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SIMILARITY, CONGRUENCE, AND HOMOTHETY FOR QUADRILATERALS

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ABSTRACT. Assume that $Q = \square ABCD$ is a general quadrilateral; namely, it is neither cyclic nor orthocentric. Denote the intersection point of ℓ_{AC} and ℓ_{BD} by R. Let O_a, O_b, O_c, and O_d (resp., N_a, N_b, N_c, and N_d) be the circumcenters (resp., nine-point centers) of $\triangle BCD$, $\triangle ACD$, $\triangle ABD$, and $\triangle ABC$, respectively. It is proved in [1] that $Q_\circ = \square O_aO_bO_cO_d$ and $Q_{\mathtt{n}} = \square N_aN_bN_cN_d$ are similar. In this paper, we prove that when $\square ABCD$ is a convex quadrilateral, Q and $Q_\circ$ are similar if and only if Q is a trapezoid and that when $\angle A > 180°$, Q and $Q_\circ$ are similar if and only if $\overrightarrow{RD} = -\overrightarrow{RB}$ and $\overrightarrow{RA} \cdot \overrightarrow{RC} = -\overrightarrow{RB} \cdot \overrightarrow{RD}$. Let $Q_{\circ\circ}$ and $Q_{\circ\mathtt{n}}$ be the circumcenter quadrilateral and the nine-point center quadrilateral of $Q_\circ$, respectively. Let $Q_{\mathtt{n}\circ}$ and $Q_{\mathtt{nn}}$ be the circumcenter quadrilateral and the nine-point center quadrilateral of $Q_{\mathtt{n}}$, respectively. We also prove that Q, $Q_{\circ\circ}$, and $Q_{\mathtt{nn}}$ are homothetic and that $Q_{\circ\mathtt{n}}$ and $Q_{\mathtt{n}\circ}$ are congruent and homothetic.

1. INTRODUCTION

This paper is a sequel to [1]. Given two quadrilaterals $\square ABCD$ and $\square A'B'C'D'$, write $\square ABCD \sim \square A'B'C'D'$ if there exists a positive number k which satisfies

$$\frac{AB}{A'B'} = \frac{BC}{B'C'} = \frac{CD}{C'D'} = \frac{AD}{A'D'} = \frac{AC}{A'C'} = \frac{BD}{B'D'} = k. \tag{1.1}$$

When $k = 1$ in (1.1), we also write $\square ABCD \cong \square A'B'C'D'$. We say that $\square ABCD$ and $\square A'B'C'D'$ are *similar* (resp., *congruent*) *quadrilaterals* if $\square ABCD \sim \square IJKL$ (resp., $\square ABCD \cong \square A'B'C'D'$), where $(I, J, K, L) = (A', B', C'D')$, $(B', C', D,' A')$, (C', D', A', B'), (D', A', B', C'), (D', C', B', A'), (C', B', A', D'), (B', A', D', C'), or (A', D', C', B').
We say two similar quadrilaterals $\square ABCD$ and $\square A'B'C'D'$ are *homothetic* or *similarly placed* if their corresponding sides are parallel. By [2, Theorem 32 on page 38], when $\square ABCD$ and $\square A'B'C'D'$ are homothetic but not congruent, the lines $\ell_{AA'}$, $\ell_{BB'}$, $\ell_{CC'}$, and $\ell_{DD'}$ intersect at a common point S so that $\square A'B'C'D'$ is a dilation of $\square ABCD$ through S.
Given a quadrilateral $\square ABCD$, let O_a, O_b, O_c, and O_d (resp., N_a, N_b, N_c, and N_d) be the circumcenters (resp., nine-point centers) of $\triangle BCD$, $\triangle ACD$, $\triangle ABD$, and $\triangle BCD$, respectively. When $\square ABCD$ is not cyclic (resp., not orthocentric), $\square O_aO_bO_cO_d$ (resp., $\square N_aN_bN_cN_d$) is a quadrilateral. A *general quadrilateral* is a quadrilateral that is neither cyclic nor orthocentric. It is proved in [1, Theorem 6.1] that if $\square ABCD$ is a general

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quadrilateral, then $\square\, O_aO_bO_cO_d \sim \square\, N_aN_bN_cN_d$. In the current paper, we determine the conditions under which a general quadrilateral and its circumcenter are similar and also prove that certain quadrilaterals generated by $\square\, ABCD$ are similar, congruent, or homothetic under the iteration by constructing circumcenter or nine-point center quadrilaterals.

Given a quadrilateral $\square\, ABCD$, let R be the intersection point of ℓ_{AC} and ℓ_{BD}. Note that $\{\overrightarrow{RB}, \overrightarrow{RC}\}$ is a basis for the canonical vector space $\mathcal{E}$ of geometric vectors associated to the plane. The vectors $\overrightarrow{RA}$ and $\overrightarrow{RD}$ are expressible as $\overrightarrow{RA} = -\alpha\overrightarrow{RC}$ and $\overrightarrow{RD} = -\beta\overrightarrow{RB}$, where $\alpha, \beta \in \mathbb{R} \setminus \{0, -1\}$. In Theorem 3.1 and Theorem 3.2, we prove that a convex general quadrilateral $\square\, ABCD$ and its circumcenter quadrilateral are similar if and only if $\square\, ABCD$ is a trapezoid and that a nonconvex general quadrilateral $\square\, ABCD$ with $\angle A > 180^\circ$ and its circumcenter quadrilateral are similar if and only if $\overrightarrow{RD} = -\overrightarrow{RB}$ and $\overrightarrow{RA} \cdot \overrightarrow{RC} = -\overrightarrow{RB} \cdot \overrightarrow{RD}$.

Assume that $Q = \square\, ABCD$ is a general quadrilateral. Let $Q_{\mathtt{o}}$ and $Q_{\mathtt{n}}$ be the circumcenter quadrilateral and the nine-point center quadrilateral of Q, respectively. Let $Q_{\mathtt{oo}}$ and $Q_{\mathtt{on}}$ be the circumcenter quadrilateral and the nine-point center quadrilateral of $Q_{\mathtt{o}}$, respectively, and let $Q_{\mathtt{no}}$ and $Q_{\mathtt{nn}}$ be the circumcenter quadrilateral and the nine-point center quadrilateral of $Q_{\mathtt{n}}$, respectively. In Theorem 4.1 and Theorem 4.2, we prove that Q, $Q_{\mathtt{oo}}$, and $Q_{\mathtt{nn}}$ are homothetic. In Theorem 4.3, we prove that $Q_{\mathtt{on}}$ and $Q_{\mathtt{no}}$ are congruent and homothetic; moreover, the center of the homothety is the center of mass of $\square\, ABCD$.

The paper is organized as follows. In §2, we collect all needed results from [1]. Theorem 3.1 and Theorem 3.2 are proved in §3. Then Theorem 4.1, Theorem 4.2, and Theorem 4.3 are proved in §4. In §4, we also introduce a natural family $\mathcal{Q}$ of quadrilaterals generated by a general quadrilateral $\square\, ABCD$. Let Λ be the set of all words in the letters o and n, including the empty word $\emptyset$. Set $Q_\emptyset = \square\, ABCD$. For each $w \in \Lambda$, $Q_{w\mathtt{o}}$ and $Q_{w\mathtt{n}}$ mean the circumcenter quadrilateral and the nine-point center quadrilateral of Q_w, respectively. If $w_1, w_2 \in \Lambda$, then Q_{w_1} and Q_{w_2} are similar whenever the lengths of the words w_1 and w_2 have the same parity. Consequently, if the length of a word w is even, then Q_w and $\square\, ABCD$ are similar, while if the length of w is odd, then Q_w and $\square\, O_aO_bO_cO_d$ are similar. If $\square\, ABCD$ and $\square\, O_aO_bO_cO_d$ are similar, then all quadrilaterals in $\mathcal{Q}$ are similar to each other.

2. Preliminaries

In this section, we collect the results from [1] which are applied to prove the main theorems in §3 and §4.

Given a quadrilateral $\square\, ABCD$, let R be the intersection point of ℓ_{AC} and ℓ_{BD}. There are $\alpha, \beta \in \mathbb{R} \setminus \{0, -1\}$ such that $\overrightarrow{RA} = -\alpha\overrightarrow{RC}$ and $\overrightarrow{RD} = -\beta\overrightarrow{RB}$. Let O_a, O_b, O_c, and O_d (resp., N_a, N_b, N_c, and N_d) be the circumcenters (resp., nine-point centers) of $\triangle BCD$, $\triangle ACD$, $\triangle ABD$, and $\triangle ABC$, respectively. By [1, Theorem 5.1], both $\square\, O_aO_bO_cO_d$ and $\square\, N_aN_bN_cN_d$ are quadrilaterals.

By [1, Definition 3.1], the *cyclic characteristic* and the *orthocentric characteristic* of $\square\, ABCD$ are defined by

$$\kappa_c = (\overrightarrow{RA} \cdot \overrightarrow{RC} - \overrightarrow{RB} \cdot \overrightarrow{RD})^2 \tag{2.1}$$

and

$$\kappa_o = (\overrightarrow{RA} \cdot \overrightarrow{RC} + \overrightarrow{RB} \cdot \overrightarrow{RD})^2 - 4(\overrightarrow{RA} \cdot \overrightarrow{RB})(\overrightarrow{RC} \cdot \overrightarrow{RD}). \tag{2.2}$$

By [1, Definition 4.1], the *normalized cyclic characteristic* and the *normalized orthocentric characteristic* of $\square\, ABCD$ are defined by

$$\bar{\kappa}_c = \frac{\kappa_c}{(\overrightarrow{RA}\cdot\overrightarrow{RC})(\overrightarrow{RB}\cdot\overrightarrow{RD})-(\overrightarrow{RA}\cdot\overrightarrow{RB})(\overrightarrow{RC}\cdot\overrightarrow{RD})} \tag{2.3}$$

and

$$\bar{\kappa}_o = \frac{\kappa_o}{(\overrightarrow{RA}\cdot\overrightarrow{RC})(\overrightarrow{RB}\cdot\overrightarrow{RD})-(\overrightarrow{RA}\cdot\overrightarrow{RB})(\overrightarrow{RC}\cdot\overrightarrow{RD})}. \tag{2.4}$$

The quadrilateral $\square\, ABCD$ is cyclic (resp., orthocentric) if and only if $\kappa_c = 0$ (resp., $\kappa_o = 0$).

Lemma 2.1 ([1, Lemma 5.1])**.** *Define $x = \overrightarrow{RB}\cdot\overrightarrow{RC}$ and $\Pi = RB^2\cdot RC^2 - x^2$. The four circumcenters of $\square\, ABCD$ are given by*

$$\begin{aligned}
\overrightarrow{RO_a} &= \frac{(1-\beta)RB^2\cdot RC^2+(\beta RB^2-RC^2)x}{2\Pi}\overrightarrow{RB} + \frac{RB^2[RC^2-\beta RB^2-(1-\beta)x]}{2\Pi}\overrightarrow{RC},\\
\overrightarrow{RO_b} &= \frac{RC^2[\beta(\alpha-1)x+\alpha RC^2-\beta^2 RB^2]}{2\beta\Pi}\overrightarrow{RB} + \frac{(\beta^2 RB^2-\alpha RC^2)x-\beta(\alpha-1)RB^2\cdot RC^2}{2\beta\Pi}\overrightarrow{RC},\\
\overrightarrow{RO_c} &= \frac{(\alpha^2 RC^2-\beta RB^2)x-\alpha(\beta-1)RB^2\cdot RC^2}{2\alpha\Pi}\overrightarrow{RB} + \frac{RB^2[\alpha(\beta-1)x+\beta RB^2-\alpha^2 RC^2]}{2\alpha\Pi}\overrightarrow{RC},\\
\overrightarrow{RO_d} &= \frac{RC^2[RB^2-\alpha RC^2-(1-\alpha)x]}{2\Pi}\overrightarrow{RB} + \frac{(1-\alpha)RB^2\cdot RC^2+(\alpha RC^2-RB^2)x}{2\Pi}\overrightarrow{RC}.
\end{aligned}$$

The four nine-point centers of $\square\, ABCD$ are given by

$$\begin{aligned}
\overrightarrow{RN_a} &= \frac{-2(1-\beta)x^2-(\beta RB^2-RC^2)x+(1-\beta)RB^2\cdot RC^2}{4\Pi}\overrightarrow{RB} + \frac{-2x^2+(1-\beta)RB^2x+RB^2\cdot RC^2+\beta RB^4}{4\Pi}\overrightarrow{RC},\\
\overrightarrow{RN_b} &= \frac{2\beta^2x^2+\beta(1-\alpha)RC^2x-\beta^2RB^2\cdot RC^2-\alpha RC^4}{4\beta\Pi}\overrightarrow{RB} + \frac{2\beta(\alpha-1)x^2+(\alpha RC^2-\beta^2RB^2)x+\beta(1-\alpha)RB^2\cdot RC^2}{4\beta\Pi}\overrightarrow{RC},\\
\overrightarrow{RN_c} &= \frac{2\alpha(\beta-1)x^2+(\beta RB^2-\alpha^2RC^2)x+\alpha(1-\beta)RB^2\cdot RC^2}{4\alpha\Pi}\overrightarrow{RB} + \frac{2\alpha^2x^2+\alpha(1-\beta)RB^2x-\alpha^2RB^2\cdot RC^2-\beta RB^4}{4\alpha\Pi}\overrightarrow{RC},\\
\overrightarrow{RN_d} &= \frac{-2x^2+(1-\alpha)RC^2x+RB^2\cdot RC^2+\alpha RC^4}{4\Pi}\overrightarrow{RB} + \frac{-2(1-\alpha)x^2-(\alpha RC^2-RB^2)x+(1-\alpha)RB^2\cdot RC^2}{4\Pi}\overrightarrow{RC}.
\end{aligned}$$

Lemma 2.2 ([1, Lemma 5.2])**.** *Define $x = \overrightarrow{RB}\cdot\overrightarrow{RC}$ and $\Pi = RB^2\cdot RC^2 - x^2$. The six vectors between circumcenters are*

$$\begin{aligned}
\overrightarrow{O_aO_b} &= \frac{-(\beta RB^2-\alpha RC^2)(RC^2+\beta x)}{2\beta\Pi}\overrightarrow{RB} + \frac{(\beta RB^2-\alpha RC^2)(\beta RB^2+x)}{2\beta\Pi}\overrightarrow{RC},\\
\overrightarrow{O_aO_c} &= \frac{-(1+\alpha)(\beta RB^2-\alpha RC^2)x}{2\alpha\Pi}\overrightarrow{RB} + \frac{(1+\alpha)(\beta RB^2-\alpha RC^2)RB^2}{2\alpha\Pi}\overrightarrow{RC},\\
\overrightarrow{O_aO_d} &= \frac{(\beta RB^2-\alpha RC^2)(RC^2-x)}{2\Pi}\overrightarrow{RB} + \frac{(\beta RB^2-\alpha RC^2)(RB^2-x)}{2\Pi}\overrightarrow{RC},\\
\overrightarrow{O_bO_c} &= \frac{(\beta RB^2-\alpha RC^2)(\alpha RC^2-\beta x)}{2\alpha\beta\Pi}\overrightarrow{RB} + \frac{(\beta RB^2-\alpha RC^2)(\beta RB^2-\alpha x)}{2\alpha\beta\Pi}\overrightarrow{RC},\\
\overrightarrow{O_bO_d} &= \frac{(1+\beta)(\beta RB^2-\alpha RC^2)RC^2}{2\beta\Pi}\overrightarrow{RB} + \frac{-(1+\beta)(\beta RB^2-\alpha RC^2)x}{2\beta\Pi}\overrightarrow{RC},\\
\overrightarrow{O_cO_d} &= \frac{(\beta RB^2-\alpha RC^2)(\alpha RC^2+x)}{2\alpha\Pi}\overrightarrow{RB} + \frac{-(\beta RB^2-\alpha RC^2)(RB^2+\alpha x)}{2\alpha\Pi}\overrightarrow{RC}.
\end{aligned}$$

The six vectors between nine-point center vectors are

$$\overrightarrow{N_aN_b} = \frac{2\beta x^2+\beta(\beta RB^2-\alpha RC^2)x-(\beta RB^2+\alpha RC^2)RC^2}{4\beta\Pi}\overrightarrow{RB} + \frac{2\alpha\beta x^2-(\beta RB^2-\alpha RC^2)x-\beta(\beta RB^2+\alpha RC^2)RB^2}{4\beta\Pi}\overrightarrow{RC},$$
$$\overrightarrow{N_aN_c} = \frac{(1+\alpha)(\beta RB^2-\alpha RC^2)x}{4\alpha\Pi}\overrightarrow{RB} + \frac{(1+\alpha)[2\alpha x^2-(\beta RB^2+\alpha RC^2)RB^2]}{4\alpha\Pi}\overrightarrow{RC},$$
$$\overrightarrow{N_aN_d} = \frac{-2\beta x^2+(\beta RB^2-\alpha RC^2)x+(\beta RB^2+\alpha RC^2)RC^2}{4\Pi}\overrightarrow{RB} + \frac{2\alpha x^2+(\beta RB^2-\alpha RC^2)x-(\beta RB^2+\alpha RC^2)RB^2}{4\Pi}\overrightarrow{RC},$$
$$\overrightarrow{N_bN_c} = \frac{-2\alpha\beta x^2+\beta(\beta RB^2-\alpha RC^2)x+\alpha(\beta RB^2+\alpha RC^2)RC^2}{4\alpha\beta\Pi}\overrightarrow{RB} + \frac{2\alpha\beta x^2+\alpha(\beta RB^2-\alpha RC^2)x-\beta(\beta RB^2+\alpha RC^2)RB^2}{4\alpha\beta\Pi}\overrightarrow{RC},$$
$$\overrightarrow{N_bN_d} = \frac{(1+\beta)[-2\beta x^2+(\beta RB^2+\alpha RC^2)RC^2]}{4\beta\Pi}\overrightarrow{RB} + \frac{(1+\beta)(\beta RB^2-\alpha RC^2)x}{4\beta\Pi}\overrightarrow{RC},$$
$$\overrightarrow{N_cN_d} = \frac{-2\alpha\beta x^2-(\beta RB^2-\alpha RC^2)x+\alpha(\beta RB^2+\alpha RC^2)RC^2}{4\alpha\Pi}\overrightarrow{RB} + \frac{-2\alpha x^2+\alpha(\beta RB^2-\alpha RC^2)x+(\beta RB^2+\alpha RC^2)RB^2}{4\alpha\Pi}\overrightarrow{RC}.$$

Lemma 2.3 ([1, Lemma 5.3]). *Assume that $\square ABCD$ is not cyclic. Let R_o be the intersection point of $\ell_{O_aO_c}$ and $\ell_{O_bO_d}$, i.e.,*

$$\overrightarrow{RR_o} = \frac{(1-\beta)RB^2\cdot RC^2-(1-\alpha)RC^2x}{2\Pi}\overrightarrow{RB} + \frac{(1-\alpha)RB^2\cdot RC^2-(1-\beta)RB^2x}{2\Pi}\overrightarrow{RC}.$$

Then

$$\begin{aligned} \overrightarrow{R_oO_a} &= -\alpha\overrightarrow{R_oO_c}, & \overrightarrow{R_oO_d} &= -\beta\overrightarrow{R_oO_b}, \\ R_oO_b^2 &= \tfrac{\alpha}{\beta}\cdot\tfrac{\bar\kappa_c RC^2}{4}, & R_oO_c^2 &= \tfrac{\beta}{\alpha}\cdot\tfrac{\bar\kappa_c RB^2}{4}, \\ \overrightarrow{R_oO_b}\cdot\overrightarrow{R_oO_c} &= \tfrac{\bar\kappa_c}{4}(\overrightarrow{RB}\cdot\overrightarrow{RC}), & \tfrac{\overrightarrow{R_oO_b}\cdot\overrightarrow{R_oO_c}}{R_oO_b\cdot R_oO_c} &= \operatorname{sgn}(\alpha\beta)\tfrac{\overrightarrow{RB}\cdot\overrightarrow{RC}}{RB\cdot RC}. \end{aligned}$$

Consequently, if $\bar\kappa_c'$ and $\bar\kappa_o'$ represent the normalized cyclic characteristic and the normalized orthocentric characteristic of $\square O_aO_bO_cO_d$, respectively, then $\bar\kappa_c' = \bar\kappa_c$ and $\bar\kappa_o' = \bar\kappa_o$.

Lemma 2.4 ([1, Lemma 5.4]). *Assume that $\square ABCD$ is not orthocentric. Let R_n be the intersection point of $\ell_{N_aN_c}$ and $\ell_{N_bN_d}$, i.e.,*

$$\overrightarrow{RR_n} = \frac{(1-\beta)RB^2\cdot RC^2+(1-\alpha)RC^2x-2(1-\beta)x^2}{4\Pi}\overrightarrow{RB} + \frac{(1-\alpha)RB^2\cdot RC^2+(1-\beta)RB^2x-2(1-\alpha)x^2}{4\Pi}\overrightarrow{RC}.$$

Then

$$\begin{aligned} \overrightarrow{R_nN_a} &= -\alpha\overrightarrow{R_nN_c}, & \overrightarrow{R_nN_d} &= -\beta\overrightarrow{R_nN_b}, \\ R_nN_b^2 &= \tfrac{\alpha}{\beta}\cdot\tfrac{\bar\kappa_o RC^2}{16}, & R_nN_c^2 &= \tfrac{\beta}{\alpha}\cdot\tfrac{\bar\kappa_o RB^2}{16}, \\ \overrightarrow{R_nN_b}\cdot\overrightarrow{R_nN_c} &= \tfrac{\bar\kappa_o}{16}(\overrightarrow{RB}\cdot\overrightarrow{RC}), & \tfrac{\overrightarrow{R_nN_b}\cdot\overrightarrow{R_nN_c}}{R_nN_b\cdot R_nN_c} &= \operatorname{sgn}(\alpha\beta)\tfrac{\overrightarrow{RB}\cdot\overrightarrow{RC}}{RB\cdot RC}. \end{aligned}$$

Consequently, if $\bar\kappa_c''$ and $\bar\kappa_o''$ represent the normalized cyclic characteristic and the normalized orthocentric characteristic of $\square N_aN_bN_cN_d$, respectively, then $\bar\kappa_c'' = \bar\kappa_c$ and $\bar\kappa_o'' = \bar\kappa_o$.

Lemma 2.5 ([1, Lemma 6.1]). *Given any pairwise distinct $I, J, K, L \in \{A, B, C, D\}$, we have $O_iO_j^2 = \frac{c_{IJ}\bar\kappa_c}{4}KL^2$ and $N_iN_j^2 = \frac{c_{IJ}\bar\kappa_o}{16}KL^2$, where*

$$c_{AB} = \frac{\alpha}{\beta}, \quad c_{BC} = \frac{1}{\alpha\beta}, \quad c_{AC} = \frac{\beta(1+\alpha)^2}{\alpha(1+\beta)^2}, \quad c_{CD} = \frac{\beta}{\alpha}, \quad c_{AD} = \alpha\beta, \quad \text{and} \quad c_{BD} = \frac{\alpha(1+\beta)^2}{\beta(1+\alpha)^2}.$$

3. Quadrilaterals and Their Circumcenter Quadrilaterals

In this section, we find the conditions under which a noncyclic quadrilateral $\square ABCD$ and its circumcenter quadrilateral $\square O_aO_bO_cO_d$ are similar. By [1, Theorem 2.3], we have $\square ABCD \cong \square O_aO_bO_cO_d$ if $\square ABCD$ is orthocentric. We assume that $\square ABCD$ is a general quadrilateral.

A *trapezoid* is a quadrilateral with at least one pair of parallel sides. A trapezoid $ABCD$ is a convex quadrilateral, so its diagonals $\overline{AC}$ and $\overline{BD}$ intersect at a point R. See Figure 1. In this case, we have $\overrightarrow{RA} = -\alpha\overrightarrow{RC}$ and $\overrightarrow{RD} = -\beta\overrightarrow{RB}$, where α and β are positive numbers.

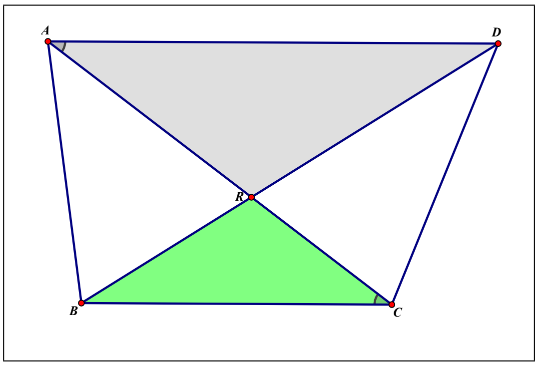

Figure 1. Trapezoid $ABCD$

Lemma 3.1. *(1) $\square ABCD$ is a trapezoid with $\overline{AD}//\overline{BC}$ if and only if $\alpha = \beta$.*
(2) $\square ABCD$ is a trapezoid with $\overline{AB}//\overline{CD}$ if and only if $\alpha = \frac{1}{\beta}$.

Proof. (1) Suppose that $\square ABCD$ is a trapezoid with $\overline{AD}//\overline{BC}$. Then $\triangle RBC \sim \triangle RDA$ so that $\alpha = \frac{RA}{RC} = \frac{RD}{RB} = \beta$.
Conversely, suppose that $\alpha = \beta$ for $\square ABCD$. By [1, Lemma 3.1], α and β are positive numbers. Since $\frac{RA}{RC} = \alpha = \beta = \frac{RD}{RB}$ and $\angle ARD = \angle CRB$, we get $\triangle RBC \sim \triangle RDA$. Then we have $\angle RAD = \angle RCB$, so $\overline{AD}//\overline{BC}$
The statement (2) can be proved similarly. □

Theorem 3.1. *Let $\square ABCD$ be a convex general quadrilateral. Then $\square ABCD$ and its circumcenter quadrilateral $\square O_aO_bO_cO_d$ are similar if and only if $\square ABCD$ is a trapezoid.*

Proof. Since the general quadrilateral $\square ABCD$ is not cyclic, we have $\angle A + \angle C \neq 180°$ and $\angle B + \angle D \neq 180°$. As labelled in Figure 2, let K, L, M, and N be the midpoints of $\overline{AB}$, $\overline{BC}$, $\overline{CD}$, and $\overline{AD}$, respectively. Consider the circles $\mathcal{C}(AO_c)$, $\mathcal{C}(BO_d)$, $\mathcal{C}(CO_a)$, and $\mathcal{C}(DO_b)$ with diameters $\overline{AO_c}$, $\overline{BO_d}$, $\overline{CO_a}$, and $\overline{DO_b}$, respectively. By the convexity of $\square ABCD$,

$$\angle A + \angle O_c = 180°, \quad \angle B + \angle O_d = 180°, \quad \angle C + \angle O_a = 180°, \quad \text{and} \quad \angle D + \angle O_b = 180°.$$

$\Rightarrow$) At least one of the angles of $\square ABCD$ is not equal to $90°$, since $\square ABCD$ is not cyclic. Without loss of generality, we assume that $\angle C \neq 90°$. Then $\angle O_a = 180° - \angle C \neq 90°$.

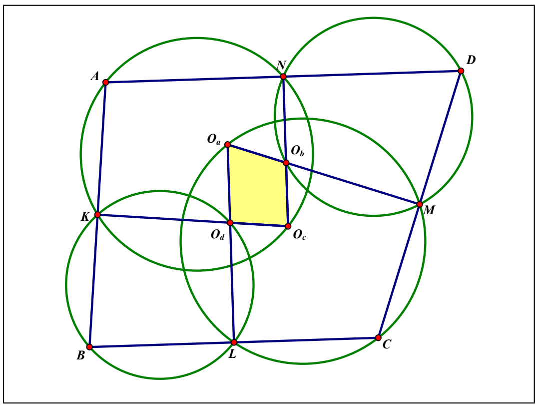

Figure 2. Convex General $\square ABCD$ and Its Circumcenter Quadrilateral

Assume one of the eight possible correspondences between $\square ABCD$ and $\square O_aO_bO_cO_d$ is a similarity correspondence, say, $\square WXYZ \sim \square O_aO_bO_cO_d$. Then $W \notin \{A, C\}$. For otherwise, if $W = A$, then $\angle A = \angle O_a = 180^\circ - \angle C$ implies that $\angle A + \angle C = 180^\circ$, a contradiction; on the other hand, if $W = C$, then $\angle C = \angle O_a = 180^\circ - \angle C$ implies that $\angle C = 90^\circ$, a contradiction. Consequently, we have $W = B$ or D.
If $W = B$, then $\angle B = \angle O_a = 180^\circ - \angle C$, or equivalently, $\angle B + \angle C = 180^\circ$. So $\square ABCD$ is a trapezoid with $\overline{AB}//\overline{CD}$.
If $W = D$, then $\angle D = \angle O_a = 180^\circ - \angle C$, or equivalently, $\angle C + \angle D = 180^\circ$. So $\square ABCD$ is a trapezoid with $\overline{AD}//\overline{BC}$.
$\Leftarrow$) Assume that $\square ABCD$ is a trapezoid with $AD//BC$. By Lemma 3.1, we get $\alpha = \beta$. Since $\triangle RBC \sim \triangle RDA$, we also have $AD = \alpha BC$. By Lemma 2.5,

$$O_aO_b^2 = \tfrac{\alpha\bar{\kappa}_c}{4\beta}CD^2 = \tfrac{\bar{\kappa}_c}{4}CD^2,$$
$$O_bO_c^2 = \tfrac{\bar{\kappa}_c}{4\alpha\beta}AD^2 = \tfrac{\bar{\kappa}_c}{4\alpha^2}AD^2 = \tfrac{\bar{\kappa}_c}{4}BC^2,$$
$$O_cO_d^2 = \tfrac{\beta\bar{\kappa}_c}{4\alpha}AB^2 = \tfrac{\bar{\kappa}_c}{4}AB^2,$$
$$O_aO_d^2 = \tfrac{\alpha\beta\bar{\kappa}_c}{4}BC^2 = \tfrac{\alpha^2\bar{\kappa}_c}{4}BC^2 = \tfrac{\bar{\kappa}_c}{4}AD^2,$$
$$O_aO_c^2 = \tfrac{\beta(1+\alpha)^2\bar{\kappa}_c}{4\alpha(1+\beta)^2}BD^2 = \tfrac{\bar{\kappa}_c}{4}BD^2,$$
$$O_bO_d^2 = \tfrac{\alpha(1+\beta)^2\bar{\kappa}_c}{4\beta(1+\alpha)^2}AC^2 = \tfrac{\bar{\kappa}_c}{4}AC^2.$$

So $\square ABCD \sim \square O_dO_cO_bO_a$.
Assume that $\square ABCD$ is a trapezoid with $AB//CD$. By Lemma 3.1, we get $\beta = \frac{1}{\alpha}$. An analogous argument can prove that $\square ABCD \sim \square O_bO_aO_dO_c$. $\square$

Theorem 3.2. *Let $\square ABCD$ be a nonconvex general quadrilateral with the interior angle at vertex A greater than 180°. Then $\square ABCD$ and its circumcenter quadrilateral $\square O_aO_bO_cO_d$ are similar if and only if*

$$\overrightarrow{RD} = -\overrightarrow{RB} \quad \text{and} \quad \overrightarrow{RA}\cdot\overrightarrow{RC} = -\overrightarrow{RB}\cdot\overrightarrow{RD}.$$

Proof. Since $\square ABCD$ is nonconvex and $\angle BAD > 180^\circ$, we have $\alpha < 0$ and $\beta > 0$.

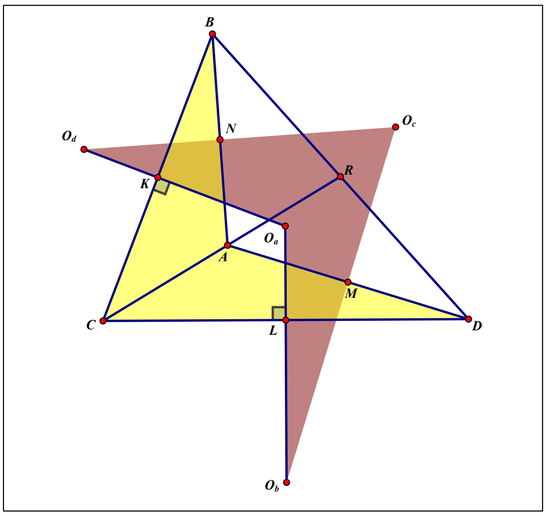

Figure 3. Nonconvex Quadrilateral $\square ABCD$ and It Circumcenter Quadrilateral

$\Rightarrow$) Assume that $\square ABCD$ and $\square O_aO_bO_cO_d$ are similar, say, $\square WXYZ \sim \square O_aO_bO_cO_d$. Since $\angle O_bO_aO_d > 180^\circ$, we have $W = A$ and $Y = C$; that is, either $\square ADCB \sim \square O_aO_bO_cO_d$ or $\square ABCD \sim \square O_aO_bO_cO_d$. See Figure 3.

Case 1: $\square ADCB \sim \square O_aO_bO_cO_d$

By Lemma 2.3, we have $\overrightarrow{R_oO_d} = -\beta\overrightarrow{R_oO_b}$. Since $\square ADCB \sim \square O_aO_bO_cO_d$, we also have $\overrightarrow{RB} = -\beta\overrightarrow{RD}$. Then $\overrightarrow{RD} = -\beta\overrightarrow{RB}$ and $\overrightarrow{RB} = -\beta\overrightarrow{RD}$ force $\beta = 1$, i.e., $\overrightarrow{RD} = -\overrightarrow{RB}$. Note that $\triangle RBC \sim \triangle R_oO_dO_c$ and $R_oO_b = R_oO_d$. By Lemma 2.3,

$$\begin{aligned}
\frac{R_oO_d^2}{RB^2} = \frac{R_oO_c^2}{RC^2} &\Rightarrow \frac{R_oO_b^2}{RB^2} = \frac{R_oO_c^2}{RC^2} \\
&\Rightarrow \frac{\alpha\bar{\kappa}_cRC^2}{4RB^2} = \frac{\bar{\kappa}_cRB^2}{4\alpha RC^2} \\
&\Rightarrow \alpha^2RC^4 = RB^4 \\
&\Rightarrow -\alpha RC^2 = RB^2 \\
&\Rightarrow \overrightarrow{RA}\cdot\overrightarrow{RC} = -\overrightarrow{RB}\cdot\overrightarrow{RD}.
\end{aligned}$$

Case 2: $\square ABCD \sim \square O_aO_bO_cO_d$

Note that $\triangle RBC \sim \triangle R_oO_bO_c$. By Lemma 2.3,

$$\begin{aligned}
\frac{R_oO_b^2}{RB^2} = \frac{R_oO_c^2}{RC^2} &\Rightarrow \frac{\alpha\bar{\kappa}_cRC^2}{4\beta RB^2} = \frac{\beta\bar{\kappa}_cRB^2}{4\alpha RC^2} \\
&\Rightarrow \alpha^2RC^4 = \beta^2RB^4 \\
&\Rightarrow -\alpha RC^2 = \beta RB^2 \\
&\Rightarrow \overrightarrow{RA}\cdot\overrightarrow{RC} = -\overrightarrow{RB}\cdot\overrightarrow{RD}
\end{aligned}$$

and

$$\frac{\overrightarrow{R_oO_b}\cdot\overrightarrow{R_oO_c}}{R_oO_b\cdot R_oO_c} = \frac{\overrightarrow{RB}\cdot\overrightarrow{RC}}{RB\cdot RC} \Rightarrow \operatorname{sgn}(\alpha\beta)\frac{\overrightarrow{RB}\cdot\overrightarrow{RC}}{RB\cdot RC} = \frac{\overrightarrow{RB}\cdot\overrightarrow{RC}}{RB\cdot RC}$$
$$\Rightarrow -\frac{\overrightarrow{RB}\cdot\overrightarrow{RC}}{RB\cdot RC} = \frac{\overrightarrow{RB}\cdot\overrightarrow{RC}}{RB\cdot RC}$$
$$\Rightarrow \overrightarrow{RB}\cdot\overrightarrow{RC} = 0$$
$$\Rightarrow (\overrightarrow{RA}\cdot\overrightarrow{RB})(\overrightarrow{RC}\cdot\overrightarrow{RD}) = 0.$$

Then $\kappa_o = (\overrightarrow{RA}\cdot\overrightarrow{RC}+\overrightarrow{RB}\cdot\overrightarrow{RD})^2 - 4(\overrightarrow{RA}\cdot\overrightarrow{RB})(\overrightarrow{RC}\cdot\overrightarrow{RD}) = 0$, so $\square ABCD$ is orthocentric, a contradiction. This case cannot occur.

$\Leftarrow$) Since $\square ABCD$ is a nonconvex quadrilateral with $\overrightarrow{RD} = -\overrightarrow{RB}$ and $\overrightarrow{RA}\cdot\overrightarrow{RC} = -\overrightarrow{RB}\cdot\overrightarrow{RD}$, we have $\alpha < 0$ and $\beta = 1$. By Lemma 2.3,

$$\overrightarrow{R_oO_a} = -\alpha\overrightarrow{R_oO_c},$$
$$\overrightarrow{R_oO_d} = -\beta\overrightarrow{R_oO_b} = -\overrightarrow{R_oO_b},$$
$$R_oO_b^2 = \frac{\alpha\bar{\kappa}_c RB^2}{4\beta} = \frac{\alpha\bar{\kappa}_c RC^2}{4},$$
$$R_oO_c^2 = \frac{\beta\bar{\kappa}_c RB^2}{4\alpha} = \frac{\bar{\kappa}_c RB^2}{4\alpha},$$
$$\frac{\overrightarrow{R_oO_b}\cdot\overrightarrow{R_oO_c}}{R_oO_b\cdot R_oO_c} = \operatorname{sgn}(\alpha\beta)\frac{\overrightarrow{RB}\cdot\overrightarrow{RC}}{RB\cdot RC} = -\frac{\overrightarrow{RB}\cdot\overrightarrow{RC}}{RB\cdot RC}.$$

Next, we check the four similarities: (1) $\triangle RDC \sim \triangle R_oO_bO_c$, (2) $\triangle RAD \sim \triangle R_oO_aO_b$, (3) $\triangle RCB \sim \triangle R_oO_cO_d$, and (4) $\triangle RAB \sim \triangle R_oO_aO_d$. All of these similarities follow from Lemma 2.3 and the hypotheses that $RD = \beta RB = RB$ and

$$-\alpha RC^2 = \overrightarrow{RA}\cdot\overrightarrow{RC} = -\overrightarrow{RB}\cdot\overrightarrow{RD} = \beta RB^2 = RB^2.$$

They imply that $\square ADCB \sim \square O_aO_bO_cO_d$ with similarity factor $\frac{\sqrt{-\bar{\kappa}_c}}{2}$.

(1) $\triangle RDC \sim \triangle R_oO_bO_c$

Since

$$\frac{R_oO_b^2}{RD^2} = \frac{\alpha\bar{\kappa}_c RC^2}{4\beta RB^2} = \frac{\alpha\bar{\kappa}_c RC^2}{4RB^2} = -\frac{\bar{\kappa}_c}{4} = \frac{\bar{\kappa}_c RB^2}{4\alpha RC^2} = \frac{\beta\bar{\kappa}_c RB^2}{4\alpha RC^2} = \frac{R_oO_c^2}{RC^2} \Rightarrow \frac{R_oO_b}{RD} = \frac{\sqrt{-\bar{\kappa}_c}}{2} = \frac{R_oO_c}{RC}$$

and

$$\frac{\overrightarrow{R_oO_b}\cdot\overrightarrow{R_oO_c}}{R_oO_b\cdot R_oO_c} = \operatorname{sgn}(\alpha\beta)\frac{\overrightarrow{RB}\cdot\overrightarrow{RC}}{RB\cdot RC} = -\frac{\overrightarrow{RB}\cdot\overrightarrow{RC}}{RB\cdot RC} = \frac{\overrightarrow{RC}\cdot\overrightarrow{RD}}{RC\cdot RD} \Rightarrow \angle DRC = \angle O_bR_oO_c,$$

we get $\triangle RDC \sim \triangle R_oO_bO_c$.

In addition, we have

$$\frac{O_bO_c}{CD} = \frac{R_oO_b}{RD} = \frac{\sqrt{-\bar{\kappa}_c}}{2},$$
$$\frac{O_aO_c}{AC} = \frac{R_oO_c - R_oO_a}{RC - RA} = \frac{R_oO_c + \alpha R_oO_c}{RC + \alpha RC} = \frac{(1+\alpha)R_oO_c}{(1+\alpha)RC} = \frac{R_oO_c}{RC} = \frac{\sqrt{-\bar{\kappa}_c}}{2},$$
$$\frac{O_bO_d}{BD} = \frac{R_oO_b + R_oO_d}{RB + RD} = \frac{R_oO_b + \beta R_oO_b}{RB + \beta RB} = \frac{(1+\beta)R_oO_b}{(1+\beta)RD} = \frac{R_oO_b}{RD} = \frac{\sqrt{-\bar{\kappa}_c}}{2}.$$

(2) $\triangle RAD \sim \triangle R_oO_aO_b$

Since

$$\frac{R_oO_a^2}{RA^2} = \frac{\alpha^2 R_oO_c^2}{\alpha^2 RC^2} = \frac{R_oO_c^2}{RC^2} = -\frac{\bar{\kappa}_c}{4} \Rightarrow \frac{R_oO_a}{RA} = \frac{\sqrt{-\bar{\kappa}_c}}{2} \Rightarrow \frac{R_oO_a}{RA} = \frac{R_oO_b}{RD}$$

and

$$\frac{\overrightarrow{R_oO_a}\cdot\overrightarrow{R_oO_b}}{R_oO_a\cdot R_oO_b} = \frac{-\alpha(\overrightarrow{R_oO_b}\cdot\overrightarrow{R_oO_c})}{-\alpha(R_oO_b\cdot R_oO_c)} = \frac{\overrightarrow{R_oO_b}\cdot\overrightarrow{R_oO_c}}{R_oO_b\cdot R_oO_c} = \operatorname{sgn}(\alpha\beta)\frac{\overrightarrow{RB}\cdot\overrightarrow{RC}}{RB\cdot RC} = -\frac{\overrightarrow{RB}\cdot\overrightarrow{RC}}{RB\cdot RC}$$
$$= \frac{\overrightarrow{RC}\cdot\overrightarrow{RD}}{RC\cdot RD} = \frac{-\frac{1}{\alpha}(\overrightarrow{RA}\cdot\overrightarrow{RD})}{-\frac{1}{\alpha}(RA\cdot RD)} = \frac{\overrightarrow{RA}\cdot\overrightarrow{RD}}{RA\cdot RD} \Rightarrow \angle ARD = \angle O_aR_oO_b,$$

we get $\triangle RAD \sim \triangle R_oO_aO_b$.
In addition,

$$\frac{O_aO_b}{AD} = \frac{R_oO_b}{RD} = \frac{\sqrt{-\bar{\kappa}_c}}{2}.$$

(3) $\triangle RCB \sim \triangle R_oO_cO_d$
Since

$$\frac{R_oO_d^2}{RB^2} = \frac{\beta^2 R_oO_b^2}{\frac{1}{\beta^2}RD^2} = \frac{R_oO_b^2}{RD^2} = -\frac{\bar{\kappa}_c}{4} \Rightarrow \frac{R_oO_d}{RB} = \frac{\sqrt{-\bar{\kappa}_c}}{2} \Rightarrow \frac{R_oO_d}{RB} = \frac{R_oO_c}{RC}$$

and

$$\frac{\overrightarrow{R_oO_c}\cdot\overrightarrow{R_oO_d}}{R_oO_c\cdot R_oO_d} = \frac{-\beta(\overrightarrow{R_oO_b}\cdot\overrightarrow{R_oO_c})}{\beta(R_oO_b\cdot R_oO_c)} = -\frac{\overrightarrow{R_oO_b}\cdot\overrightarrow{R_oO_c}}{R_oO_b\cdot R_oO_c} = -\operatorname{sgn}(\alpha\beta)\frac{\overrightarrow{R_oO_b}\cdot\overrightarrow{R_oO_c}}{R_oO_b\cdot R_oO_c} = \frac{\overrightarrow{RB}\cdot\overrightarrow{RC}}{RB\cdot RC} \Rightarrow \angle BRC = \angle O_dR_oO_c,$$

we get $\triangle RCB \sim \triangle R_oO_cO_d$.
In addition,

$$\frac{O_cO_d}{BC} = \frac{R_oO_c}{RC} = \frac{\sqrt{-\bar{\kappa}_c}}{2}.$$

(4) $\triangle RAB \sim \triangle R_oO_aO_d$
Since

$$\frac{R_oO_a^2}{RA^2} = \frac{\alpha^2 R_oO_c^2}{\alpha^2 RC^2} = \frac{R_oO_c^2}{RC^2} = -\frac{\bar{\kappa}_c}{4} = \frac{R_oO_b^2}{RD^2} = \frac{\frac{1}{\beta^2}R_oO_d^2}{\beta^2 RB^2} = \frac{R_oO_d^2}{RB^2} \Rightarrow \frac{R_oO_a}{RA} = \frac{\sqrt{-\bar{\kappa}_c}}{2} = \frac{R_oO_d}{RB}$$

and

$$\frac{\overrightarrow{R_oO_a}\cdot\overrightarrow{R_oO_d}}{R_oO_a\cdot R_oO_d} = \frac{\alpha\beta(\overrightarrow{R_oO_b}\cdot\overrightarrow{R_oO_c})}{-\alpha\beta(R_oO_b\cdot R_oO_c)} = -\frac{\overrightarrow{R_oO_b}\cdot\overrightarrow{R_oO_c}}{R_oO_b\cdot R_oO_c} = -\operatorname{sgn}(\alpha\beta)\frac{\overrightarrow{RB}\cdot\overrightarrow{RC}}{RB\cdot RC} = \frac{\overrightarrow{RB}\cdot\overrightarrow{RC}}{RB\cdot RC}$$
$$= \frac{-\frac{1}{\alpha}(\overrightarrow{RA}\cdot\overrightarrow{RB})}{-\frac{1}{\alpha}(RA\cdot RB)} = \frac{\overrightarrow{RA}\cdot\overrightarrow{RB}}{RA\cdot RB} \Rightarrow \angle ARB = \angle O_aR_oO_d,$$

we get $\triangle RAB \sim \triangle R_oO_aO_d$.
In addition,

$$\frac{O_aO_d}{AB} = \frac{R_oO_a}{RA} = \frac{\sqrt{-\bar{\kappa}_c}}{2}.$$

Finally, the equations

$$\frac{O_aO_b}{AD} = \frac{O_aO_c}{AC} = \frac{O_aO_d}{AB} = \frac{O_bO_c}{CD} = \frac{O_bO_d}{BD} = \frac{O_cO_d}{BC} = \frac{\sqrt{-\bar{\kappa}_c}}{2}$$

lead to $\square\, ADCB \sim \square\, O_aO_bO_cO_d$. □

Corollary 3.1. *Assume that $\triangle BCD$ is a triangle such that $C \notin \mathcal{C}(BD)$, the circle with diameter $\overline{BD}$, and $C \notin \ell_{BD}$. Let A be the inverse of C in $\mathcal{C}(BD)$. Then $\square\, ABCD$ is a nonconvex quadrilateral that is similar to its circumcenter quadrilateral.*

Proof. Without loss of generality, we assume that $\square\, ABCD$ is not an orthocentric qudrilateral; otherwise, $\square\, O_aO_bO_cO_d$ and $\square\, ABCD$ are congruent by [1, Theorem 2.3].
Since R is the midpoint of $\overline{BD}$, we have $\overrightarrow{RD} = -\overrightarrow{RB}$. Next, by definition of inversion in a circle, A is the point on the ray beginning at R and passing through C such that $RA \cdot RC = RB^2 = RB \cdot RD$, i.e., $\overrightarrow{RA} \cdot \overrightarrow{RC} = -\overrightarrow{RB} \cdot \overrightarrow{RD}$. Also the points A and C lie on the same side of ℓ_{BD}, so $\square\, ABCD$ is a nonconvex quadrilateral. Since $\square\, ABCD$ is neither cyclic nor orthocentric, it is a general quadrilateral. If C lies outside $\mathcal{C}(BD)$, then the

interior angle of $\square ABCD$ at vertex A is greater than $180°$, while the interior angle at vertex C is greater than $180°$ when C lies inside $\mathcal{C}(BD)$. By Theorem 3.2, $\square ABCD$ and $\square O_aO_bO_cO_d$ are similar. Figure 4 illustrates the construction of $\square ADCB$. $\square$

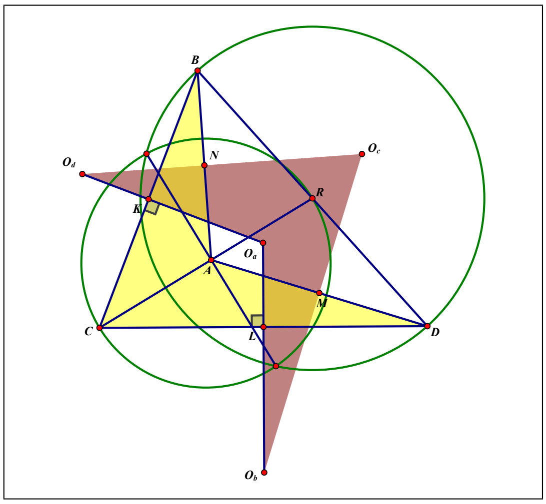

Figure 4. Constructing a Nonconvex $\square ABCD$ Similar to Its Circumcenter Quadrilateral

4. Homothetic and Congruent Quadrilaterals Under Iteration

Let $\square ABCD$ be a general quadrilateral. As earlier, $\square O_aO_bO_cO_d$ represents the circumcenter quadrilateral of $\square ABCD$. Denote the circumcenter quadrilateral of $\square O_aO_bO_cO_d$ by $\square O'_aO'_bO'_cO'_d$.
Set $Q = \square ABCD$, $Q_\circ = \square O_aO_bO_cO_d$, and $Q_{\circ\circ} = \square O'_aO'_bO'_cO'_d$. We call $Q_\circ$ the *iteration of Q by circumcenter quadrilateral*, and $Q_{\circ\circ}$ the *iteration of $Q_\circ$ by circumcenter quadrilateral*. By [1, Theorem 5.1], $Q_{\circ\circ}$ is a well-defined quadrilateral.
Next, we argue that corresponding sides of Q and $Q_{\circ\circ}$ are parallel and hence

$$\angle A = \angle O'_a, \quad \angle B = \angle O'_b, \quad \angle C = \angle O'_c, \quad \text{and} \quad \angle D = \angle O'_d. \tag{4.1}$$

Let K, L, M, and N be the midpoints of $\overline{AB}$, $\overline{BC}$, $\overline{CD}$, and $\overline{AD}$, respectively. We have $\ell_{O_cO_d} = \ell_{O_cK}$ and $\ell_{O_bO_c} = \ell_{O_cN}$. Then $\ell_{O'_aO'_b} \perp \ell_{O_cO_d}$ and $\ell_{O_cK} \perp \ell_{AB}$ imply that $\ell_{O'_aO'_b} // \ell_{AB}$ and $\ell_{O'_aO'_d} \perp \ell_{O_bO_c}$ and $\ell_{O_cN} \perp \ell_{AD}$ imply $\ell_{O'_aO'_d} // \ell_{AD}$. In the same way, we argue that $\ell_{IJ} // \ell_{O'_iO'_j}$ for all distinct $i, j \in \{a, b, c, d\}$. The equations in (4.1) follow immediately. Note that the convexity of Q in the above argument is not relevant. That said, the two figures in Figure 5 illustrate $\angle A = \angle O'_a$ in both the convex and nonconvex cases.

In the same way, we can define the iterations $Q_{\text{n}} = \square N_aN_bN_cN_d$ and $Q_{\text{nn}} = \square N'_aN'_bN'_cN'_d$ of Q by nine-point center quadrilaterals. Since $Q_\circ \sim Q_{\text{n}}$, we get

$$\angle A = \angle N'_a, \quad \angle B = \angle N'_b, \quad \angle C = \angle N'_c, \quad \text{and} \quad \angle D = \angle N'_d. \tag{4.2}$$

Both (4.1) and (4.2) suggest that $\square O'_aO'_bO'_cO'_d$ and $\square N'_aN'_bN'_cN'_d$ could be similar to $\square ABCD$. The next two theorems establish this fact in a strong sense.

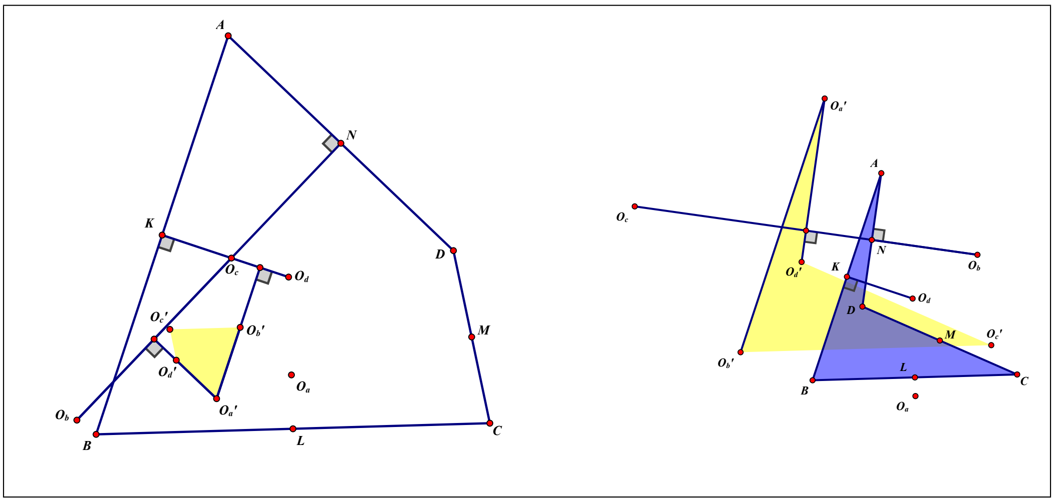

Figure 5. Two Iterations by Circumcenter Quadrilateral

Two similar quadrilaterals $\square\, ABCD$ and $\square\, A'B'C'D'$ are said to be *similarly placed* or *homothetic* if $\square\, ABCD \sim \square\, A'B'C'D'$ and their corresponding sides are parallel. In this case, if $\square\, ABCD \ncong \square\, A'B'C'D'$, then the four lines $\ell_{AA'}$, $\ell_{BB'}$, $\ell_{CC'}$, and $\ell_{DD'}$ are concurrent at a point S and $\square\, A'B'C'D'$ is obtained by dilating $\square\, ABCD$ through S by either $\pm$ the homothetic ratio $\frac{A'B'}{AB} = \frac{B'C'}{BC} = \frac{C'D'}{CD} = \frac{D'A'}{DA}$. See [2, Theorem 32 on page 38].

Theorem 4.1. *If $Q = \square\, ABCD$ is a noncyclic quadrilateral, then $Q_{\circ\circ} = O'_aO'_bO'_cO'_d$, the iteration of $Q_\circ = O_aO_bO_cO_d$ by circumcenter quadrilateral, and Q are homothetic.*

Proof. As observed earlier, $\overline{O'_iO'_j} // \overline{IJ}$ for all distinct $i, j \in \{a, b, c, d\}$. To compete the proof, we only need to check that $\square\, ABCD \sim \square\, O'_aO'_bO'_cO'_d$.
Next, we prove that

$$\frac{O'_aO'_b}{AB} = \frac{O'_bO'_c}{BC} = \frac{O'_cO'_d}{CD} = \frac{O'_dO'_a}{DA} = \frac{O'_aO'_c}{AC} = \frac{O'_bO'_d}{BD} = \frac{|\bar{\kappa}_c|}{4}.$$

Apply Lemma 2.5 first to $\square\, O_aO_bO_cO_d$ and then to $\square\, ABCD$ as follows:

$$(O'_aO'_b)^2 = \tfrac{\alpha\bar{\kappa}_c}{4\beta} O_cO_d^2 = \tfrac{\alpha\bar{\kappa}_c}{4\beta} \cdot \tfrac{\beta\bar{\kappa}_c}{4\alpha} AB^2 = \tfrac{\bar{\kappa}_c^2}{16} AB^2.$$

By analogy, $(O'_iO'_j)^2 = \frac{\bar{\kappa}_c^2}{16} IJ^2$for all distinct $i, j \in \{a, b, c, d\}$. So $\square\, ABCD \sim \square\, O'_aO'_bO'_cO'_d$ and $\frac{\bar{\kappa}_c}{4}$ is the homothetic ratio. □

Theorem 4.2. *If $Q = \square\, ABCD$ is a nonorthocentric quadrilateral, then $Q_{\mathrm{nn}} = \square\, N'_aN'_bN'_cN'_d$, the iteration of $Q_{\mathrm{n}} = \square\, N_aN_bN_cN_d$ by nine-point center quadrilateral, and Q are homothetic.*

Proof. Let R_n be the intersection point of $\ell_{N_aN_c}$ and $\ell_{N_bN_d}$. To argue that $\square\, N'_aN'_bN'_cN'_d$ and $\square\, ABCD$ are homothetic, we will prove that for all distinct $i, j \in \{a, b, c, d\}$,

$$\overrightarrow{N'_iN'_j} = \tfrac{\bar{\kappa}_o}{16}\, \overrightarrow{IJ}, \tag{4.3}$$

which implies that $\overline{N'_iN'_j} // \overline{IJ}$ for all distinct $i, j \in \{a, b, c, d\}$ and $\square\, N'_aN'_bN'_cN'_d \sim \square\, ABCD$.

Set $x = \overrightarrow{RB} \cdot \overrightarrow{RC}$ and $\Pi = RB^2 \cdot RC^2 - x^2$. The following equations hold:

$$\begin{aligned}
\overrightarrow{R_nN_a'} &= \tfrac{4\alpha\beta(1-\beta)x^2-2\alpha\beta(1-\alpha)RC^2x-\alpha(1-\beta)(\beta RB^2+\alpha RC^2)RC^2}{16\alpha\beta\Pi}\overrightarrow{RB} \\
&\quad + \tfrac{4\alpha\beta x^2-2\alpha\beta(1-\beta)RB^2x-(\beta RB^2+\alpha RC^2)(\beta RB^2+\alpha^2 RC^2)}{16\alpha\beta\Pi}\overrightarrow{RC}, \\
\overrightarrow{R_nN_b'} &= \tfrac{-4\alpha\beta^2x^2-2\alpha\beta(1-\alpha)RC^2x+\beta(\beta RB^2+\alpha RC^2)(RB^2+\alpha RC^2)}{16\alpha\beta\Pi}\overrightarrow{RB} \\
&\quad + \tfrac{4\alpha\beta(1-\alpha)x^2-2\alpha\beta(1-\beta)RB^2x-\beta(1-\alpha)(\beta RB^2+\alpha RC^2)RB^2}{16\alpha\beta\Pi}\overrightarrow{RC}, \\
\overrightarrow{R_nN_c'} &= \tfrac{4\alpha\beta(1-\beta)x^2-2\alpha\beta(1-\alpha)RC^2x-\alpha(1-\beta)(\beta RB^2+\alpha RC^2)RC^2}{16\alpha\beta\Pi}\overrightarrow{RB} \\
&\quad + \tfrac{-4\alpha^2\beta x^2-2\alpha\beta(1-\beta)RB^2x+\alpha(\beta RB^2+\alpha RC^2)(\beta RB^2+RC^2)}{16\alpha\beta\Pi}\overrightarrow{RC}, \\
\overrightarrow{R_nN_d'} &= \tfrac{4\alpha\beta x^2-2\alpha\beta(1-\alpha)RC^2x-(\beta RB^2+\alpha RC^2)(\beta^2 RB^2+\alpha RC^2)}{16\alpha\beta\Pi}\overrightarrow{RB} \\
&\quad + \tfrac{4\alpha\beta(1-\alpha)x^2-2\alpha\beta(1-\beta)RB^2x-\beta(1-\alpha)(\beta RB^2+\alpha RC^2)RB^2}{16\alpha\beta\Pi}\overrightarrow{RC}.
\end{aligned} \tag{4.4}$$

Proving the equations in (4.4) takes some work. Accepting these equations for the moment, observe that

$$\overrightarrow{N_a'N_b'} = \overrightarrow{R_nN_b'} - \overrightarrow{R_nN_a'} = \tfrac{\kappa_o}{16\alpha\beta\Pi}\overrightarrow{RB} + \tfrac{\alpha\kappa_o}{16\alpha\beta\Pi}\overrightarrow{RC} = \tfrac{\bar{\kappa}_0}{16}(\overrightarrow{RB} + \alpha\overrightarrow{RC}) = \tfrac{\bar{\kappa}_0}{16}\overrightarrow{AB}.$$

In the same way, the remaining five cases in (4.3) hold as well.
To justify (4.4), we proceed as follows.
First, apply Lemma 2.1 to the pair of $\square\, N_aN_bN_cN_d$ and $\square\, N_a'N_b'N_c'N_d'$ as follows: Set $R = R_n$, $N_a = N_a'$, $B = N_b$, and $C = N_c$. Define $x' = \overrightarrow{R_nN_b} \cdot \overrightarrow{R_nN_c}$, $\Pi' = R_nN_b^2 \cdot R_nN_c^2 - (x')^2$, and

$$\begin{aligned}
f_a &= \tfrac{(1-\beta)R_nN_b^2 \cdot R_nN_c^2-(\beta R_nN_b^2-R_nN_c^2)x'-2(1-\beta)(x')^2}{4\Pi'}, \\
g_a &= \tfrac{R_nN_b^2 \cdot R_nN_c^2+\beta R_nN_b^4+(1-\beta)R_nN_b^2x'-2(x')^2}{4\Pi'}.
\end{aligned}$$

By Lemma 2.4, $x' = \frac{\bar{\kappa}_o}{16}x$ and $\Pi' = \frac{\bar{\kappa}_o^2}{16^2}\Pi$. So

$$\begin{aligned}
f_a &= \tfrac{(1-\beta)RB^2 \cdot RC^2-(\alpha RC^2-\frac{\beta}{\alpha}RB^2)x-2(1-\beta)x^2}{4\Pi}, \\
g_a &= \tfrac{RB^2 \cdot RC^2+\frac{\alpha^2}{\beta}RC^4+\frac{\alpha(1-\beta)}{\beta}RC^2x-2x^2}{4\Pi}.
\end{aligned}$$

Then

$$\overrightarrow{R_nN_a'} = f_a\overrightarrow{R_nN_b} + g_a\overrightarrow{R_nN_c}.$$

Note that

$$\overrightarrow{RR_n} = \overrightarrow{RN_a} + \tfrac{\alpha}{1+\alpha}\overrightarrow{N_aN_c}.$$

Then

$$\overrightarrow{R_nN_b} = \overrightarrow{RN_b} - \overrightarrow{RR_n} = \overrightarrow{RN_b} - \overrightarrow{RN_a} - \tfrac{\alpha}{1+\alpha}\overrightarrow{N_aN_c} = \overrightarrow{N_aN_b} - \tfrac{\alpha}{1+\alpha}\overrightarrow{N_aN_c}.$$

In the same way, we get $\overrightarrow{R_nN_c} = \overrightarrow{N_aN_c} - \frac{\alpha}{1+\alpha}\overrightarrow{N_aN_c}$. In turn,

$$\overrightarrow{R_nN_a'} = f_a\overrightarrow{N_aN_b} + \tfrac{g_a}{1+\alpha}\overrightarrow{N_aN_c}.$$

Finally, apply Lemma 2.2 to the vectors $\overrightarrow{N_aN_b}$ and $\overrightarrow{N_aN_c}$. Define

$$\begin{aligned} f_{ab} &= \frac{2\beta x^2+\beta(\beta RB^2-\alpha RC^2)x-(\beta RB^2+\alpha RC^2)RC^2}{4\beta\Pi}, \\ g_{ab} &= \frac{2\alpha\beta x^2-(\beta RB^2-\alpha RC^2)x-\beta(\beta RB^2+\alpha RC^2)RB^2}{4\beta\Pi}, \\ f_{ac} &= \frac{(1+\alpha)(\beta RB^2-\alpha RC^2)x}{4\alpha\Pi}, \\ g_{ac} &= \frac{(1+\alpha)[2\alpha x^2-(\beta RB^2+\alpha RC^2)RB^2]}{4\alpha\Pi}. \end{aligned}$$

Then

$$\overrightarrow{R_nN'_a} = (f_a g_{ab} + \tfrac{g_a f_{ac}}{1+\alpha})\overrightarrow{RB} + (f_a f_{ab} + \tfrac{g_a g_{ac}}{1+\alpha})\overrightarrow{RC}.$$

We can show that

$$\begin{aligned} f_a g_{ab} + \tfrac{g_a f_{ac}}{1+\alpha} &= \frac{4\alpha\beta(1-\beta)x^2-2\alpha\beta(1-\alpha)RC^2x-\alpha(1-\beta)(\beta RB^2+\alpha RC^2)RC^2}{16\alpha\beta\Pi}, \\ f_a f_{ab} + \tfrac{g_a g_{ac}}{1+\alpha} &= \frac{4\alpha\beta x^2-2\alpha\beta(1-\beta)RB^2x-(\beta RB^2+\alpha RC^2)(\beta RB^2+\alpha^2 RC^2)}{16\alpha\beta\Pi}. \end{aligned}$$

The remaining three equations in (4.4) are checked in the same manner. □

Theorem 4.3. *Let $\square ABCD$ be a general quadrilateral. Let $Q_{\rm no} = \square O''_aO''_bO''_cO''_d$, the iteration of $Q_{\rm n} = \square N_aN_bN_cN_d$ by circumcenter quadrilateral. Let $Q_{\rm on} = \square N''_aN''_bN''_cN''_d$, the iteration of $Q_{\rm o} = \square O_aO_bO_cO_d$ by nine-point center quadrilateral. Then $Q_{\rm no}$ and $Q_{\rm on}$ are congruent and homothetic; moreover, the center of the homothety is the center of mass of $\square ABCD$.*

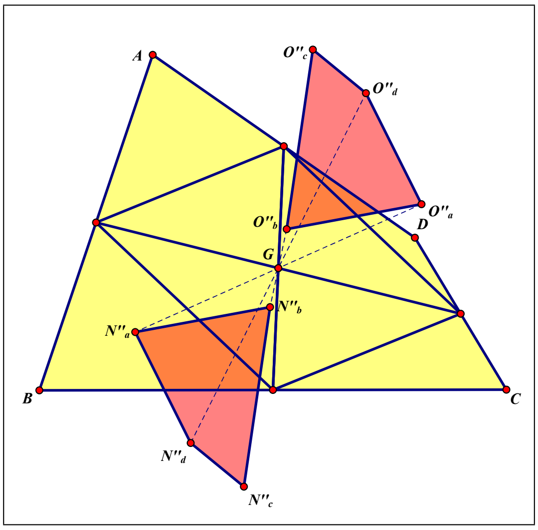

Figure 6. Two Iterations by Circumcenter and Nine-Point Center Quadrilaterals

Proof. To prove that $Q_{\rm no}$ and $Q_{\rm on}$ are congruent and homothetic, it suffices to check that $\overrightarrow{O''_iO''_j} = -\overrightarrow{N''_iN''_j}$ for all distinct $i, j \in \{a, b, c, d\}$.

First, we express the vectors $\overrightarrow{R_nN''_a}$, $\overrightarrow{R_nN''_b}$, $\overrightarrow{R_nN''_c}$, and $\overrightarrow{R_nN''_d}$ as the linear combinations of $\overrightarrow{RB}$ and $\overrightarrow{RC}$. Apply Lemma 2.1 to the pair of $Q_{\rm o}$ and $Q_{\rm on}$. This means replace $\overrightarrow{RN_a}$,

$\overrightarrow{RN_b}$, $\overrightarrow{RN_c}$, $\overrightarrow{RN_d}$, $\overrightarrow{RB}$ and $\overrightarrow{RC}$ in Lemma 2.1 with $\overrightarrow{R_oN_a''}$, $\overrightarrow{R_oN_b''}$, $\overrightarrow{R_oN_c''}$, $\overrightarrow{R_oN_d''}$, $\overrightarrow{R_oO_b}$, and $\overrightarrow{R_oO_c}$, respectively. Recall that $x = \overrightarrow{RB} \cdot \overrightarrow{RC}$ and $\Pi = RB^2 \cdot RC^2 - x^2$. Using Lemma 2.3, we get

$$\begin{aligned}
\overrightarrow{R_oN_a''} &= -\tfrac{(\beta RB^2-\alpha RC^2)[\alpha(1-\beta)RC^2+2\beta x]}{8\alpha\beta\Pi}\overrightarrow{RB} + \tfrac{(\beta RB^2-\alpha RC^2)[(\beta RB^2+\alpha^2 RC^2)+2\alpha(1-\beta)x]}{8\alpha\beta\Pi}\overrightarrow{RC},\\
\overrightarrow{R_oN_b''} &= \tfrac{(\beta RB^2-\alpha RC^2)[(RB^2+\alpha RC^2)-2(1-\alpha)x]}{8\alpha\Pi}\overrightarrow{RB} + \tfrac{(\beta RB^2-\alpha RC^2)[(1-\alpha)RB^2-2\alpha x]}{8\alpha\Pi}\overrightarrow{RC},\\
\overrightarrow{R_oN_c''} &= -\tfrac{(\beta RB^2-\alpha RC^2)[(1-\beta)RC^2-2\beta x]}{8\beta\Pi}\overrightarrow{RB} - \tfrac{(\beta RB^2-\alpha RC^2)[(\beta RB^2+RC^2)-2(1-\beta)x]}{8\beta\Pi}\overrightarrow{RC},\\
\overrightarrow{R_oN_d''} &= -\tfrac{(\beta RB^2-\alpha RC^2)[(\beta^2 RB^2+\alpha RC^2)+2\beta(1-\alpha)x]}{8\alpha\beta\Pi}\overrightarrow{RB} + \tfrac{(\beta RB^2-\alpha RC^2)[\beta(1-\alpha)RB^2+2\alpha x]}{8\alpha\beta\Pi}\overrightarrow{RC}.
\end{aligned}$$

Next, apply Lemma 2.1 to the pair of $Q_{\mathtt{n}}$ and $Q_{\mathtt{no}}$. Replace $\overrightarrow{RN_a}$, $\overrightarrow{RN_b}$, $\overrightarrow{RN_c}$, $\overrightarrow{RN_d}$, $\overrightarrow{RB}$ and $\overrightarrow{RC}$ with $\overrightarrow{R_nO_a''}$, $\overrightarrow{R_nO_b''}$, $\overrightarrow{R_nO_c''}$, $\overrightarrow{R_nO_d''}$, $\overrightarrow{R_nN_b}$, and $\overrightarrow{R_nN_c}$, respectively. Using Lemma 2.4, we get

$$\begin{aligned}
\overrightarrow{R_nO_a''} &= -\tfrac{\alpha(1-\beta)(\beta RB^2+\alpha RC^2)RC^2-2\beta(\beta RB^2-\alpha^2 RC^2)x}{8\alpha\beta\Pi}\overrightarrow{RB} - \tfrac{(\beta RB^2-\alpha^2 RC^2)(\beta RB^2+\alpha RC^2)-2\alpha^2(1-\beta)RC^2 x}{8\alpha\beta\Pi}\overrightarrow{RC},\\
\overrightarrow{R_nO_b''} &= -\tfrac{(RB^2-\alpha RC^2)(\beta RB^2+\alpha RC^2)-2\beta(1-\alpha)RB^2 x}{8\alpha\Pi}\overrightarrow{RB} - \tfrac{(1-\alpha)(\beta RB^2+\alpha RC^2)RB^2-2\alpha(RB^2-\alpha RC^2)x}{8\alpha\Pi}\overrightarrow{RC},\\
\overrightarrow{R_nO_c''} &= -\tfrac{(1-\beta)(\beta RB^2+\alpha RC^2)RC^2+2\beta(\beta RB^2-RC^2)x}{8\beta\Pi}\overrightarrow{RB} + \tfrac{(\beta RB^2-RC^2)(\beta RB^2+\alpha RC^2)+2\alpha(1-\beta)RC^2 x}{8\beta\Pi}\overrightarrow{RC},\\
\overrightarrow{R_nO_d''} &= \tfrac{(\beta^2 RB^2-\alpha RC^2)(\beta RB^2+\alpha RC^2)+2\beta^2(1-\alpha)RB^2 x}{8\alpha\beta\Pi}\overrightarrow{RB} - \tfrac{\beta(1-\alpha)(\beta RB^2+\alpha RC^2)RB^2+2\alpha(\beta^2 RB^2-\alpha RC^2)x}{8\alpha\beta\Pi}\overrightarrow{RC}.
\end{aligned}$$

For all distinct $i, j \in \{a, b, c, d\}$, $\overrightarrow{O_i''O_j''} = \overrightarrow{R_nO_j''} - \overrightarrow{R_nO_i''}$ and $\overrightarrow{N_i''N_j''} = \overrightarrow{R_oN_j''} - \overrightarrow{R_oN_i''}$. Using the above formulas, we have $\overrightarrow{O_i''O_j''} = -\overrightarrow{N_i''N_j''}$ for all distinct $i, j \in \{a, b, c, d\}$. Finally, by Lemma 2.3 and Lemma 2.4, the homothety center is given by the vector

$$\begin{aligned}
RS &= \tfrac{1}{2}(\overrightarrow{RN_a''} + \overrightarrow{RO_a''})\\
&= \tfrac{1}{2}(\overrightarrow{R_oN_a''} + \overrightarrow{R_nO_a''}) + \tfrac{1}{2}(\overrightarrow{RR_o} + \overrightarrow{RR_n})\\
&= \tfrac{1-\beta}{4}\overrightarrow{RB} + \tfrac{1-\alpha}{4}\overrightarrow{RC}.
\end{aligned}$$

On the other hand, the center of mass G of $\square\, ABCD$ is given by the vector equation

$$\begin{aligned}
\overrightarrow{RG} &= \tfrac{1}{4}(\overrightarrow{RA} + \overrightarrow{RB} + \overrightarrow{RC} + \overrightarrow{RD})\\
&= \tfrac{1}{4}(\overrightarrow{RB} + \overrightarrow{RD}) + \tfrac{1}{4}(\overrightarrow{RA} + \overrightarrow{RC})\\
&= \tfrac{1-\beta}{4}\overrightarrow{RB} + \tfrac{1-\alpha}{4}\overrightarrow{RC}.
\end{aligned}$$

It is well known that the center of mass of a quadrilateral is the intersection of the diagonals of its midpoint parallelogram. Figure 6 illustrates this fact. □

Use the letters o and n to represent the circumcenter and the nine-point center, respectively. Define Λ to be the set of all words consisting of o and n including the empty word $\emptyset$. Given $w = \mathtt{w}_1\mathtt{w}_2\cdots\mathtt{w}_k$ with $\mathtt{w}_i \in \{\mathtt{o}, \mathtt{n}\}$, we call k the *length* of w, written $\ell(w) = k$. Write $\ell_{\mathtt{o}}(w)$ for the number of times o occurring in w and $\ell_{\mathtt{n}}(w)$ for the number of times

$\mathtt{n}$ occurring in w. For example,

$$Q_{\mathtt{o}} = \square\, O_a O_b O_c O_d, \qquad Q_{\mathtt{n}} = \square\, N_a N_b N_c N_d, \qquad Q_{\mathtt{oo}} = \square\, O'_a O'_b O'_c O'_d,$$
$$Q_{\mathtt{nn}} = \square\, N'_a N'_b N'_c N'_d, \qquad Q_{\mathtt{on}} = \square\, N''_a N''_b N''_c N''_d, \qquad Q_{\mathtt{no}} = \square\, O''_a O''_b O''_c O''_d.$$

Set $Q_{\emptyset} = \square\, ABCD$. Given $w = \mathtt{w}_1 \mathtt{w}_2 \cdots \mathtt{w}_k \in \Lambda$ and $\mathtt{w}_{k+1} \in \{\mathtt{o}, \mathtt{n}\}$, set $w' = \mathtt{w}_1 \mathtt{w}_2 \cdots \mathtt{w}_k \mathtt{w}_{k+1}$ and

$$Q_{w'} = \begin{cases} \text{the circumcenter quadrilateral of } Q_w & \text{if } \mathtt{w}_{k+1} = \mathtt{o}, \\ \text{the nine-point center quadrilateral of } Q_w & \text{if } \mathtt{w}_{k+1} = \mathtt{n}. \end{cases} \tag{4.5}$$

Then $\mathcal{Q} = \{Q_w \mid w \in \Lambda\}$ is a family of quadrilaterals generated by $\square\, ABCD$, where all quadrilaterals have the same normalized cyclic and orthocentric characteristics by [1, Theorem 5.1].

We use the sides of $\square\, ABCD$ to label the corresponding sides of Q_w; for example, we write $Q_{\mathtt{on}}(\overline{IJ}) = \overline{N''_i N''_j}$ for any distinct $I, J \in \{A, B, C, D\}$.

(1) Let $w \in \Lambda$. By Lemma 2.5,

$$|Q_w(\overline{IJ})|^2 = \begin{cases} \frac{\bar{\kappa}_c^{\ell_{\mathtt{o}}(w)} \bar{\kappa}_o^{\ell_{\mathtt{n}}(w)}}{4^{\ell_{\mathtt{o}}(w)+2\ell_{\mathtt{n}}(w)}} IJ^2 & \text{if } \ell(w) \text{ is even}, \\ \frac{c_{IJ}\, \bar{\kappa}_c^{\ell_{\mathtt{o}}(w)} \bar{\kappa}_o^{\ell_{\mathtt{n}}(w)}}{4^{\ell_{\mathtt{o}}(w)+2\ell_{\mathtt{n}}(w)}} KL^2 & \text{if } \ell(w) \text{ is odd} \end{cases} \tag{4.6}$$

for all distinct $I, J \in \{A, B, C, D\}$,.

(2) If $w_1, w_2 \in \Lambda$ satisfy $\ell(w_1) \equiv \ell(w_2) \pmod 2$, then Q_{w_1} and Q_{w_2} are similar; moreover, if $\ell_o(w_1) = \ell_o(w_2)$ and $\ell_n(w_1) = \ell_n(w_2)$, then Q_{w_1} and Q_{w_2} are congruent by (4.6).

Consequently, if $Q_{\emptyset}$ is not similar to $Q_{\mathtt{o}}$, then $\mathcal{Q}$ partitions into two sets, namely, $[Q_{\emptyset}]$ and $[Q_{\mathtt{o}}]$, the similarity classes represented by $Q_{\emptyset}$ and $Q_{\mathtt{o}}$, respectively; otherwise, $\mathcal{Q}$ does not partition according to the parity of word length, since $Q_{\emptyset}$ is similar to $Q_{\mathtt{o}}$.

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