Global Journal of Advanced Research
on Classical and Modern Geometries
ISSN: 2284-5569, Vol.9, (2025), Issue 1, pp.84-99

A NOTE ON $phv-$SEMI-SLANT SUBMERSIONS

SUSHIL KUMAR, SUMEET KUMAR AND MURAT POLAT

ABSTRACT. In this article, we describe and study the concept of pointwise h-v-semi slant submersions ($phv-$ semi slant submersions, In short) and the almost $phv-$ semi-slant submersions whose base manifolds are almost quaternionic Hermitian manifolds. We examine geometric properties of such submersions: the integrability of distributions, the condition for such maps to be totally geodesic, the geometry of foliations, etc. Furthermore, we provide some illustrative examples of the almost $phv-$semi-slant submersions.

1. INTRODUCTIONS

In differential geometry, the geometry of submanifolds is a very productive field. One of the ways to obtain a submanifold by using a smooth map between Riemann manifolds is the subject of submersion. The study of Riemannian submersions originated from the work of Gray [8] and O'Neill [18]. Then, Riemann submersions were discussed with the help of differentiable structures of manifolds. In 1976, Watson [36] described almost Hermitian submersions and investigated a kind of structural problems among base manifolds and total manifolds.
In [29], Şahin defined the notion of anti-invariant Riemannian submersion from an almost Hermitian manifold (AHM) to a Riemannian manifold (RM). He also investigated such submersions from Kahlerian manifolds to Riemannian manifolds [32]. Further, he investigated slant submersions [31] and semi-invariant submersions [30]. Recently, many new Riemann submersions between different types of manifolds have been identified and studied; such as hemi-slant submersions [[1],[35]], quasi-hemi-slant submersions [[17],[26]], conformal anti-invariant submersions [14], quasi bi-slant submersions [[24],[25],[27],[28]], conformal hemi-slant submersions [15], conformal quasi bi-slant submersions [16], H-slant submersions [19], H-semi-slant submersions [21], Almost h-conformal semi-invariant submersion [23],V-semi-slant submersions [20], h-v-semi-slant submersions [22], pointwise semi-slant submersion [34] etc.
We note that the notion of Riemannian submersions is most inventive topic presently due to its wide applications in physics, mechanics and robotics. Such as: in Kaluza-Klein theory ([12],[5]), the Yang-Mills theory [6], Space-time [13] Supersymmetry theory [7], robotic chains [3], Betti Numbers [9],[10] etc. Latest developments regarding submersion theory can be found in the book [33].

2020 *Mathematics Subject Classification.* 58D17, 53C12, 58C25, 53C26, 53D25, 53B35.
Key words and phrases. Pointwise submersions, Pointwise H-V-semi-slant submersions, Quaternionic Hermitian manifolds.

This article is organized as follows: In Sec. 2; we give the fundamental equations and concepts of Riemannian submersions. In Sec. 3; we present the definition of the *phv*− semi slant submersion, almost *phv*− semi slant submersion and investigate their geometric properties. In this section, the necessary and sufficient conditions for integrability of distributions, totally geodesicness and some interesting results are also obtained. In the last section, we supply proper examples of the almost *phv*− semi slant submersions.

2. Preliminaries

Let $\Psi : (M_q, g_{M_q}) \rightarrow (M_r, g_{M_r})$ be a Riemannian submersion between *RM*s [33].
For any vector fields Z_1, Z_2 on M_q, the O'Neill's tensors [18] $\mathcal{T}$ and $\mathcal{A}$ define by

$$\mathcal{A}_{Z_1} Z_2 = \mathcal{H}\nabla_{\mathcal{H}Z_1}\mathcal{V}Z_2 + \mathcal{V}\nabla_{\mathcal{H}Z_1}\mathcal{H}Z_2, \tag{2.1}$$

$$\mathcal{T}_{Z_1} Z_2 = \mathcal{H}\nabla_{\mathcal{V}Z_1}\mathcal{V}Z_2 + \mathcal{V}\nabla_{\mathcal{V}Z_1}\mathcal{H}Z_2 \tag{2.2}$$

where ∇ is the Levi-Civita connection of g_{M_q}. We know that, on the tangent bundle of M_q, $\mathcal{T}_{Z_1}$ and $\mathcal{A}_{Z_1}$ are skew-symmetric operators which reversing the horizontal and vertical distributions.
From (2.1) and (2.2), we obtain

$$\nabla_{Y_1} Y_2 = \mathcal{T}_{Y_1} Y_2 + \mathcal{V}\nabla_{Y_1} Y_2, \tag{2.3}$$

$$\nabla_{Y_1} U_1 = \mathcal{T}_{Y_1} U_1 + \mathcal{H}\nabla_{Y_1} U_1, \tag{2.4}$$

$$\nabla_{U_1} Y_1 = \mathcal{A}_{U_1} Y_1 + \mathcal{V}\nabla_{U_1} Y_1, \tag{2.5}$$

$$\nabla_{U_1} V_2 = \mathcal{H}\nabla_{U_1} V_2 + \mathcal{A}_{U_1} V_2, \tag{2.6}$$

for $Y_1, Y_2 \in \Gamma(\ker \Psi_*)$ and $U_1, V_2 \in \Gamma(\ker \Psi_*)^{\perp}$, where $\mathcal{H}\nabla_{Y_1} U_1 = \mathcal{A}_{U_1} Y_1$, if U_1 is basic vector field. We can state that $\mathcal{T}$ plays role on the fibers as the second fundamental form, while $\mathcal{A}$ plays role on the horizontal distribution and measures the obstruction to the integrability of this distribution [4].
Now, we remind that the representation of second fundamental form of a map between two *RM*s. Let (M_q, g_{M_q}) and (M_r, g_{M_r}) be *RM*s and $\Psi : (M_q, g_{M_q}) \rightarrow (M_r, g_{M_r})$ be a map then the second fundamental form of Ψ is defined as

$$(\nabla \Psi_*)(V_1, V_2) = \nabla^{\Psi}_{V_1} \Psi_*(V_2) - \Psi_*(\nabla^{N_1}_{V_1} V_2) \tag{2.7}$$

for $V_1, V_2 \in \Gamma(TM_q)$, where ∇^{Ψ} is the pullback connection and we state for convenience by ∇ the Riemannian connections of the metrics g_{M_q} and g_{M_r}.
We also remind that a differentiable map Ψ between two *RM*s is totally geodesic if

$$(\nabla \Psi_*)(V_1, V_2) = 0, \text{ for all } V_1, V_2 \in \Gamma(TM_q). \tag{2.8}$$

Lemma 2.1. [11] *Let (M_q, g_{M_q}) and (M_r, g_{M_r}) are RMs. If $\Psi : (M_q, g_{M_q}) \rightarrow (M_r, g_{M_r})$ be a Riemannian submersion, then for vertical vector fields U_1, U_2 any horizontal vector fields Y_1, Y_2, then we obtain*
(i) $(\nabla \Psi_*)(Y_1, Y_2) = 0$,
(ii) $(\nabla \Psi_*)(U_1, U_2) = -\Psi_*(\mathcal{T}_{U_1} U_2) = -\Psi_*(\nabla_{U_1} U_2)$,
(iii) $(\nabla \Psi_*)(Y_1, U_1) = -\Psi_*(\nabla_{Y_1} U_1) = -\Psi_*(\mathcal{A}_{Y_1} U_1)$.

Let M_q be a $4k$-dimensional differentiable manifold, g_{M_q} be a Riemannian metric and F be a rank 3 subbundle of $End(TM_q)$ such that for any point $q \in M_q$ with its some neighborhood V, there exists a local basis $\{P_1, P_2, P_3\}$ of sections of F on V satisfying for all $\beta \in \{1,2,3\}$

$$P_\beta^2 = -id, \ P_\beta P_{\beta+1} = -P_{\beta+1}P_\beta = P_{\beta+2}, \tag{2.9}$$

$$g_{M_q}(P_\beta Y_1, P_\beta Y_2) = g_{M_q}(Y_1, Y_2), \tag{2.10}$$

for $\forall Y_1, Y_2 \in \Gamma(TM_q)$, where the indices belong to $\{1,2,3\}$ modulo 3. Then (M_q, F, g_{M_q}) is an almost quaternionic Hermitian manifold (AQHM) ([2], [11]). The basis $\{P_1, P_2, P_3\}$ is called a quaternionic Hermitian basis. (M_q, F, g_{M_q}) is a quaternionic Kähler manifold if there exist locally defined 1-forms $\omega_1, \omega_2, \omega_3$ such that for $\beta \in \{1,2,3\}$

$$\nabla_{Y_1} P_\beta = \omega_{\beta+2}(Y_1)P_{\beta+1} - \omega_{\beta+1}(Y_1)P_{\beta+2} \tag{2.11}$$

where $Y_1 \in \Gamma(TM_q)$ and the indices belong to $\{1,2,3\}$ modulo 3. (N_1, F, g_{M_q}) is a hyperkähler manifold (HKM) if $\nabla P_\beta = 0$ for $\beta \in \{1,2,3\}$. Moreover, g_{M_q} is called a hyperkähler metric and $\{P_1, P_2, P_3, g_{M_q}\}$ is called a hyperkähler structure on M_q.
Let (M_q, F_1, g_{M_q}) and (M_r, F_2, g_{M_r}) be $AQHM$s. If given a point $x \in M_q$, for any $P \in (F_1)_x$ there is $P' \in (F_2)_{\pi(x)}$ that satisfies

$$\pi_* \circ P = P' \circ \pi_*,$$

then a map $\Psi : M_q \to M_r$ is called a (F_1, F_2)-holomorphic map.
A Riemannian submersion $\Psi : M_q \to M_r$, which is an (F_1, F_2)-holomorphic map is called a quaternionic submersion. Besides, if (M_q, F_1, g_{M_q}) is a quaternionic Kähler manifold then Ψ is called a quaternionic Kähler submersion. This is also called hyperkähler submersion.
Now, we remind some basic definitions that we will use later.

Definition 2.1. [33] *Let $\Psi : (M_q, g_{M_q}, P) \to (M_r, g_{M_r})$ be a Riemannian submersion from an AHM to a RM. If the vertical distribution is invariant with respect to the complex structure P, i.e., $P(\ker \Psi_*) = \ker \Psi_*$, then Ψ is an invariant Riemannian submersion.*

Definition 2.2. [33] *Let $\Psi : (M_q, g_{M_q}, P) \to (M_r, g_{M_r})$ be a Riemannian submersion from an AHM to a RM. If the Wirtinger angle $\vartheta(Z)$ between PZ and the space $(\ker \pi_*)_q$ is independent of the choice of the nonzero vector $Z \in (\ker \pi_*)_q$ at each given point $q \in M_q$, then Ψ is a pointwise slant submersion. Additionally, the angle ϑ can be considered as a function on M_q which is called the slant function of the pointwise slant submersion.*

Definition 2.3. *A Riemannian submersion $\Psi : (M_q, g_{M_q}, P) \to (M_r, g_{M_r})$ is called a pointwise semi-slant submersion* [33] *if there is a distribution $\mathcal{D} \subset (\ker \Psi_*)$ such that*

$$\ker \Psi_* = \mathcal{D} \oplus \mathcal{D}_1, P(\mathcal{D}) = \mathcal{D},$$

and for $q \in M_q$ and $Y_1 \in (\mathcal{D}_1)_q$, the angle $\vartheta = \vartheta(Y_1)$ between PY_1 and the space $(\mathcal{D}_1)_q$ is independent of the choice of the nonzero vector Y_1, where $\mathcal{D}_1$ is the orthogonal complement of $\mathcal{D}$ in $\ker \Psi_*$. The angle ϑ is called pointwise semi-slant function of the slant submersion.

Definition 2.4. [22] *Let (M_q, F, g_{M_q}) be an $AQHM$ and (M_r, g_{M_r}) a RM. A Riemannian submersion $\Psi : (M_q, F, g_{M_q}) \to (M_r, g_{M_r})$ is called a h-v-semi-slant submersion if given a point $p \in M_q$, with a neighborhood V, there exists a quaternionic Hermitian basis $\{I, P, L\}$ of* *sections of F on V such that for any $J \in \{I, P, L\}$, there is a distribution $D_1 \subset (\ker \Psi_*)^\perp$ on V such that*

$$(\ker \Psi_*)^\perp = D_1 \oplus D_2, J(D_1) = D_1$$

and the angle $\vartheta_J = \vartheta_J(X)$ between JX and the space D_2 is constant for nonzero $X \in (D_2)_p$ and $p \in V$ where D_2 is the orthogonal complement of D_1 in $(\ker \Psi_*)^\perp$. We call such a basis $\{I, P, L\}$ a h-v-semi-slant basis and the angles $\{\vartheta_I, \vartheta_P, \vartheta_L\}$ h-v-semi-slant angles.

Moreover, the map $\Psi : (M_q, F, g_{M_q}) \to (M_r, g_{M_r})$ is called a strictly h-v-semi-slant submersion, the angle ϑ a strictly h-v-semi-slant angle and $\{I, P, L\}$ a strictly h-v-semi-slant basis if $\vartheta = \vartheta_I = \vartheta_P = \vartheta_L$.

Definition 2.5. *Let (M_q, F, g_{M_q}) be an AQHM and (M_r, g_{M_r}) a RM. A Riemannian submersion $\Psi : (M_q, F, g_{M_q}) \to (M_r, g_{M_r})$ is called an almost h-v-semi-slant submersion if given a point $p \in M_q$ with a neighborhood V, there exists a quaternionic Hermitian basis $\{I, P, L\}$ of sections of F on V such that for each $J \in \{I, P, L\}$, there is a distribution $D_1^J \subset (\ker \Psi_*)^\perp$ on V such that*

$$(\ker \Psi_*)^\perp = D_1^J \oplus D_2^J, J(D_1^J) = D_1^J$$

and the angle $\vartheta_J = \vartheta_J(X)$ between and the space is constant for nonzero and, where D_2^J is the orthogonal complement of D_1^J in $(\ker \Psi_*)^\perp$. Where the angles $\{\vartheta_I, \vartheta_P, \vartheta_L\}$ is almost h-v-semi-slant angles and a basis $\{I, P, L\}$ is an almost h-v-semi-slant basis.

3. *phv*− SEMI-SLANT SUBMERSIONS

Definition 3.1. *A Riemannian submersion $\Psi : (M_q, F, g_{M_q}) \to (M_r, g_{M_r})$ is called a phv− semi-slant submersion if given a point $q \in M_q$ with a neighborhood V, there exists a quaternionic Hermitian basis $\{I, P, L\}$ of sections of F on V so that for any $J \in \{I, P, L\}$, there is a distribution $D_1 \subset (\ker \Psi_*)^\perp$ on V such that*

$$(\ker \Psi_*)^\perp = D_1 \oplus D_2, J(D_1) = D_1$$

and the angle $\vartheta_J = \vartheta_J(X)$ between JX and the space $(D_2)_p$ is independent of the choice of nonzero vector $X \in (D_2)_p$ and $p \in V$, where D_2 is the orthogonal complementary distribution of D_1 in $(\ker \Psi_*)^\perp$.

The angles $\{\vartheta_I, \vartheta_P, \vartheta_L\}$ are called *phv*− semi-slant functions and $\{I, P, L\}$ is called a *phv*− semi-slant basis. Moreover, if

$$\vartheta_I = \vartheta_P = \vartheta_L = \vartheta,$$

then $\Psi : (M_q, F, g_{M_q}) \to (M_r, g_{M_r})$ is called a strictly *phv*− semi-slant submersion, the function ϑ is called a strictly *phv*− semi-slant function, $\{I, P, L\}$ is called a strictly *phv*− semi-slant basis.

Definition 3.2. *Let (M_q, F, g_{M_q}) be an AQHM and (M_r, g_{M_r}) a RM. A Riemannian submersion $\Psi : (M_q, F, g_{M_q}) \to (M_r, g_{M_r})$ is called an almost phv− semi-slant submersion if given a point $p \in M_q$ with a neighborhood V, there exists a quaternionic Hermitian basis $\{I, P, L\}$ of sections of F on V such that for any $J \in \{I, P, L\}$, there is a distribution $D^J \subset (\ker \Psi_*)^\perp$ on V such that*

$$(\ker \Psi_*)^\perp = D_1^J \oplus D_2^J, J(D_1^J) = D_1^J,$$

and the angle $\vartheta_J = \vartheta_J(X)$ *between* JX *and the space* $(D_2^J)_p$ *is is independent of the choice of the nonzero vector* $X \in (D_2^J)_p$ *and* $p \in V$*, where* D_2^J *is the orthogonal complement of* D_2^J *in* $(\ker \Psi_*)^\perp$.

We call such a basis $\{I, P, L\}$ an almost $phv-$ semi-slant basis and the angles $\{\vartheta_I, \vartheta_P, \vartheta_L\}$ almost $phv-$ semi-slant function.
Let Ψ be a $phv-$ semi-slant submersions from an $AQHM$ (M_q, I, P, L, g_{M_q}) onto a RM (M_r, g_{M_r}). Then, we have

$$TM_q = \ker \Psi_* \oplus (\ker \Psi_*)^\perp. \tag{3.1}$$

Now, for any vector field $U_1 \in \Gamma(\ker \Psi_*)^\perp$, we put

$$U_1 = W_J U_1 + Q_J U_1, \tag{3.2}$$

where W_J and Q_J are projection morphisms of $\Gamma(\ker \Psi_*)^\perp$ onto D_1^J and D_2^J, respectively.
For $X_1 \in (\Gamma \ker \Psi_*)$, we set

$$JX_1 = \phi_J X_1 + \omega_J X_1, \tag{3.3}$$

where $\phi_J X_1 \in (\Gamma \ker \Psi_*)$ and $\omega_J X_1 \in (\Gamma \ker \Psi_*)^\perp$.
Also for any non-zero vector field $Y_1 \in \Gamma(\ker \Psi_*)^\perp$, we have

$$JY_1 = B_J Y_1 + C_J Y_1, \tag{3.4}$$

where $B_J Y_1 \in \Gamma(\ker \Psi_*)$ and $C_J Y_1 \in \Gamma(\ker \Psi_*)^\perp$.

Lemma 3.1. *Let* Ψ *be an almost phv− semi-slant submersion from an* $AQHM$ (M_q, I, P, L, g_{M_q}) *onto a* RM (M_r, g_{M_r}) *such that* (I, P, L) *is an almost phv− semi-slant basis. Then, we have*

$$\phi_J^2 X_1 + B_J \omega_J X_1 = -X_1, \omega_J \phi_J X_1 + C_J \omega_J X_1 = 0,$$

$$\omega_J B_J X_2 + C_J^2 X_2 = -X_2, \phi_J B_J X_2 + B_J C_J X_2 = 0,$$

for all $X_1 \in \Gamma(\ker \Psi_*)$ *and* $Y_2 \in \Gamma(\ker \Psi_*)^\perp$ *and* $J \in \{I, P, L\}$.

Proof. Using equations (2.9), (3.3) and (3.4) , we have Lemma (3.1). □

The proof of the following Lemma is the same as Lemma in [22] therefore, we omit its proof.

Lemma 3.2. *Let* Ψ *be an almost phv− semi-slant submersion from an* $AQHM$ (M_q, F, g_{M_q}) *onto a* RM (M_r, g_{M_r}) *such that* (I, P, L) *is an almost phv− semi-slant basis. Then, we have*
(i) $C_J^2 V = -(\cos^2 \vartheta_J) V$,
(ii) $g_{M_q}(C_J V, C_J Z) = \cos^2 \vartheta_J \, g_{M_q}(V, Z)$,
(iii) $g_{M_q}(B_J V, B_J Z) = \sin^2 \vartheta_J \, g_{M_q}(V, Z)$,
for all $V, Z \in \Gamma(D_2^J)$*, where* $J \in \{I, , P, L\}$.

Lemma 3.3. *Let* Ψ *be a phv− semi-slant submersion from a* HKM (M_q, I, P, L, g_{M_q}) *onto a* RM (M_r, g_{M_r}) *such that* (I, P, L) *is an almost phv− semi-slant basis. Then, we have*

$$\mathcal{V}\nabla_{Y_1} \phi_J Y_2 + \mathcal{T}_{Y_1} \omega_J Y_2 = \phi_J \mathcal{V}\nabla_{Y_1} Y_2 + B_J \mathcal{T}_{Y_1} Y_2, \tag{3.5}$$

$$\mathcal{T}_{Y_1} \phi_J Y_2 + \mathcal{H}\nabla_{Y_1} \omega_J Y_2 = \omega_J \mathcal{V}\nabla_{Y_1} Y_2 + C_J \mathcal{T}_{Y_1} Y_2, \tag{3.6}$$

$$\mathcal{V}\nabla_{X_1} B_J X_2 + \mathcal{A}_{X_1} C_J X_2 = \phi_J \mathcal{A}_{X_1} X_2 + B_J \mathcal{H}\nabla_{X_1} X_2, \tag{3.7}$$

$$\mathcal{A}_{X_1}B_JX_2 + \mathcal{H}\nabla_{X_1}C_JX_2 = \omega_J\mathcal{A}_{X_1}X_2 + C_J\mathcal{H}\nabla_{X_1}X_2, \tag{3.8}$$
$$\mathcal{V}\nabla_{Y_1}B_JX_1 + \mathcal{T}_{Y_1}C_JX_1 = \phi_J\mathcal{T}_{Y_1}X_1 + B_J\mathcal{H}\nabla_{Y_1}X_1, \tag{3.9}$$
$$\mathcal{T}_{Y_1}B_JX_1 + \mathcal{H}\nabla_{Y_1}C_JX_1 = \omega_J\mathcal{T}_{Y_1}X_1 + C_J\mathcal{H}\nabla_{Y_1}X_1 \tag{3.10}$$
$$\mathcal{V}\nabla_{X_1}\phi_JY_1 + \mathcal{A}_{X_1}\omega_JY_1 = B_J\mathcal{A}_{X_1}Y_1 + \phi_J\mathcal{V}\nabla_{X_1}Y_1, \tag{3.11}$$
$$\mathcal{A}_{X_1}\phi_JY_1 + \mathcal{H}\nabla_{X_1}\omega_JY_1 = C_J\mathcal{A}_{X_1}Y_1 + \omega_J\mathcal{V}\nabla_{X_1}Y_1 \tag{3.12}$$

for any $Y_1, Y_2 \in \Gamma(\ker \Psi_*)$ *and* $X_1, X_2 \in \Gamma(\ker \Psi_*)^{\perp}$.

Proof. Using equations (2.3)−(2.6), (3.3) and (3.4), we get equations (3.5)−(3.12). □

Now, we define

$$(\nabla_{U_1}\phi_J)U_2 = \mathcal{V}\nabla_{U_1}\phi_JU_2 - \phi_J\mathcal{V}\nabla_{U_1}U_2, \tag{3.13}$$
$$(\nabla_{U_1}\omega_J)U_2 = \mathcal{H}\nabla_{U_1}\omega_JU_2 - \omega_J\mathcal{V}\nabla_{U_1}U_2, \tag{3.14}$$
$$(\nabla_{X_1}C_J)X_2 = \mathcal{H}\nabla_{X_1}C_JX_2 - C_J\mathcal{H}\nabla_{X_1}X_2, \tag{3.15}$$
$$(\nabla_{X_1}B_J)X_2 = \mathcal{V}\nabla_{X_1}B_JX_2 - B_J\mathcal{H}\nabla_{X_1}X_2, \tag{3.16}$$

for any $U_1, U_2 \in \Gamma(\ker \Psi_*)$ and $X_1, X_2 \in \Gamma(\ker \Psi_*)^{\perp}$.

Lemma 3.4. *Let* Ψ *be an almost phv− semi-slant submersion from a HKM* (M_q, I, P, L, g_{M_q}) *onto a RM* (M_r, g_{M_r}) *such that* (I, P, L) *is an almost phv− semi-slant basis. Then, we have*

$$(\nabla_{X_1}\phi_J)X_2 = B_J\mathcal{T}_{X_1}X_2 - \mathcal{T}_{X_1}\omega_JX_2,$$
$$(\nabla_{X_1}\omega_J)X_2 = C_J\mathcal{T}_{X_1}X_2 - \mathcal{T}_{X_1}\phi_JX_2,$$
$$(\nabla_{Y_1}C_J)Y_2 = \omega_J\mathcal{A}_{Y_1}Y_2 - \mathcal{A}_{Y_1}B_JY_2,$$
$$(\nabla_{Y_1}B_J)Y_2 = \phi_J\mathcal{A}_{Y_1}Y_2 - \mathcal{A}_{Y_1}C_JY_2$$

for any vectors $X_1, X_2 \in \Gamma(\ker \Psi_*)$ *and* $Y_1, Y_2 \in \Gamma(\ker \Psi_*)^{\perp}$.

Proof. Using equations (3.5)−(3.8) and (3.13)−(3.16) we get all equations of Lemma (3.4). □

Theorem 3.1. *Let* Ψ *be a phv− semi-slant submersion from a HKM* (M_q, I, P, L, g_{M_q}) *onto a RM* (M_r, g_{M_r}) *such that* (I, P, L) *is a phv− semi-slant basis. Then, it can be said that the following conditions are valid:*

(a) D_1^I *is integrable.*

(b)

$$C_I(\mathcal{H}\nabla_{Y_2}IY_1 - \mathcal{H}\nabla_{Y_1}IY_2) = \omega_I(A_{Y_1}IY_2 - A_{Y_2}IY_1),$$

for $Y_1, Y_2 \in \Gamma(D_1^I)$.

(c)

$$C_P(\mathcal{H}\nabla_{Y_2}PY_1 - \mathcal{H}\nabla_{Y_1}PY_2) = \omega_P(\mathcal{A}_{Y_1}PY_2 - \mathcal{A}_{Y_2}PY_1),$$

for $Y_1, Y_2 \in \Gamma(D_1^P)$.

(d)

$$C_L(\mathcal{H}\nabla_{Y_2}LY_1 - \mathcal{H}\nabla_{Y_1}LY_2) = \omega_L(\mathcal{A}_{Y_1}LY_2 - \mathcal{A}_{Y_2}LY_1),$$

for $Y_1, Y_2 \in \Gamma(D_1^L)$.

Proof. For $Y_1, Y_2 \in \Gamma(D_1^J), Z_1 \in \Gamma(D_2^J)$ and $J \in \{I, P, L\}$. Using equations (2.6), (2.10), (3.3) and (3.4), we have

$$\begin{aligned} & g_{M_q}([Y_1, Y_2], Z_1) \\ = \;& g_{M_q}(\nabla_{Y_1} JY_2, JZ_1) - g_{M_q}(\nabla_{Y_2} JY_1, JZ_1), \\ = \;& g_{M_q}(\mathcal{A}_{Y_1} JY_2 - \mathcal{A}_{Y_2} JY_1, JZ_1) - g_{M_q}(\mathcal{H}\nabla_{Y_2} JY_1 - \mathcal{H}\nabla_{Y_1} JY_2, JZ_1), \\ = \;& -g_{M_q}(\omega_J(\mathcal{A}_{Y_1} JY_2 - \mathcal{A}_{Y_2} JY_1), Z_1) + g_{M_q}(C_J(\mathcal{H}\nabla_{Y_2} JY_1 - \mathcal{H}\nabla_{Y_1} JY_2), Z_1) \end{aligned}$$

which completes the proof. □

Theorem 3.2. *Let Ψ be a phv− semi-slant submersion from a HKM (M_q, I, P, L, g_{M_q}) onto a RM (M_r, g_{M_r}) such that (I, P, L) is a phv− semi-slant basis. Then, it can be said that the following conditions are valid:*
(a)slant distribution D_2^J is integrable.
(b)

$$\begin{aligned} & g_{M_q}(\mathcal{A}_{Z_1} B_I Z_2 - \mathcal{A}_{Z_2} B_I Z_1, IU_1) \\ = \;& g_{M_q}(\mathcal{A}_{Z_1} B_I C_I Z_2 - \mathcal{A}_{Z_2} B_I C_I Z_1, U_1), \end{aligned}$$

for all $Z_1, Z_2 \in \Gamma(D_1^I)$ and $U_1 \in \Gamma(D_2^I)$.
(c)

$$\begin{aligned} & g_{M_q}(\mathcal{A}_{Z_1} B_P Z_2 - \mathcal{A}_{Z_2} B_P Z_1, PU_1) \\ = \;& g_{M_q}(\mathcal{A}_{Z_1} B_P C_P Z_2 - \mathcal{A}_{Z_2} B_P C_P Z_1, U_1), \end{aligned}$$

for all $Z_1, Z_2 \in \Gamma(D_1^P)$ and $U_1 \in \Gamma(D_2^P)$.
(d)

$$\begin{aligned} & g_{M_q}(\mathcal{A}_{Z_1} B_L Z_2 - \mathcal{A}_{Z_2} B_L Z_1, LU_1) \\ = \;& g_{M_q}(\mathcal{A}_{Z_1} B_L C_L Z_2 - \mathcal{A}_{Z_2} B_L C_L Z_1, U_1), \end{aligned}$$

for all $Z_1, Z_2 \in \Gamma(D_1^L)$ and $U_1 \in \Gamma(D_2^L)$.

Proof. For $Z_1, Z_2 \in \Gamma(D_2^J)$ and $U_1 \in \Gamma D_1^J)$, we have

$$g_{M_q}([Z_1, Z_2], U_1) = g_{M_q}(\nabla_{X_1} X_2, U_1) - g_{M_q}(\nabla_{Y_2} Y_1, U_1).$$

Using equations (2.5), (2.6), (2.10), (3.4) and Lemma 3.2, we get

$$\begin{aligned} & g_{M_q}([Z_1, Z_2], U_1) \\ = \;& g_{M_q}(\nabla_{Z_1} JZ_2, JU_1) - g_{M_q}(\nabla_{Z_2} JZ_1, JU_1), \\ = \;& g_{M_q}(\nabla_{Z_1} B_J Z_2, JU_1) + g_{M_q}(\nabla_{Z_1} C_J Z_2, JU_1) - g_{M_q}(\nabla_{Z_2} B_J Z_1, JU_1) - \\ & g_{M_q}(\nabla_{Z_2} C_J Z_1, JU_1), \\ = \;& g_{M_q}(\mathcal{A}_{Z_1} B_J Z_2 + \mathcal{V}\nabla_{Z_1} B_J Z_2, JU_1) + \cos^2\vartheta_J \, g_{M_q}(\nabla_{Z_1} Z_2, U_1) - \\ & 2\sin\vartheta_J \cos\vartheta_J \, Z_1(\vartheta_J) \, g_{M_q}(Z_2, U_1) - g_{M_q}(\nabla_{Z_1} B_J C_J Z_2, U_1) - \\ & g_{M_q}(\mathcal{A}_{Z_2} B_J Z_1 + \mathcal{V}\nabla_{Z_2} B_J Z_1, JU_1) - \cos^2\vartheta_J \, g_{M_q}(\nabla_{Z_2} Z_1, U_1) + \\ & 2\sin\vartheta_J \cos\vartheta_J \, Z_2(\vartheta_J) \, g_{M_q}(Z_1, U_1) + g_{M_q}(\nabla_{Z_2} B_J C_J Z_1, U_1). \end{aligned}$$

Now, we have

$$\begin{aligned} & \sin^2\vartheta_J\, g_{M_q}([Z_1,Z_2],U_1) \\ = \; & g_{M_q}(\mathcal{A}_{Z_1}B_JZ_2-\mathcal{A}_{Z_2}B_JZ_1,JU_1)- \\ & g_{M_q}(\mathcal{A}_{Z_1}B_JC_JZ_2-\mathcal{A}_{Z_2}B_JC_JZ_1,U_1), \end{aligned}$$

which completes the proof. □

Theorem 3.3. *Let Ψ be a phv− semi-slant from a HKM (M_q,I,P,L,g_{M_q}) onto a RM (M_r,g_{M_r}) such that (I,P,L) is a phv− semi-slant basis. Then, it can be said that the following conditions are valid:*
(a) the horizontal distribution $(\ker\Psi_*)^\perp$ defines a totally geodesic.
(b)

$$\begin{aligned} & \sin^2\vartheta_I\, g_{M_q}([Y_1,Z_1],Y_2)-\cos^2\vartheta_I\, g_{M_q}(\mathcal{H}\nabla_{Z_1}W_IY_1,Y_2) \\ = \; & -g_{M_q}(\mathcal{H}\nabla_{Z_1}IW_IY_1,Y_2)-g_{M_q}(\mathcal{T}_{Z_1}IW_IY_1,Y_2)- \\ & g_{M_q}(\mathcal{V}\nabla_{Z_1}B_IQ_IY_1,B_IY_2)-g_{M_q}(\mathcal{T}_{Z_1}B_IQ_IY_1,C_IY_2)+ \\ & g_{M_q}(\mathcal{T}_{U_1}B_IC_IQ_IY_1,Y_2)+\sin 2\vartheta_I\; Z_1[\vartheta_I]g_{M_q}(Q_IY_1,Q_IY_2). \end{aligned}$$

for $Y_1,Y_2\in\Gamma(\ker\Psi_*)^\perp$ and $Z_1\in\Gamma(\ker\Psi_*)$.
(c)

$$\begin{aligned} & \sin^2\vartheta_P\, g_{M_q}([Y_1,Z_1],Y_2)-\cos^2\vartheta_P\, g_{M_q}(\mathcal{H}\nabla_{Z_1}W_PY_1,Y_2) \\ = \; & -g_{M_q}(\mathcal{H}\nabla_{Z_1}PW_PY_1,Y_2)-g_{M_q}(\mathcal{T}_{Z_1}PW_PY_1,Y_2)- \\ & g_{M_q}(\mathcal{V}\nabla_{Z_1}B_PQ_PY_1,B_PY_2)-g_{M_q}(\mathcal{T}_{Z_1}B_PQ_PY_1,C_PY_2)+ \\ & g_{M_q}(\mathcal{T}_{U_1}B_PC_PQ_PY_1,Y_2)+\sin 2\vartheta_P\; Z_1[\vartheta_P]g_{M_q}(Q_PY_1,Q_PY_2). \end{aligned}$$

for $Y_1,Y_2\in\Gamma(\ker\Psi_*)^\perp$ and $Z_1\in\Gamma(\ker\Psi_*)$.
(d)

$$\begin{aligned} & \sin^2\vartheta_L\, g_{M_q}([Y_1,Z_1],Y_2)-\cos^2\vartheta_L\, g_{M_q}(\mathcal{H}\nabla_{Z_1}W_LY_1,Y_2) \\ = \; & -g_{M_q}(\mathcal{H}\nabla_{Z_1}LW_LY_1,Y_2)-g_{M_q}(\mathcal{T}_{Z_1}LW_LY_1,Y_2)- \\ & g_{M_q}(\mathcal{V}\nabla_{Z_1}B_LQ_LY_1,B_LY_2)-g_{M_q}(\mathcal{T}_{Z_1}B_LQ_LY_1,C_LY_2)+ \\ & g_{M_q}(\mathcal{T}_{U_1}B_LC_LQ_LY_1,Y_2)+\sin 2\vartheta_L\; Z_1[\vartheta_L]g_{M_q}(Q_LY_1,Q_LY_2). \end{aligned}$$

for $Y_1,Y_2\in\Gamma(\ker\Psi_*)^\perp$ and $Z_1\in\Gamma(\ker\Psi_*)$.

Proof. For $Y_1,Y_2\in\Gamma(\ker\Psi_*)^\perp$ and $Z_1\in\Gamma(\ker\Psi_*)$, using equations (2.3), (2.4), (2.10), (3.2), (3.4) and Lemma 3.2, we get

$$\begin{aligned} g_{M_q}(\nabla_{Y_1}Y_2,Z_1) &= -g_{M_q}([Y_1,Z_1],Y_2)-g_{M_q}(\nabla_{Z_1}Y_1,Y_2), \\ &= -g_{M_q}([Y_1,Z_1],Y_2)-g_{M_q}(\nabla_{Z_1}JW_JY_1,JY_2)- \\ & \quad g_{M_q}(\nabla_{Z_1}B_JQ_JY_1,JY_2)+g_{M_q}(\nabla_{Z_1}B_JC_JQ_JY_1,Y_2)- \\ & \quad \cos^2\vartheta_J\, g_{M_q}(\nabla_{Z_1}Q_JY_1,Y_2)+\sin 2\vartheta_J\; Z_1[\vartheta_J]g_{M_q}(Q_JY_1,Y_2). \end{aligned}$$

Now, we obtain

$$\begin{aligned} & \sin^2\vartheta_J\, g_{M_q}(\nabla_{X_1}X_2, U_1) \\ = & -\sin^2\vartheta_J\, g_{M_q}([Y_1, Z_1], Y_2) + \cos^2\vartheta_J\, g_{M_q}(\mathcal{H}\nabla_{Z_1}W_JY_1, Y_2) - \\ & -g_{M_q}(\mathcal{H}\nabla_{Z_1}JW_JY_1, Y_2) - g_{M_q}(\mathcal{T}_{Z_1}JW_JY_1, Y_2) - \\ & -g_{M_q}(\mathcal{V}\nabla_{Z_1}B_JQ_JY_1, B_JY_2) - g_{M_q}(\mathcal{T}_{Z_1}B_JQ_JY_1, C_JY_2) + \\ & -g_{M_q}(\mathcal{T}_{U_1}B_JC_JQ_JY_1, Y_2) + \sin 2\vartheta_J\, Z_1[\vartheta_J]g_{M_q}(Q_JY_1, Q_JY_2). \end{aligned}$$

□

Theorem 3.4. *Let Ψ be a phv− semi-slant from a HKM (M_q, I, P, L, g_{M_q}) onto a RM (M_r, g_{M_r}) such that (I, P, L) is a phv− semi-slant basis. Then, it can be said that the following conditions are valid:*
(a) the vertical distribution $(\ker\Psi_*)$ defines a totally geodesic.
(b)

$$\begin{aligned} & g_{M_q}(\mathcal{V}\nabla_{U_1}U_2, B_IC_IW_1) \\ = & g_{M_q}(\mathcal{V}\nabla_{U_1}\phi_IU_2, B_IW_1) + g_{M_q}(\mathcal{T}_{U_1}\omega_IU_2, B_IW_1), \end{aligned}$$

for $U_1, U_2 \in \Gamma(\ker\Psi_*)$ and $W_1 \in \Gamma(\ker\Psi_*)^\perp$.
(c)

$$\begin{aligned} & g_{M_q}(\mathcal{V}\nabla_{U_1}U_2, B_PC_PW_1) \\ = & g_{M_q}(\mathcal{V}\nabla_{U_1}\phi_PU_2, B_PW_1) + g_{M_q}(\mathcal{T}_{U_1}\omega_PU_2, B_PW_1), \end{aligned}$$

for $U_1, U_2 \in \Gamma(\ker\Psi_*)$ and $W_1 \in \Gamma(\ker\Psi_*)^\perp$.
(d)

$$\begin{aligned} & g_{M_q}(\mathcal{V}\nabla_{U_1}U_2, B_LC_LW_1) \\ = & g_{M_q}(\mathcal{V}\nabla_{U_1}\phi_LU_2, B_LW_1) + g_{M_q}(\mathcal{T}_{U_1}\omega_LU_2, B_LW_1), \end{aligned}$$

for $U_1, U_2 \in \Gamma(\ker\Psi_*)$ and $W_1 \in \Gamma(\ker\Psi_*)^\perp$.

Proof. For $U_1, U_2 \in \Gamma(\ker\Psi_*)$ and $W_1 \in \Gamma(\ker\Psi_*)^\perp$, using equations (2.3), (2.4), (2.10), (3.2) and (3.4), we have

$$\begin{aligned} & g_{M_q}(\nabla_{U_1}U_2, W_1) \\ = & g_{M_q}(\nabla_{U_1}JU_2, JW_1), \\ = & g_{M_q}(\mathcal{V}\nabla_{U_1}\phi_JU_2, B_JW_1) + g_{M_q}(\mathcal{T}_{U_1}\omega_JU_2, B_JW_1) + \\ & \cos^2\vartheta_J\, g_{M_q}(\nabla_{U_1}U_2, W_1) - g_{M_q}(\mathcal{V}\nabla_{U_1}U_2, B_JC_JW_1). \end{aligned}$$

Now, we get

$$\begin{aligned} & \sin^2\vartheta_J\, g_{M_q}(\nabla_{U_1}U_2, W_1) \\ = & g_{M_q}(\mathcal{V}\nabla_{U_1}\phi_JU_2, B_JW_1) + g_{M_q}(\mathcal{T}_{U_1}\omega_JU_2, B_JW_1) \\ & -g_{M_q}(\mathcal{V}\nabla_{U_1}U_2, B_JC_JW_1). \end{aligned}$$

□

Theorem 3.5. *Let Ψ be a phv− semi-slant submersion from a HKM (M_q, I, P, L, g_{M_q}) onto a RM (M_r, g_{M_r}) such that (I, P, L) is a phv− semi-slant basis. Then, it can be said that the following conditions are valid:*
(a) *the invariant distribution D_1^J defines a totally geodesic.*
(b)

$$\begin{aligned} g_{M_q}(\mathcal{A}_{X_1} IX_2, B_I Y_1) &= g_{M_q}(\mathcal{A}_{X_1} X_2, B_I C_I Y_1), \\ g_{M_q}(\mathcal{A}_{X_1} IX_2, \phi_I Y_2) &= -g_{M_q}(\mathcal{H}\nabla_{X_1} IX_2, \omega_I Y_2), \end{aligned}$$

for $X_1, X_2 \in \Gamma(D_1^I), Y_1 \in \Gamma(D_2^I)$ and $Y_2 \in \Gamma(\ker \Psi_*)$.
(c)

$$\begin{aligned} g_{M_q}(\mathcal{A}_{X_1} PX_2, B_P Y_1) &= g_{M_q}(\mathcal{A}_{X_1} X_2, B_P C_P Y_1), \\ g_{M_q}(\mathcal{A}_{X_1} PX_2, \phi_P Y_2) &= -g_{M_q}(\mathcal{H}\nabla_{X_1} PX_2, \omega_P Y_2), \end{aligned}$$

for $X_1, X_2 \in \Gamma(D_1^P), Y_1 \in \Gamma(D_2^P)$ and $Y_2 \in \Gamma(\ker \Psi_*)$.
(d)

$$\begin{aligned} g_{M_q}(\mathcal{A}_{X_1} LX_2, B_L Y_1) &= g_{M_q}(\mathcal{A}_{X_1} X_2, B_L C_L Y_1), \\ g_{M_q}(\mathcal{A}_{X_1} LX_2, \phi_L Y_2) &= -g_{M_q}(\mathcal{H}\nabla_{X_1} LX_2, \omega_L Y_2), \end{aligned}$$

for $X_1, X_2 \in \Gamma(D_1^L), Y_1 \in \Gamma(D_2^L)$ and $Y_2 \in \Gamma(\ker \Psi_*)$.

Proof. For $X_1, X_2 \in \Gamma(D_1^J), Y_1 \in \Gamma(D_2^J)$ and $Y_2 \in \Gamma(\ker \Psi_*)$. Using equations (2.6), (2.10), (3.4) and Lemma 3.2, we have

$$\begin{aligned} & g_{M_q}(\nabla_{X_1} X_2, U_1) \\ = \ & g_{M_q}(\nabla_{X_1} JX_2, B_J Y_1) - g_{M_q}(\nabla_{X_1} X_2, C_J^2 Y_1) - g_{M_q}(\nabla_{X_1} X_2, B_J C_J Y_1), \\ = \ & g_{M_q}(\mathcal{A}_{X_1} JX_2, B_J Y_1) + \cos^2 \vartheta_J \, g_{M_q}(\nabla_{X_1} X_2, Y_1) - g_{M_q}(\mathcal{A}_{X_1} X_2, B_J C_J Y_1). \end{aligned}$$

Now, we get

$$\begin{aligned} & \sin^2 \vartheta_J \, g_{M_q}(\nabla_{X_1} X_2, Y_1) \\ = \ & g_{M_q}(\mathcal{A}_{X_1} JX_2, B_J Y_1) - g_{M_q}(\mathcal{A}_{X_1} X_2, B_J C_J Y_1). \end{aligned}$$

Now, again using equations (2.6), (2.10) and (3.3), we have

$$\begin{aligned} g_{M_q}(\nabla_{X_1} X_2, Y_2) &= g_{M_q}(\nabla_{X_1} JX_2, JY_2), \\ &= g_{M_q}(\nabla_{X_1} JX_2, \phi_J Y_2 + \omega_J Y_2), \\ &= g_{M_q}(\mathcal{A}_{X_1} JX_2, \phi_J Y_2) + g_{M_q}(\mathcal{H}\nabla_{X_1} JX_2, \omega_J Y_2), \end{aligned}$$

which completes the proof. □

Theorem 3.6. *Let Ψ be a phv− semi-slant submersion from a HKM (M_q, I, P, L, g_{M_q}) onto a RM (M_r, g_{M_r}) such that (I, P, L) is a phv− semi-slant basis. Then, it can be said that the following conditions are valid:*
(a) *the slant distribution D_2^J defines a totally geodesic.*
(b)

$$g_{M_q}(\mathcal{A}_{X_1} B_I X_2, IY_1) = g_{M_q}(\mathcal{A}_{X_1} B_I C_I X_2, Y_1),$$

$$\begin{aligned}
&\sin^2\vartheta_I\, g_{M_q}([X_1,Y_2],X_2)\\
=\;& -g_{M_q}(\mathcal{T}_{Y_2}B_IX_1,C_IX_2)-g_{M_q}(\mathcal{V}\nabla_{Y_2}B_IX_1,B_IX_2)+\\
&\sin 2\vartheta_I\; Y_2[\vartheta_I]\; g_{M_q}(X_1,X_2)+g_{M_q}(\mathcal{T}_{Y_2}B_IC_IX_1,X_2),
\end{aligned}$$

for $X_1, X_2 \in \Gamma(D_1^I), Y_1 \in \Gamma(D_2^I)$ *and* $Y_2 \in \Gamma(\ker\Psi_*)^\perp$.

(c)

$$g_{M_q}(\mathcal{A}_{X_1}B_PX_2,PY_1)=g_{M_q}(\mathcal{A}_{X_1}B_PC_PX_2,Y_1),$$

$$\begin{aligned}
&\sin^2\vartheta_P\, g_{M_q}([X_1,Y_2],X_2)\\
=\;& -g_{M_q}(\mathcal{T}_{Y_2}B_PX_1,C_PX_2)-g_{M_q}(\mathcal{V}\nabla_{Y_2}B_PX_1,B_PX_2)+\\
&\sin 2\vartheta_P\; Y_2[\vartheta_P]\; g_{M_q}(X_1,X_2)+g_{M_q}(\mathcal{T}_{Y_2}B_PC_PX_1,X_2),
\end{aligned}$$

for $X_1, X_2 \in \Gamma(D_1^P), Y_1 \in \Gamma(D_2^P)$ *and* $Y_2 \in \Gamma(\ker\Psi_*)^\perp$.

(d)

$$g_{M_q}(\mathcal{A}_{X_1}B_LX_2,LY_1)=g_{M_q}(\mathcal{A}_{X_1}B_LC_LX_2,Y_1),$$

$$\begin{aligned}
&\sin^2\vartheta_L\, g_{M_q}([X_1,Y_2],X_2)\\
=\;& -g_{M_q}(\mathcal{T}_{Y_2}B_LX_1,C_LX_2)-g_{M_q}(\mathcal{V}\nabla_{Y_2}B_LX_1,B_LX_2)+\\
&\sin 2\vartheta_L\; Y_2[\vartheta_L]\; g_{M_q}(X_1,X_2)+g_{M_q}(\mathcal{T}_{Y_2}B_LC_LX_1,X_2),
\end{aligned}$$

for $X_1, X_2 \in \Gamma(D_1^L), Y_1 \in \Gamma(D_2^L)$ *and* $Y_2 \in \Gamma(\ker\Psi_*)^\perp$.

Proof. For $X_1, X_2 \in \Gamma(D_1^J), Y_1 \in \Gamma(D_2^J)$ and $Y_2 \in \Gamma(\ker\Psi_*)^\perp$. Using equations (2.5), (2.10), (3.4) and Lemma 3.2, we have

$$\begin{aligned}
&g_{M_q}(\nabla_{X_1}X_2,Y_1)\\
=\;& g_{M_q}(\nabla_{X_1}JX_2,JY_1),\\
=\;& g_{M_q}(\nabla_{X_1}B_JX_2,JY_1)+\cos^2\vartheta_J\, g_{M_q}(\nabla_{X_1}X_2,Y_1)-g_{M_q}(\nabla_{X_1}B_JC_JX_2,Y_1).
\end{aligned}$$

Now, we have

$$\begin{aligned}
&\sin^2\vartheta_J\, g_{M_q}(\nabla_{X_1}X_2,Y_1)\\
=\;& g_{M_q}(\mathcal{A}_{X_1}B_JX_2,JY_1)-g_{M_q}(\mathcal{A}_{X_1}B_JC_JX_2,Y_1).
\end{aligned}$$

Next, from equations (2.4), (2.10), (3.4) and Lemma 3.2, we have

$$\begin{aligned}
&g_{M_q}(\nabla_{X_1}X_2,Y_2)\\
=\;& -g_{M_q}([X_1,Y_2],X_2)-g_{M_q}(\nabla_{Y_2}X_1,X_2),\\
=\;& -g_{M_q}([X_1,Y_2],X_2)-g_{M_q}(\nabla_{Y_2}B_JX_1,JX_2)-\cos^2\vartheta_J\, g_{M_q}(\nabla_{Y_2}X_1,X_2)\\
&+\sin 2\vartheta_J\; Y_2[\vartheta_J]g_{M_q}(X_1,X_2)+g_{M_q}(\nabla_{Y_2}B_JC_JX_1,X_2).
\end{aligned}$$

Now, we have

$$\begin{aligned}
&\sin^2\vartheta_J\, g_{M_q}(\nabla_{X_1}X_2,Y_2)\\
=\;& -\sin^2\vartheta_J\, g_{M_q}([X_1,Y_2],X_2)-g_{M_q}(\mathcal{T}_{Y_2}B_JX_1,C_JX_2)-\\
&g_{M_q}(\mathcal{V}\nabla_{Y_2}B_JX_1,B_JX_2)+\sin 2\vartheta_J\; Y_2[\vartheta]g_{M_q}(X_1,X_2)+\\
&g_{M_q}(\mathcal{T}_{Y_2}B_JC_JX_1,X_2).
\end{aligned}$$

□

Theorem 3.7. *Let Ψ be a phv− semi-slant submersion from a HKM (M_q, I, P, L, g_{M_q}) onto a RM (M_r, g_{M_r}) such that (I, P, L) is a phv− semi-slant basis. Then, it can be said that the following conditions are valid:*
(a) *Ψ is a totally geodesic map.*
(b)

$$C_I\mathcal{T}_{U_1}\phi_I U_2 + \omega_I\mathcal{V}\nabla_{U_1}\phi_I U_2 + C_I\mathcal{H}\nabla_{U_1}\omega_I U_2 + \omega_I\mathcal{T}_{U_1}\omega_I U_2 = 0,$$

$$C_I\mathcal{H}\nabla_{U_1} IZ_1 + \omega_I\mathcal{T}_{U_1} IZ_1 = 0,$$

$$C_I\mathcal{T}_{U_1}B_I X_1 + \omega_I\mathcal{V}\nabla_{U_1}B_I X_1 + \mathcal{T}_{U_1}B_I C_I X_1 - \cos^2\vartheta_I\ \mathcal{H}\nabla_{U_1}X_1 + \sin 2\vartheta_I\ U_1[\vartheta_I]X_1 = 0,$$

for $U_1, U_2 \in \Gamma(\ker\Psi_*)$, $Z_1 \in \Gamma(D_1^I)$ and $X_1 \in \Gamma(D_2^I)$.
(c)

$$C_P\mathcal{T}_{U_1}\phi_P U_2 + \omega_P\mathcal{V}\nabla_{U_1}\phi_P U_2 + C_P\mathcal{H}\nabla_{U_1}\omega_P U_2 + \omega_P\mathcal{T}_{U_1}\omega_P U_2 = 0,$$

$$C_P\mathcal{H}\nabla_{U_1} PZ_1 + \omega_P\mathcal{T}_{U_1} PZ_1 = 0,$$

$$C_P\mathcal{T}_{U_1}B_P X_1 + \omega_P\mathcal{V}\nabla_{U_1}B_P X_1 + \mathcal{T}_{U_1}B_P C_P X_1 - \cos^2\vartheta_P\ \mathcal{H}\nabla_{U_1}X_1 + \sin 2\vartheta_P\ U_1[\vartheta_P]X_1 = 0,$$

for $U_1, U_2 \in \Gamma(\ker\Psi_*)$, $Z_1 \in \Gamma(D_1^P)$ and $X_1 \in \Gamma(D_2^P)$.
(d)

$$C_L\mathcal{T}_{U_1}\phi_L U_2 + \omega_L\mathcal{V}\nabla_{U_1}\phi_L U_2 + C_L\mathcal{H}\nabla_{U_1}\omega_L U_2 + \omega_L\mathcal{T}_{U_1}\omega_L U_2 = 0,$$

$$C_L\mathcal{H}\nabla_{U_1} LZ_1 + \omega_L\mathcal{T}_{U_1} LZ_1 = 0,$$

$$C_L\mathcal{T}_{U_1}B_L X_1 + \omega_L\mathcal{V}\nabla_{U_1}B_L X_1 + \mathcal{T}_{U_1}B_L C_L X_1 - \cos^2\vartheta_L\ \mathcal{H}\nabla_{U_1}X_1 + \sin 2\vartheta_L\ U_1[\vartheta_L]X_1 = 0,$$

for $U_1, U_2 \in \Gamma(\ker\Psi_*)$, $Z_1 \in \Gamma(D_1^L)$ and $X_1 \in \Gamma(D_2^L)$.

Proof. Since Ψ is a Riemannian submersion, we get

$$(\nabla\Psi_*)(W_1, W_2) = 0,$$

for $W_1, W_2 \in \Gamma(\ker\Psi_*)^\perp$.
For $U_1, U_2 \in \Gamma(\ker\Psi_*)$, using (2.3), (2.4), (2.7), (3.3) and (3.4), we get

$$\begin{aligned} & (\nabla\Psi_*)(U_1, U_2) \\ = \ & -\Psi_*(\nabla_{U_1}U_2), \\ = \ & \Psi_*(J\nabla_{U_1}JU_2), \\ = \ & \Psi_*(J\mathcal{T}_{U_1}\phi_J U_2 + J\mathcal{V}\nabla_{U_1}\phi_J U_2 + J\mathcal{H}\nabla_{U_1}\omega_J J_2 + J\mathcal{T}_{U_1}\omega_J U_2), \\ = \ & \Psi_*(B_J\mathcal{T}_{U_1}\phi_J U_2 + C_J\mathcal{T}_{U_1}\phi_J U_2 + \phi_J\mathcal{V}\nabla_{U_1}\phi_J U_2 + \omega_J\mathcal{V}\nabla_{U_1}\phi_J U_2 + \\ & B_J\mathcal{H}\nabla_{U_1}\omega_J U_2 + C_J\mathcal{H}\nabla_{U_1}\omega_J U_2 + \phi_J\mathcal{T}_{U_1}\omega_J U_2 + \omega_J\mathcal{T}_{U_1}\omega_J U_2). \end{aligned}$$

For $U_1 \in \Gamma(\ker F_*)$ and $Z_1 \in \Gamma(D_1^J)$, using (2.4), (2.7), (3.3) and (3.4), we have

$$\begin{aligned} & (\nabla\Psi_*)(U_1, Z_1) \\ = \ & -\Psi_*(\nabla_{U_1}Z_1), \\ = \ & \Psi_*(J\nabla_{U_1}JZ_1), \\ = \ & \Psi_*(B_J\mathcal{H}\nabla_{U_1}JZ_1 + C_J\mathcal{H}\nabla_{U_1}JZ_1 + \phi_J\mathcal{T}_{U_1}JZ_1 + \omega_J\mathcal{T}_{U_1}JZ_1). \end{aligned}$$

For $U_1 \in \Gamma(\ker \Psi_*)$ and $X_1 \in \Gamma(D_2^J)$, using (2.3), (2.4), (2.7), (3.3), (3.4) and Lemma 3.2, we obtain

$$\begin{aligned}
& (\nabla \Psi_*)(U_1, X_1) \\
= \;& -\Psi_*(\nabla_{U_1} X_1), \\
= \;& \Psi_*(J \nabla_{U_1} J X_1), \\
= \;& \Psi_*(J \nabla_{U_1} B_J X_1 + \nabla_{U_1} B_J C_J X_1 + \nabla_{U_1} C_J^2 X_1), \\
= \;& \Psi_*(B_J \mathcal{T}_{U_1} B_J X_1 + C_J \mathcal{T}_{U_1} B_J X_1 + \phi_J \mathcal{V} \nabla_{U_1} B_J X_1 + \omega_J \mathcal{V} \nabla_{U_1} B_J X_1 + \\
& \mathcal{T}_{U_1} B_J C_J X_1 + \mathcal{V} \nabla_{U_1} B_J C_J X_1 - \cos^2 \vartheta_J \; \mathcal{H} \nabla_{U_1} X_1 - \cos^2 \vartheta_J \mathcal{T}_{U_1} X_1 + \\
& \sin 2\vartheta_J \; U_1[\vartheta_J] X_1).
\end{aligned}$$

□

4. Example

Let R^{4t} be a Euclidean space with coordinates $(u_1, u_2, ..., u_{4t})$. Complex structures I, P, L on R^{4t} can be chosen as:

$$\begin{aligned}
I(\frac{\partial}{\partial u_{4r+1}}) &= \frac{\partial}{\partial u_{4r+2}}, I(\frac{\partial}{\partial u_{4r+2}}) = -\frac{\partial}{\partial u_{4r+1}}, I(\frac{\partial}{\partial u_{4r+3}}) = \frac{\partial}{\partial u_{4r+4}}, \\
I(\frac{\partial}{\partial u_{4r+4}}) &= -\frac{\partial}{\partial u_{4r+3}}, P(\frac{\partial}{\partial u_{4r+1}}) = \frac{\partial}{\partial u_{4r+3}}, P(\frac{\partial}{\partial u_{4r+2}}) = -\frac{\partial}{\partial u_{4r+4}}, \\
P(\frac{\partial}{\partial u_{4r+3}}) &= -\frac{\partial}{\partial u_{4r+1}}, P(\frac{\partial}{\partial u_{4r+4}}) = \frac{\partial}{\partial u_{4r+2}}, L(\frac{\partial}{\partial u_{4r+1}}) = \frac{\partial}{\partial u_{4r+4}}, \\
L(\frac{\partial}{\partial u_{4r+2}}) &= \frac{\partial}{\partial u_{4r+3}}, L(\frac{\partial}{\partial u_{4r+3}}) = -\frac{\partial}{\partial u_{4r+2}}, L(\frac{\partial}{\partial u_{4r+4}}) = -\frac{\partial}{\partial u_{4r+1}},
\end{aligned}$$

for $r \in \{0, 1, 2, ..., ..., t-1\}$.
Then we easily check that (I, P, L) is a hyperkähler structure on R^{4t}, where $\langle , \rangle$ indicates the Euclidean metric on R^{4t}.

Example 4.1. *We describe a map* $\Psi : R^8 \to R^6$ *by*

$$\Psi(u_1, u_2, ..., u_8) = (\frac{u_1 - u_3}{\sqrt{2}}, u_4, u_5, u_6, u_7, u_8).$$

Then, we have

$$\ker \Psi_* = \left\langle \frac{\partial}{\partial u_1} + \frac{\partial}{\partial u_3}, \frac{\partial}{\partial u_2} \right\rangle,$$

$$(\ker \Psi_*)^\perp = \left\langle \frac{\partial}{\partial u_1} - \frac{\partial}{\partial u_3}, \frac{\partial}{\partial u_4}, \frac{\partial}{\partial u_5}, \frac{\partial}{\partial u_6}, \frac{\partial}{\partial u_7}, \frac{\partial}{\partial u_8} \right\rangle,$$

It is easy to see that

$$D_1^I = \left\langle \frac{\partial}{\partial u_5}, \frac{\partial}{\partial u_6}, \frac{\partial}{\partial u_7}, \frac{\partial}{\partial u_8} \right\rangle, D_2^I = \left\langle \frac{\partial}{\partial u_1} - \frac{\partial}{\partial u_3}, \frac{\partial}{\partial u_4} \right\rangle,$$

$$\begin{aligned}
D_1^P &= \left\langle \frac{\partial}{\partial u_5}, \frac{\partial}{\partial u_6}, \frac{\partial}{\partial u_7}, \frac{\partial}{\partial u_8} \right\rangle, D_2^P = \left\langle \frac{\partial}{\partial u_1} - \frac{\partial}{\partial u_3}, \frac{\partial}{\partial u_4} \right\rangle, \\
D_1^L &= \left\langle \frac{\partial}{\partial u_5}, \frac{\partial}{\partial u_6}, \frac{\partial}{\partial u_7}, \frac{\partial}{\partial u_8} \right\rangle, D_2^L = \left\langle \frac{\partial}{\partial u_1} - \frac{\partial}{\partial u_3}, \frac{\partial}{\partial u_4} \right\rangle.
\end{aligned}$$

Thus the map Ψ *is an almost phv*− *semi-slant submersion with the almost phv*− *semi-slant angle* $\{\vartheta_I = \frac{\pi}{4}, \vartheta_P = \frac{\pi}{2}, \vartheta_L = \frac{\pi}{4}\}$.

Example 4.2. *Let* (R^8, g_{R^8}, P) *be a hyperKähler manifold endowed with usual metric* g_{R^8} *and* (R^3, g_{R^3}) *be a RM endowed with Riemannian metric*

$$\begin{bmatrix} \sin^2 u_1 + \cos^2 u_5 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

.

Now, we describe a map $\Psi : R^8 \to R^4$ *by*

$$\Psi(u_1, u_2, ..., u_8) = (\cos u_1 - \sin u_5, u_6, u_7, u_8,).$$

Then, we have

$$\ker \Psi_* = \left\langle (\cos u_5 \frac{\partial}{\partial u_1} - \sin u_1 \frac{\partial}{\partial u_5}), \frac{\partial}{\partial u_2}, \frac{\partial}{\partial u_3}, \frac{\partial}{\partial u_4} \right\rangle,$$

$$(\ker \Psi_*)^\perp = \left\langle (\sin u_1 \frac{\partial}{\partial u_1} + \cos u_5 \frac{\partial}{\partial u_5}), \frac{\partial}{\partial u_6}, \frac{\partial}{\partial u_7}, \frac{\partial}{\partial u_8} \right\rangle,$$

It is easy to see that

$$\begin{aligned} D_1^I &= \left\langle \frac{\partial}{\partial u_7}, \frac{\partial}{\partial u_8} \right\rangle, D_2^I = \left\langle (\sin u_1 \frac{\partial}{\partial u_1} + \cos u_5 \frac{\partial}{\partial u_5}), \frac{\partial}{\partial u_6} \right\rangle, \\ D^P &= \left\langle \frac{\partial}{\partial u_6}, \frac{\partial}{\partial u_8} \right\rangle, D_1^P = \left\langle (\sin u_1 \frac{\partial}{\partial u_1} + \cos u_5 \frac{\partial}{\partial u_5}), \frac{\partial}{\partial u_7} \right\rangle, \\ D^L &= \left\langle \frac{\partial}{\partial u_6}, \frac{\partial}{\partial u_7} \right\rangle, D_1^L = \left\langle (\sin u_1 \frac{\partial}{\partial u_1} + \cos u_5 \frac{\partial}{\partial u_5}), \frac{\partial}{\partial u_8} \right\rangle. \end{aligned}$$

Thus, the map Ψ *is a phv*− *semi-slant submersion with the almost phv*− *semi-slant function* $\{\vartheta_I = \vartheta_P = \vartheta_L = u_5\}$.

References

[1] M. A. Akyol and Y. Gündüzalp, *Hemi-slant submersions from almost product Riemannian manifolds*, Gulf J. Math. **4** (2016), no. 3, 15-27.

[2] D. V. Alekseevsky and S. Marchiafava, *Almost complex submanifolds of quaternionic manifolds*, In: Proceedings of the Colloquium on Differential Geometry, Debrecen (Hungary), (2000), 25-30, Inst. Math. Inform. Debrecen. (2001) 23-38.

[3] C. Altafini, *Redundant robotic chains on Riemannian submersions*, IEEE Transaction on Robotics and Automation. **20** (2004), no. 2, 335-340.

[4] P. Baird and J. C. Wood, *Harmonic morphisms between Riemannian manifolds*, Oxford science publications. (2003).

[5] J. P. Bourguignon and H. B. Lawson, *A mathematician's visit to Kaluza-Klein theory*, Rend. Semin. Mat. Torino Fasc. Spec. (1989), 143-163.

[6] J. P. Bourguignon and H. B. Lawson, *Stability and isolation phenomena for Yang-mills fields*, Commum. Math. Phys. **79** (1981), 189-230.

[7] V. Cortes, C. Mayer, T. Mohaupt and F. Saueressig, Special geometry of Euclidean supersymmetry, Vector multiplets, J. High Energy Phys., 03 (2004), 028.

[8] A. Gray, *Pseudo-Riemannian almost product manifolds and submersions*, J. Math. Mech. **16** (1967), 715-737.

[9] D. Guan, On Riemann-Roch Formula and Bounds of the Betti Numbers of Irreducible Compact HyperkÂ¨ahler Manifold- n =4, preprint 1999.

[10] D. Guan, On the Betti numbers of irreducible compact hyperkähler manifolds of complex dimension four. Mathematical Research Letters. 2001 Sep;8(5):663-9.

[11] S. Ianus, R. Mazzocco and G. E. Vilcu, *Riemannian submersions from quaternionic manifolds*, Acta. Appl. Math. **104** (2008), 83-89.

[12] S. Ianus and M. Visinescu, *Kaluza-Klein theory with scalar fields and generalized Hopf manifolds*, Class. Quantum Gravity. **4** (1987), 1317-1325.

[13] S. Ianus and M. Visinescu, *Space-time compactification and Riemannian submersions*, In: The Mathematical Heritage of C. F. Gauss, Rassias, G. (ed.), World Scientific, River Edge. (1991), 358-371.

[14] S. Kumar and P. Rajendra, *Conformal anti-invariant submersions from Kenmotsu manifolds onto Riemannian manifold*, Ital. J. Pure Appl. Math. **40** (2018), 474-500.

[15] S. Kumar, S. Kumar, S. Pandey and R. Prasad, *Conformal hemi-slant submersions from almost Hermitian manifolds*, Commun. Korean Math. Soc. **35** (2020), no. 3, 999-1018.

[16] S. Kumar, AT. Vanli, S. Kumar, R. Prasad, Conformal quasi bi-slant submersions, Annals of the Alexandru Ioan Cuza University-Mathematics. 2022 Jul 1;68(2).

[17] S. Longwap, F. Massamba and N. E. Homti, *On Quasi-Hemi-Slant Riemannian submersion*, J. Adv. Math. Comp. Sci., **34** (2019), no. 1, 1-14.

[18] B. O'Neill, *The fundamental equations of a submersion*, Mich. Math. J. **13** (1966), 458-469.

[19] K. S. Park, *H-slant submersions*, Bull. Korean Math. Soc. **49** (2012), no. 2, 329-338.

[20] K. S. Park, *V-semi-slant submersions*, Commun. Korean Math. Soc. **36** (2021), no. 1, 173-187.

[21] K. S. Park, *H-semi-slant submersions from almost quaternionic Hermitian manifolds*, Taiwan. J. Math. **18** (2014), no. 6, 1909-1926.

[22] K. S. Park, *h-v-semi-slant submersions from almost quaternionic Hermitian manifolds*, Bull. Korean Math. Soc. 53 (2016), No. 2, pp. 441-460.

[23] K. S. Park, *Almost h-conformal semi-invariant submersion from almost quaternionic Hermitian manifolds*, Hacettepe J. Math. Stat. **49** (2020), no. 5, 1804-1824.

[24] R. Prasad, S. S. Shukla and S. Kumar, *On Quasi-bi-slant submersions*, Mediterr. J. Math. **16** (2019) 155.

[25] R. Prasad, P. K. Singh, and S. Kumar, *On quasi bi-slant submersions from Sasakian manifolds onto Riemannian manifolds*, Afr. Mat. **32** (2021), no. 3, 403-417.

[26] R. Prasad, S. Kumar, S. Kumar S, AT. Vanli, On Quasi-Hemi-Slant Riemannian Maps, Gazi University Journal of Science. 2021 Jun 1; 34(2): 477-91.

[27] R. Prasad, M. A. Akyol, S. Kumar and P. K. Singh, Quasi bi-slant submersions in contact geometry, Cubo, **24** (2022), no. 01, 01–20.

[28] R. Prasad, M. A. Akyo, P. K. Singh and S. Kumar, On Quasi Bi-Slant Submersions from Kenmotsu Manifolds onto any Riemannian Manifolds, J. Math. Ext. **16** (2022) no. 6, 1-25.

[29] B. Sahin, *Anti-invariant Riemannian submersions from almost Hermitian manifolds*, Cent. Eur. J. Math. **8** (2010), no. 3, 437-447.

[30] B. Sahin, *Semi-invariant submersions from almost Hermitian manifolds*, Canad. Math. Bull. **56** (2013), No. 1, 173-183.

[31] B. Sahin, *Slant submersions from Almost Hermitian manifolds*, Bull. Math. Soc. Sci. Math. Roum. **54** (2011), no. 1, 93-105.

[32] B. Sahin, *Riemannian submersion from almost Hermitian manifolds*, Taiwan. J. Math. **17** (2013), no. 2, 629-659.

[33] B. Sahin, *Riemannian submersions, Riemannian maps in Hermitian geometry, and their applications*, Elsevier, Academic Press (2017).

[34] S. A. Sepet and H. G. Bozok, *Pointwise semi-slant submersion*, Differential Geometry-Dynamical System **22** (2020), pp 1-10.

[35] H. M. Tastan, B. Sahin and S. Yanan, *Hemi-slant submersions*, Mediterr. J. Math. **13** (2016), no. 4, 2171-2184.

[36] B. Watson, *Almost Hermitian submersions*, J. Differential Geom. **11** (1976), no. 1, 147-165.

ADDRESS
SHRI JAI NARAIN POST GRADUATE COLLEGE, LUCKNOW-INDIA,
MOTIHARI COLLEGE OF ENGINEERING, MOTIHARI, BIHAR ENGINEERING UNIVERSITY, PATNA, BIHAR, INDIA,
DEPARTMENT OF MATHEMATICS, FACULTY OF SCIENCE, DICLE UNIVERSITY, SUR, DIYARBAK-21280, TURKEY.
Email address: sushilmath20@gmail.com, itssumeetkumar@gmail.com and murat.polat@dicle.edu.tr