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A NEW PROOF OF THE BARROW AND ERDOS-MORDELL INEQUALITY

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ABSTRACT. In this article, we will introduce the new proof to the Barrow and Erdos-Mordell inequality.

1. INTRODUCTION

The famous Erdos-Mordell inequality is a beautiful geometric inequality, which can be stated as follows:

Theorem 1.1 (Erdos-Mordell inequality)**.** *From a point P inside a given triangle ABC, the perpendiculars PD, PE, PF are drawn to its sides. Then*

$$PA + PB + PC \geq 2(PD + PE + PF).$$

Equality holds if and only if the triangle ABC is equilateral and the point P is its center.

Barrow's inequality is a strengthened version of the Erdos - Mordell inequality in which:

Theorem 1.2 (Barrow inequality)**.** *Consider the triangle ABC and point P lying inside this triangle. Draw the inner bisectors of angles $\angle BPC$, $\angle CPA$, and $\angle APB$ meeting the sides BC, CA, and AB of triangle ABC at X, Y, Z, respectively. Then*

$$PA + PB + PC \geq 2(PX + PY + PZ).$$

Equality holds if and only if the triangle ABC is equilateral and the point P is its center.

The proofs of theorem 1.1 and 1.2 can be referenced at [1-6]. And addition, some their development and generalizations can be referenced at [7-16].
We now go to the new proof of these above in-equalties by using the strict lemmas of two above inequalities.

Key words and phrases. The Barrow inequality, the Erdos-Mordell inequality.

2. BUILDING THE CONCERNED RESULTS

Lemma 2.1. *Given a triangle ABC and point P lying inside this triangle. Let B_C belong to the ray BC and C_B belong to the ray CB such that $\angle BPB_C = \angle CPC_B = \angle BAC$. The pairs of points B_C, B_A and C_A, C_B are defined similarly. Then we have the inequality:*

$$PA + PB + PC \geq 2\sqrt{PA_B \cdot PB_A} + 2\sqrt{PB_C \cdot PC_B} + 2\sqrt{PC_A \cdot PA_C}.$$

Proof. We have:

$$\begin{aligned}\angle A_BAA_C + \angle A_BPA_C &= \angle BAC + \angle APA_B + \angle APA_C \\ &= \angle BAC + \angle ACB + \angle ABC = 180^\circ.\end{aligned}$$

It follows AA_BPA_C inscribed in a circle, hence $\angle AA_BA_C = \angle APA_C = \angle ABC$ so $A_BA_C \parallel BC$. From that, we have that two triangles AA_BA_C and ABC are similar according the a-a case.
Hence, if let $BC = a, CA = b, AB = c$ then $\frac{AA_B}{A_BA_C} = \frac{AB}{BC} = \frac{c}{a}$ and $\frac{AA_C}{A_CA_B} = \frac{b}{a}$.
Applying the *Ptolemy* theorem to the inscribed quadrilateral AA_BPA_C, we have:

$$AP \cdot A_BA_C = AA_B \cdot PA_C + AA_C \cdot PA_B \Longrightarrow PA = PA_C\frac{AA_B}{A_BA_C} + PA_B\frac{AA_C}{A_BA_C}.$$

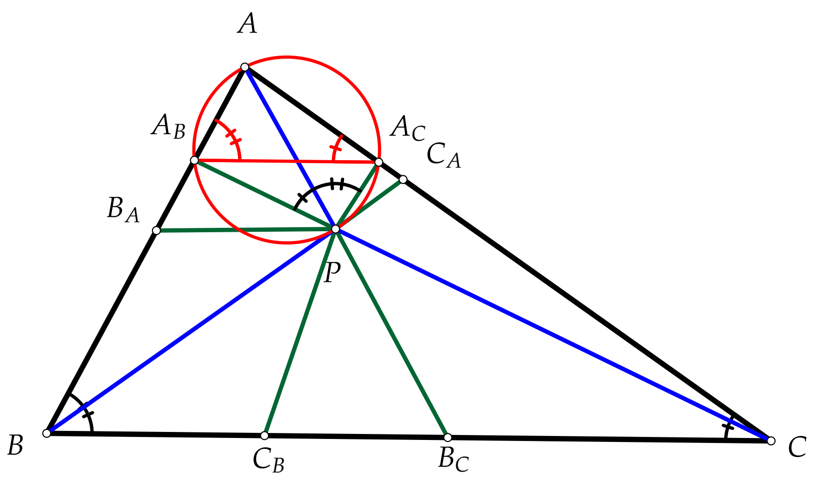

FIGURE 1

From that, we obtain

$$PA = PA_B\frac{b}{a} + PA_C\frac{c}{a}. \tag{2.1}$$

Similarly, we also have

$$PB = PB_C\frac{c}{b} + PB_A\frac{a}{b}; \tag{2.2}$$

$$PC = PC_A\frac{a}{c} + PC_B\frac{b}{c}. \tag{2.3}$$

Adding (2.1), (2.2) and (2.3) side-by-side, then applying the *AM-GM* inequaltiy to two positive real numbers, we have:

$$\begin{aligned} PA + PB + PC &= \left(PA_B\frac{b}{a} + PB_A\frac{a}{b}\right) + \left(PB_C\frac{c}{b} + PC_B\frac{b}{c}\right) + \left(PC_A\frac{a}{c} + PA_C\frac{c}{a}\right) \\ &\geq 2\sqrt{PA_B\frac{b}{a}\cdot PB_A\frac{a}{b}} + 2\sqrt{PB_C\frac{c}{b}\cdot PC_B\frac{b}{c}} + 2\sqrt{PC_A\frac{a}{c}\cdot PA_C\frac{c}{a}} \\ &\geq 2\sqrt{PA_B\cdot PB_A} + 2\sqrt{PB_C\cdot PC_B} + 2\sqrt{PC_A\cdot PA_C}. \end{aligned}$$

□

Lemma 2.2. *Give the triangle ABC having the inner bisector AD. Then we have the relation as follows:*

$$AD^2 = AB\cdot AC - DB\cdot DC.$$

Proof. Draw the circumscribed circle of triangle ABC meeting AD at $E \neq A$ again. We have that two triangles ABD and AEC are similar according the a-a case, so $\frac{AB}{AD} = \frac{AE}{AC}$. It follows $AB\cdot AC = AD\cdot AE$.

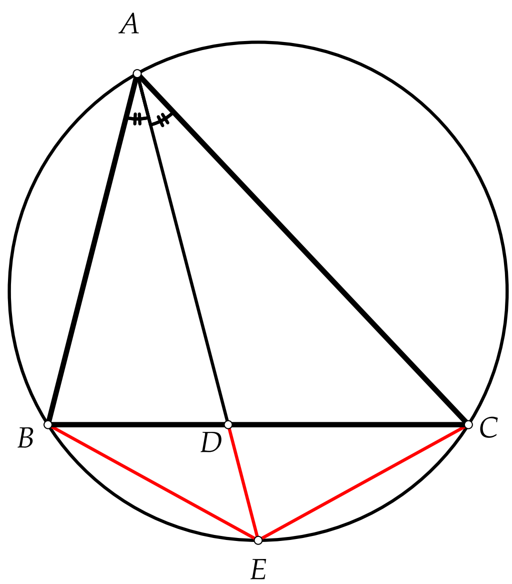

FIGURE 2

On the other hand, applying the Intersecting chords' theorem, we have:

$$DA \cdot DE = DB \cdot DC.$$

It follows that

$$AB \cdot AC = AD \cdot AE = AD \cdot (AD + DE) = AD^2 + AD \cdot DE = AD^2 + BD \cdot CD.$$

The lemma is proved. □

Lemma 2.3. *Given a triangle ABC and point P lies inside this triangle. The line, which is symmetric to PA with respect to the inner bisector of angle $\angle BPC$, meets the circumscribed circle of triangle PBC at two points P and A'. Similarly to points B' and C'. Then $PA' + PB' + PC' \geq 2(PA + PB + PC)$.*

Proof. Let $x = \angle BPC, y = \angle CPA$ and $z = \angle APB$.

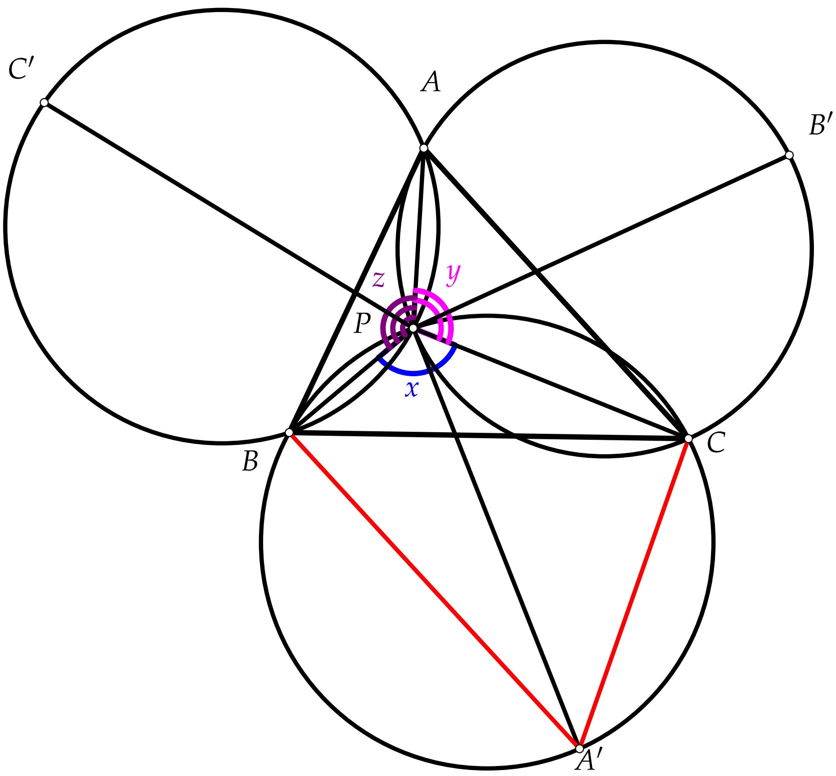

FIGURE 3

We have:

$\angle A'BC = \angle A'PC$ (since A' lies on the circumscribed circle of triangle BCP)

$= 180^\circ - \angle APB$ (since AP is isogonal conjugate of $A'P$ with respect to angle $\angle BPC$).

Hence $\sin\angle A'BC = \sin\angle APB = \sin z$.
Similarly: $\sin\angle A'CB = \sin y$. Note that $\sin\angle BA'C = \sin\angle BPC = \sin x$. Hence, applying the Sines theorem to the triangle $A'BC$, we have:

$$\frac{A'B}{BC} = \frac{\sin\angle A'BC}{\sin\angle BA'C} = \frac{\sin y}{\sin x}. \tag{2.4}$$

Similarly, we also have:

$$\frac{A'C}{A'B} = \frac{\sin z}{\sin x}. \tag{2.5}$$

Applying the *Ptolemy* theorem to the inscribed quadrilateral $PBA'C$, we have:

$$PA' \cdot BC = A'C \cdot PB + A'B \cdot PC \Rightarrow PA' = \frac{A'C}{BC} \cdot PB + \frac{A'B}{BC} \cdot PC. \tag{2.6}$$

From (2.4), (2.5) and (2.6), we have

$$PA' = \frac{\sin z}{\sin x} \cdot PB + \frac{\sin y}{\sin x} \cdot PC. \tag{2.7}$$

Similarly, we also have:

$$PB' = \frac{\sin x}{\sin y} \cdot PC + \frac{\sin z}{\sin y} \cdot PA; \tag{2.8}$$

and

$$PC' = \frac{\sin y}{\sin z} \cdot PA + \frac{\sin x}{\sin z} \cdot PB. \tag{2.9}$$

Adding (2.7), (2.8) and (2.9) side-by-side, then applying note on the inequalities $\frac{u}{v} + \frac{v}{u} \geq 2$ for every positive numbers u and v, we obtain:

$$\begin{aligned} PA' + PB' + PC' &= \left(\frac{\sin y}{\sin z} + \frac{\sin z}{\sin y}\right) PA + \left(\frac{\sin z}{\sin x} + \frac{\sin x}{\sin z}\right) PB + \left(\frac{\sin x}{\sin y} + \frac{\sin y}{\sin x}\right) PC \\ &\geq 2PA + 2PB + 2PC = 2\left(PA + PB + PC\right). \end{aligned}$$

The equality happens if and only if $\sin x = \sin y = \sin z$, that is $\angle BPC = \angle CPA = \angle APB = 120^\circ$ or P is the *Fermat-Torricelli* point of triangle ABC. □

3. PROOF OF THE INEQUALITY

Proof of theorem 1.1.

Proof. Draw the chord PD' of the circumscribed circle of triangle PEF that is symmetric to PD with respect to the inner bisector of $\angle EPF$. Similarly to E' and F'.

Clearly, AP is the diameter of the circumscribed circle of triangle PEF so

$$PA \geq PD'. \tag{3.1}$$

Equality holds if and only if $PD \perp EF$ or $EF \parallel BC$.
Similarly, we also have:

$$PB \geq PE'; \tag{3.2}$$

$$PC \geq PF'. \tag{3.3}$$

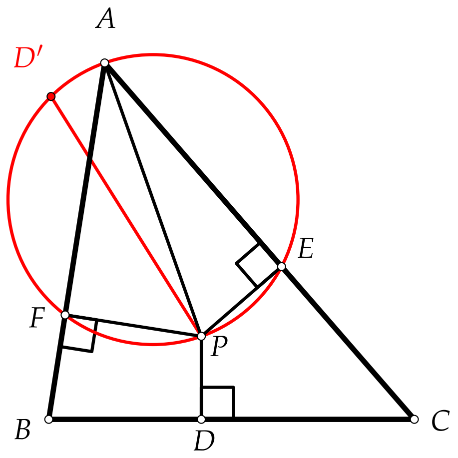

FIGURE 4

From (3.1), (3.2), (3.3) and the lemma 2.3, we obtain

$$PA + PB + PC \geq PD' + PE' + PF' \geq 2PD + 2PE + 2PF.$$

Equality holds if and only if two triangles DEF and ABC have corresponding sides that are parallel and $\angle BPC = \angle CPA = \angle APB = 120^\circ$ or the triangle ABC is equilateral and P is its center. □

Proof of theorem 1.2

Proof. Let B_C belong to the ray BC and C_B belong to the ray CB such that $\angle CPC_B = \angle BPB_C = \angle BAC$; Similarly to A_B, B_A and C_A, A_C.
Clearly, two angles $\angle BPC$ and $\angle B_CPC_B$ have the common inner bisector.

Applying to the lemma 2.2, we have $PB_C \cdot PC_P = PX^2 + XB_C \cdot XC_B \geq PX^2$.
Equality holds if and only if $B_C \equiv C_B$, that is $\angle BPC = 2\angle BAC$. Hence

$$\sqrt{PB_C \cdot PC_B} \geq PX. \tag{3.4}$$

Similarly, we also have:

$$\sqrt{PC_A \cdot PA_C} \geq PY; \tag{3.5}$$

$$\sqrt{PA_B \cdot PB_A} \geq PZ. \tag{3.6}$$

From (3.4), (3.5), (3.6) and the lemma 2.1, we follow

$$\begin{aligned} PA + PB + PC &\geq 2\sqrt{PB_C \cdot PC_B} + 2\sqrt{PC_A \cdot PA_C} + 2\sqrt{PA_B \cdot PB_A} \\ &\geq 2PX + 2PY + 2PZ. \end{aligned}$$

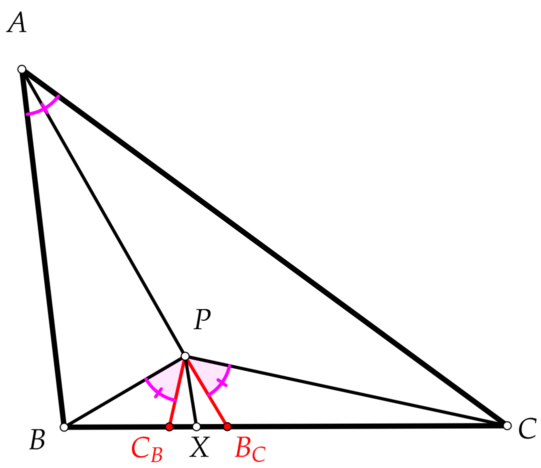

FIGURE 5

Equality holds if and only if $\angle BPC = 2\angle BAC$, $\angle CPA = 2\angle ABC$ and $\angle APB = 2\angle ACB$ and triangle ABC is equilateral that is ABC is equilateral P is its center.
The theorem 1.2 is proved. ☐

4. A GENERALIZATION OF LEMMA 2.3 AND THE PROOF OF A GENERALIZED RESULT OF ERDOS-MODELL INEQUALITY

From [2], we have the result as follows:

Theorem 4.1. *Let ABC be a triangle and let P, Q, R be three points inside it such that $QR \perp BC, RP \perp CA$ and $PQ \perp AB$. Let QR meet BC at D, RP meet CA at E and PQ meet AB at F. Then*

$$PA + QB + RC \geq PE + PF + QF + QD + RD + RE.$$

We here introduce another proof based on the development of lemma 2.3 and from that we have a new proof for the theorem 4.1. This lemma is as follows:

Lemma 4.2. *Given a triangle ABC having the semi-perimeter p. Let A_1, B_1, and C_1 be the points lying on the opposite rays of rays BC, CA, and AB, respectively. Tangent lines at A of the circumscribed circle of triangle ABC meet the circumscribed circle of triangle AB_1C_1 at A_2 again. Points B_2 and C_2 are defined similarly as the definition of point A_2. Then*

$$AA_2 + BB_2 + CC_2 \geq 2(AC_1 + CB_1 + BA_1 + p).$$

Proof. Denote by $a = BC, b = CA$ and $c = AB$. We have:

$\angle ABC = \angle A_2AB_1$ (since AA_2 is tangent line of the circumscribed circle of triangle ABC)
$= \angle A_2C_1B_1)$ (since the quadrilateral $AB_1A_2C_1$ is concyclic).

Similarly: $\angle ACB = \angle A_2B_1C_1$.
Hence, we have that two triangles ABC and $A_2C_1B_1$ are similar according the a-a case.
Hence $\dfrac{A_2C_1}{B_1C_1} = \dfrac{AB}{CB} = \dfrac{c}{a}$ and $\dfrac{A_2B_1}{C_1B_1} = \dfrac{AC}{BC} = \dfrac{b}{a}$.

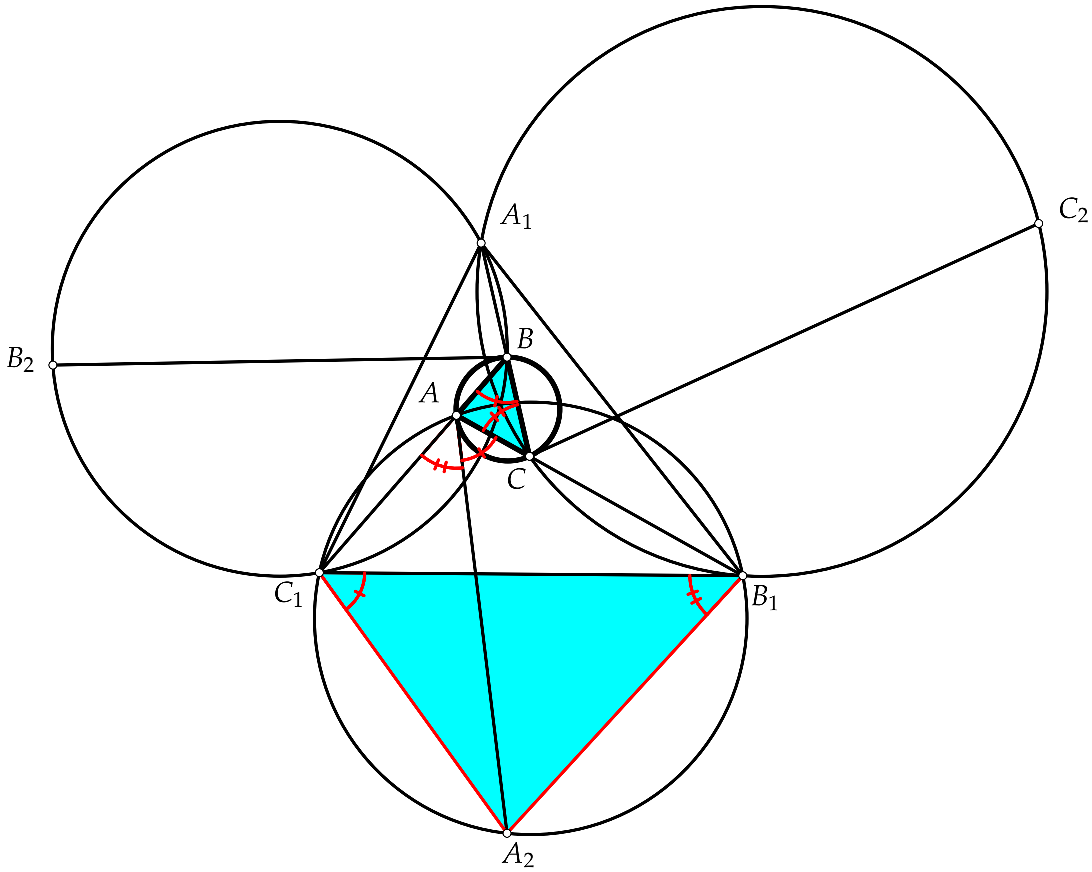

FIGURE 6

Applying the *Ptolemy* theorem to the inscribed quadrilateral $AB_1A_2C_1$, we have:

$$AA_2 \cdot B_1C_1 = AC_1 \cdot A_2B_1 + AB_1 \cdot A_2C_1 \Longrightarrow AA_2 = AC_1\frac{A_2B_1}{C_1B_1} + AB_1\frac{A_2C_1}{B_1C_1}.$$

Since $AB_1 = AC + CB_1 = b + CB_1$, it follows:

$$AA_2 = AC_1\frac{b}{a} + CB_1\frac{c}{a} + \frac{bc}{a}. \tag{4.1}$$

Similarly, we also have:

$$BB_2 = BA_1\frac{c}{b} + AC_1\frac{a}{b} + \frac{ca}{b}, \tag{4.2}$$

and

$$CC_2 = CB_1\frac{a}{c} + BA_1\frac{b}{c} + \frac{ab}{c}. \tag{4.3}$$

Adding (4.1), (4.2) and (4.3) side-by-side, note on the inequalities $\frac{u}{v} + \frac{v}{u} \geq 2$ for every positive numbers u and v, then $\frac{xy}{z} + \frac{yz}{x} + \frac{zx}{y} = \frac{1}{2}\left(\frac{y}{z} + \frac{z}{y}\right)x + \frac{1}{2}\left(\frac{z}{x} + \frac{x}{z}\right)y +$

$\frac{1}{2}\left(\frac{x}{y}+\frac{y}{x}\right)z \geq x+y+z$ for every positive numbers x, y, z, we have:

$$AA_2 + BB_2 + CC_2 = AC_1\left(\frac{a}{b}+\frac{b}{a}\right) + CB_1\left(\frac{c}{a}+\frac{a}{c}\right) + BA_1\left(\frac{c}{b}+\frac{b}{c}\right) + \left(\frac{ab}{c}+\frac{bc}{a}+\frac{ca}{b}\right)$$
$$\geq 2AC_1 + 2CB_1 + 2BA_1 + (a+b+c) = 2(AC_1 + CB_1 + BA_1 + p).$$

The lemma is proved.
Equality holds if and only if $a = b = c$, that is the triangle ABC is equilateral. □

We now prove the theorem 4.1 as follows:

Proof. Draw the circumscribed circles of triangles AEF, BFD, CDE and PQR.
Tangent lines at P of the circumscribed circle of triangle PQR meet the circumscribed circle of triangle AEF at A' again. Similarly to the points B' and C'.
Applying the lemma 4.1, we have:

$$PA' + PB' + PC' \geq 2PE + 2QF + 2RD + PQ + QR + RP. \tag{4.4}$$

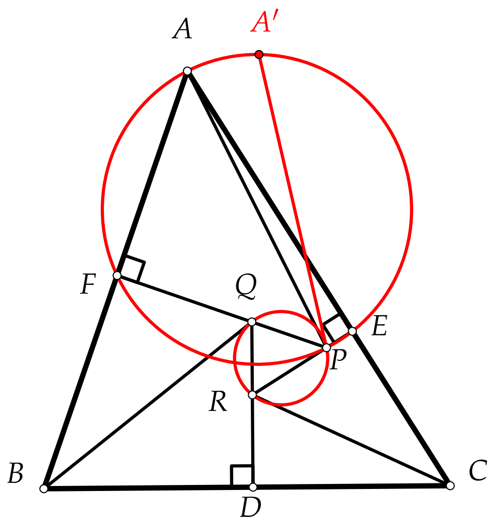

FIGURE 7

Clearly, AP is the diameter of circumscribed circle of triangle PEF so $PA \geq PA'$. Similarly to $PB \geq PB'$ and $PC \geq PC'$. Hence

$$PA + PB + PC \geq PA' + PB' + PC'. \tag{4.5}$$

From (4.4) and (4.5), it follows:

$$PA + PB + PC \geq 2PE + 2QF + 2RD + PQ + QR + RP.$$

Theorem 4.1 is proved. □

5. The similar inequality of Erdos-Modell one and its proof

Theorem 5.1. *Given a triangle ABC and point P lying inside this triangle. Line which is symmetric to AP with respect to the inner bisector of $\angle BPC$ meets BC at D; line which is symmetric to BP with respect to the inner bisector $\angle CPA$ meets CA at E; line which is symmetric to CP with respect to the inner bisector of $\angle APB$ meets BC at F. Then*

$$\frac{1}{PA}+\frac{1}{PB}+\frac{1}{PC}\le\frac{1}{2}\left(\frac{1}{PD}+\frac{1}{PE}+\frac{1}{PF}\right).$$

Proof. Denote by $\angle BPC=x, \angle CPA=y$ and $\angle APB=z$.
Draw the circumscribed circle of triangle PBC meeting AP at A' again.
We have $\angle BPD=\angle A'PC$ and $\angle CA'P=\angle CBP=\angle DAB$.
It follows that two triangles PBD and $PA'C$ is similar, hence $\dfrac{PB}{PD}=\dfrac{PA'}{PC}$, it follows $PB\cdot PC=PD\cdot PA'$.
On the other hand, $\angle A'BC=\angle A'PC=180^\circ-\angle APC=180^\circ-y$, $\angle A'CB=\angle A'PB=180^\circ-\angle APB=180^\circ-z$ and $\widehat{BA'C}=180^\circ-\angle BPC=180^\circ-x$.
Hence, applying the Law of Sine to triangle $A'BC$, we have $\dfrac{A'B}{BC}=\dfrac{\sin\angle A'BC}{\sin\angle BA'C}=\dfrac{\sin y}{\sin x}$ and $\dfrac{CA'}{BC}=\dfrac{\sin z}{\sin x}$.

Applying the *Ptolemy* theorem to the concyclic quadrilateral $BPCA'$, we have

$$\begin{aligned}&A'P\cdot BC=A'B\cdot PC+A'C\cdot PB\\ &\Longrightarrow PA'=\frac{A'B}{BC}PC+\frac{A'C}{BC}PB=\frac{\sin z}{\sin x}PC+\frac{\sin z}{\sin x}PB.\end{aligned}$$

From that, we have

$$PB\cdot PC=PD\cdot PA'=PD\frac{PC\sin z+PB\sin y}{\sin x}.$$

It follows

$$\frac{1}{PD}=\frac{1}{PB}\frac{\sin z}{\sin x}+\frac{1}{PC}\frac{\sin y}{\sin x}. \tag{5.1}$$

Similarly, we also have:

$$\frac{1}{PE}=\frac{1}{PC}\frac{\sin x}{\sin y}+\frac{1}{PA}\frac{\sin z}{\sin y}; \tag{5.2}$$

and

$$\frac{1}{PF}=\frac{1}{PA}\frac{\sin y}{\sin z}+\frac{1}{PB}\frac{\sin x}{\sin z}. \tag{5.3}$$

From (5.1), (5.2), (5.3) and applying the inequalities $\dfrac{u}{v}+\dfrac{v}{u}\ge 2$ for every positive numbers u and v, we have:

$$\begin{aligned}\frac{1}{PD}+\frac{1}{PE}+\frac{1}{PF}&=\frac{1}{PA}\left(\frac{\sin z}{\sin y}+\frac{\sin y}{\sin z}\right)+\frac{1}{PB}\left(\frac{\sin z}{\sin x}+\frac{\sin x}{\sin z}\right)+\frac{1}{PC}\left(\frac{\sin x}{\sin y}+\frac{\sin y}{\sin x}\right)\\ &\ge 2\frac{1}{PA}+2\frac{1}{PB}+2\frac{1}{PC}.\end{aligned}$$

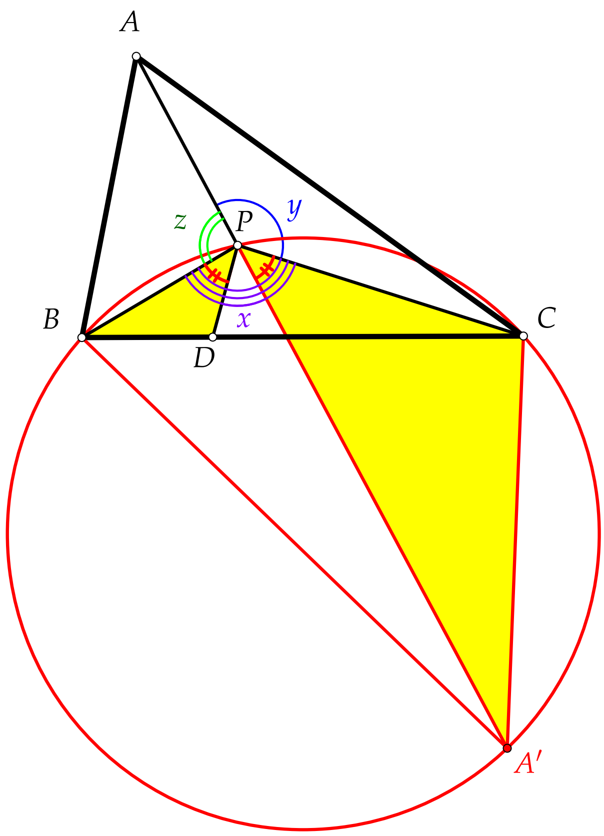

FIGURE 8

Theorem 5.1 is proved.
Equality holds if and only if $x = y = z$ or P is the *Fermat-Torricenlli* point of triangle ABC. □

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