP^2)$$
$$2PQ^2 + OH^2 = 6PO^2 + 3QO^2$$

In an equilateral triangle, we have $H = O$, so the two equations above simplify to

$$PQ^2 = 3(R^2 - HP^2)$$
$$2PQ^2 = 6HP^2 + 3R^2$$

If we now eliminate HP^2 from both equations, we obtain

$$4PQ^2 = 9R^2 \implies PQ = \frac{3}{2}R$$

But $\frac{3}{2}R$ is the common length of each of the three medians in an equilateral triangle; for example, the median from A has length

$$m_a^2 = \frac{2b^2 + 2c^2 - a^2}{4} = \frac{3}{4}a^2 \implies m_a = \frac{\sqrt{3}}{2}a = \frac{\sqrt{3}}{2}\left(2R\sin 60^\circ\right) = \frac{3}{2}R.$$

□

Corollary 2.3. *In any equilateral triangle, the length of ST equals the common length of the medians.*

Proof. From Theorem 2.6 below we have $2(PQ^2 + ST^2 - OH^2) = a^2 + b^2 + c^2$. In an equilateral triangle $H = O$ and $a = b = c$; also $PQ = \frac{3}{2}R$ from the preceding result, where R is the radius of the circumcircle. Thus:

$$2\left(\frac{9}{4}R^2 + ST^2\right) = 3a^2$$
$$ST^2 = \frac{3}{2}a^2 - \frac{9}{4}R^2$$
$$= \frac{3}{2}\left(2R\sin 60^\circ\right)^2 - \frac{9}{4}R^2$$

Thus $ST = \frac{3}{2}R$, which is the length of each of the three medians in an equilateral triangle.

□

Proposition 2.17. *The following three statements are equivalent in any triangle ABC:*

(1) *Q coincides with a vertex (A, say)*
(2) *P coincides with the midpoint of the opposite side (midpoint of BC)*
(3) *AB and AC have opposite slopes.*

In particular, in a right-triangle in which the legs (AC and BC, say) have slopes ± 1, we have that Q coincides with the orthocenter C, while P coincides with the circumcenter. (Proposition 2.14 will then give a degenerate triangle PQC, though $PQ^2 = PC^2 + QC^2$ still holds, trivially.)

Proposition 2.18. *The following two statements are equivalent in any triangle ABC:*

(1) *Q is the reflection of H over side AB*
(2) *the slope of side AB is ± 1.*

2.4. **Other properties.**

Theorem 2.3. *Let Q and P be as defined in Corollary 2.1 and Proposition 2.1. Then, given any triangle ABC with circumcenter O, the line segment joining the midpoint of PQ to the midpoint of PO is a diameter of the nine-point circle of the medial triangle.*

Proof. We first show that both the midpoint of PQ and the midpoint of PO lie on the nine-point circle of the medial triangle. Let X be the midpoint of PQ and Y the midpoint of PO. Let F be the nine-point center of the medial triangle associated with ABC, then F is the midpoint of ON. The radius of the nine-point circle of the medial triangle is $\frac{1}{4}R$, where R is the radius of the circumcircle of ABC.

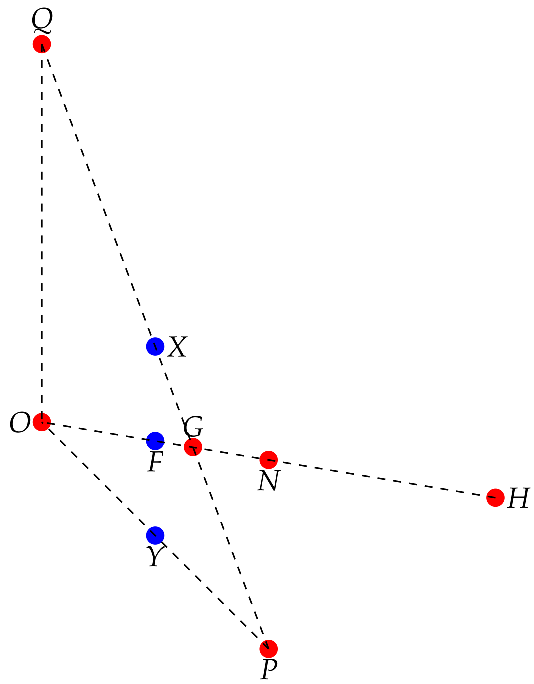

Figure 7. Points X, F, Y

We have $GF : FO = 1 : 3$ and $GX : XQ = 1 : 3$, and it follows that FX is parallel to OQ in triangle GQO; moreover, $FX = \frac{1}{4}OQ$. Since Q is on the circumcircle of triangle ABC, the segment OQ is a radius, and so $FX = \frac{1}{4}R$. This shows that X is on the nine-point circle of the medial triangle. Since O is the orthocenter of the medial triangle and P is on the nine-point circle of the parent triangle ABC, it follows that the midpoint Y of PO is on the nine-point circle of the medial triangle. Furthermore, in triangle OPQ, X and Y are the midpoints of PQ and PO, so $XY = \frac{1}{2}QO = \frac{1}{2}R$. Therefore, XY is a diameter of the nine-point circle of the medial triangle.

□

Theorem 2.4. *The image of the reflection of the orthocenter in P, and the image of the reflection of Q in the circumcenter, coincide. Moreover, this common point is the point S defined in Proposition 2.2.*

Proof. Since P is on the nine-point circle, the image of the reflection of the orthocenter in P will be on the circumcircle. The first part of the proof then follows from the fact that PO is parallel to HQ and $PO = \frac{1}{2}HQ$ in any triangle with circumcenter O and orthocenter H (see Figure 8 below). Let the common point be U. To show that $U = S$, let m_1, m_2, m_3 be the slopes of sides AB, BC, CA. By construction, the slope of QA is the negative of the slope of side BC, namely $-m_2$. Since QU is a diameter, we have that $\angle QAU$ is a right angle, and so the slope of AU is the negative reciprocal of the slope of QA, that is, $-\frac{1}{-m_2} = \frac{1}{m_2}$. Repeating for the right triangles QBU and QCU, we find that the slopes of BU and CU are $\frac{1}{m_3}$ and $\frac{1}{m_1}$ respectively. The only point with this property is the point S described in Proposition 2.2, so $U = S$. □

Theorem 2.5. *The line joining P to the midpoint of H and Q is a diameter of the nine-point circle. Moreover, the midpoint of H and Q is the point T described in Corollary 2.2.*

Proof. Since Q is on the circumcircle, the midpoint V of H and Q is on the nine-point circle. To show that PV is a diameter of the nine-point circle, consider the diagram below:

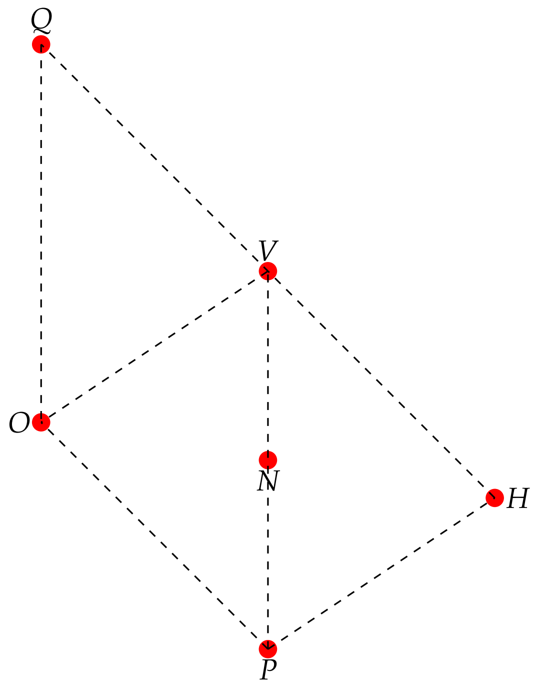

Figure 8. A diameter PV of the nine-point circle

Since PO is parallel to HQ and $PO = \frac{1}{2}HQ$, we have that $PO = HV$ and $OV = PH$, so $HPOV$ is a parallelogram. Thus, PV goes through the midpoint N of the Euler line, which is the center of the nine-point circle, showing that PV is a diameter of the nine-point circle. To see that $V = T$, let m_1, m_2, m_3 be the slopes of sides AB, BC, CA as before. First join PV to M_a, the midpoint of side BC. Then $\angle PM_aV$ is a right angle. By construction, the slope of PM_a is $-m_2$, and the slope of VM_a has to be $\frac{1}{m_2}$. Repeating for the other two midpoints shows that the slopes of VM_b and VM_c are $\frac{1}{m_3}$ and $\frac{1}{m_1}$ respectively. T is by construction the only point with this property, so $V = T$. □

Theorem 2.6. *In any triangle we have* $2(PQ^2 + ST^2 - OH^2) = a^2 + b^2 + c^2$.

Proof. By the preceding results, P is the midpoint of HS, T is the midpoint of HQ, and O is the midpoint of QS. Thus, in the diagram below

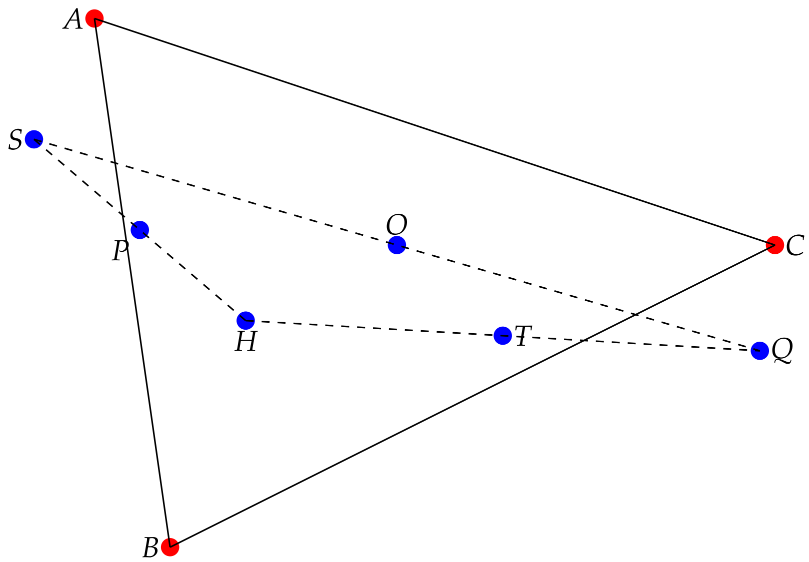

Figure 9. Medians HO, QP, ST in triangle HQS

the Euler line HO as well as QP and ST are all medians in triangle HQS. Noting that QS is a diameter, we have $QS = 2R$, where R is the radius of the circumcircle of ABC. Also, the length HO^2 is $9R^2 - a^2 - b^2 - c^2$ in any triangle with side-lengths a, b, c. Thus:

$$\begin{aligned}
HO^2 &= \frac{2HS^2 + 2HQ^2 - 4R^2}{4} \\
ST^2 &= \frac{2HS^2 + 2(4R^2) - HQ^2}{4} \\
QP^2 &= \frac{2HQ^2 + 2(4R^2) - HS^2}{4} \\
QP^2 + ST^2 - HO^2 &= \frac{20R^2 - HS^2 - HQ^2}{4} \\
&= \frac{20R^2 - \left(\frac{4HO^2+4R^2}{2}\right)}{4} \\
&= \frac{20R^2 - 2HO^2 - 2R^2}{4} \\
&= \frac{20R^2 - 2(9R^2 - a^2 - b^2 - c^2) - 2R^2}{4} \\
&= \frac{a^2 + b^2 + c^2}{2}
\end{aligned}$$

□

Proposition 2.19. *One of the following is true:* $a \times QA = a \times PM_a + b \times PM_b + c \times PM_c$, $b \times QB = a \times PM_a + b \times PM_b + c \times PM_c$, $c \times QC = a \times PM_a + b \times PM_b + c \times PM_c$.

Proof.
- First observe that one of the following is true:

$$a \times PM_a = b \times PM_b + c \times PM_c,\ b \times PM_b = a \times PM_a + c \times PM_c,\ c \times PM_c = a \times PM_a + b \times PM_b$$

The points P, M_a, M_b, M_c all lie on the nine-point circle of triangle ABC, so we can apply Ptolemy's theorem. For example, using the configuration in the diagram below

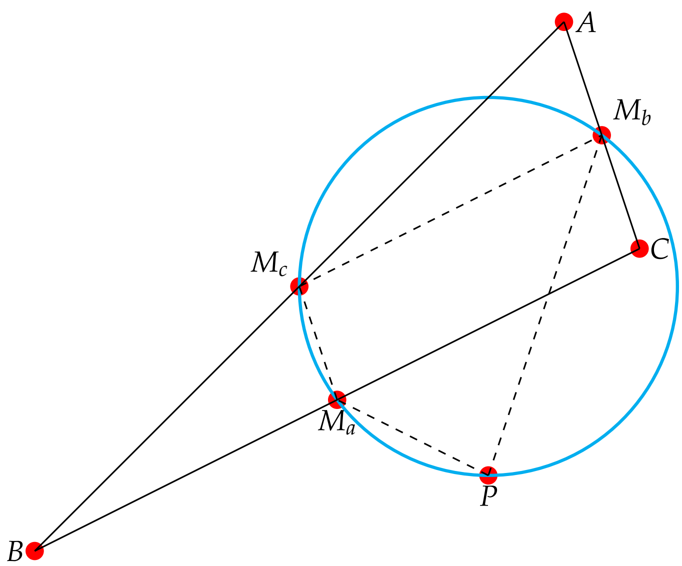

Figure 10. Point P on the nine-point circle

we have

$$PM_c \times M_aM_b = PM_b \times M_aM_c + M_bM_c \times PM_a$$
$$PM_c \times \frac{c}{2} = PM_b \times \frac{b}{2} + PM_a \times \frac{a}{2}$$
$$c \times PM_c = a \times PM_a + b \times PM_b$$

- Using the fact that $QC = 2 \times PM_c$ from Theorem 2.1 gives:

$$\begin{aligned} c \times QC &= 2c \times PM_c \\ &= c \times PM_c + c \times PM_c \\ &= c \times PM_c + (a \times PM_a + b \times PM_b) \end{aligned}$$

□

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