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TRIANGULAR INEQUALITIES RELATED TO AN INTERIOR POINT OF TETRAHEDRON

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ABSTRACT. In this article, I will use elementary methods to provide several triangular inequalities about an interior point of tetrahedron, which have certain theoretical value.

1. INTRODUCTION

Given a tetrahedron $A_1A_2A_3A_4$. Let P be any a point in the interior of the tetrahedron $A_1A_2A_3A_4$. Put $PA_i = R_i(i = 1, 2, 3, 4)$; let r_i denote the distance from the point P to the vertex A_i opposite face. Then

$$(R_1 + R_2 + R_3 + R_4)^3 \geq 432\,(r_1r_2r_3 + r_2r_3r_4 + r_3r_4r_1 + r_4r_1r_2)\,.$$

This is the famous Erdos-Mordell inequality in space [1]. Under the same conditions, reference [2] provided the following result

$$\frac{R_1}{R_1 + r_1} + \frac{R_2}{R_2 + r_2} + \frac{R_3}{R_3 + r_3} + \frac{R_4}{R_4 + r_4} \geq 3.$$

Interestingly, we also have

Theorem 1.1.

$$\sqrt[6]{\cos\frac{\angle A_2A_1A_3}{2}\cos\frac{\angle A_3A_1A_4}{2}\cos\frac{\angle A_4A_1A_2}{2}\sin\frac{\angle A_2PA_3}{2}\sin\frac{\angle A_3PA_4}{2}\sin\frac{\angle A_4PA_2}{2}}+$$
$$\sqrt[6]{\cos\frac{\angle A_1A_2A_3}{2}\cos\frac{\angle A_1A_2A_4}{2}\cos\frac{\angle A_3A_2A_4}{2}\sin\frac{\angle A_1PA_3}{2}\sin\frac{\angle A_1PA_4}{2}\sin\frac{\angle A_3PA_4}{2}}+$$
$$\sqrt[6]{\cos\frac{\angle A_1A_3A_2}{2}\cos\frac{\angle A_1A_3A_4}{2}\cos\frac{\angle A_2A_3A_4}{2}\sin\frac{\angle A_1PA_2}{2}\sin\frac{\angle A_1PA_4}{2}\sin\frac{\angle A_2PA_4}{2}}+$$
$$\sqrt[6]{\cos\frac{\angle A_1A_4A_2}{2}\cos\frac{\angle A_1A_4A_3}{2}\cos\frac{\angle A_2A_4A_3}{2}\sin\frac{\angle A_1PA_2}{2}\sin\frac{\angle A_1PA_3}{2}\sin\frac{\angle A_2PA_3}{2}}$$
$$< 2\sqrt{3}.$$

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2. PRELIMINARY KNOWLEDGE

In proving Theorem, we will need the following related results:
Lemma 1 In a triangle $A_1A_2A_3$, we have

$$2 < \cos^2\frac{A_1}{2} + \cos^2\frac{A_2}{2} + \cos^2\frac{A_3}{2} \le \frac{9}{4} \qquad (1)$$
$$\frac{3}{4} \le \sin^2\frac{A_1}{2} + \sin^2\frac{A_2}{2} + \sin^2\frac{A_3}{2} < 1 \qquad (2)$$

Proof Let R be the circumradius and r the inradius, let Δ denote the area.
Put $s = \frac{1}{2}(a_1 + a_2 + a_3) = \frac{1}{2}\sum a_k, a_1a_2a_3 = \prod a_k$. Then

$$\cos^2\frac{A_1}{2} = \frac{1+\cos A_1}{2} = \frac{1}{2}\left(1 + \frac{{a_2}^2 + {a_3}^2 - a_1^2}{2a_2a_3}\right) = \frac{s(s-a_1)}{a_2a_3},$$

similarly

$$\cos^2\frac{A_2}{2} = \frac{s(s-a_2)}{a_3a_1}, \cos^2\frac{A_3}{2} = \frac{s(s-a_3)}{a_1a_2}.$$

So that

$$\cos^2\frac{A_1}{2} + \cos^2\frac{A_2}{2} + \cos^2\frac{A_3}{2} = \frac{s(s-a_1)}{a_2a_3} + \frac{s(s-a_2)}{a_3a_1} + \frac{s(s-a_3)}{a_1a_2}$$
$$= \frac{s}{a_1a_2a_3}\left[2s^2 - \left(a_1^2 + a_2^2 + a_3^2\right)\right] = \frac{2s}{a_1a_2a_3}\left(\sum a_1a_2 - s^2\right).$$

By $\frac{a_1}{\sin A_1} = \frac{a_2}{\sin\ \mathrm{A}_2} = \frac{a_3}{\sin\ \mathrm{A}_3} = 2R$ and $\Delta = \frac{1}{2}a_1\mathrm{a}_2 \sin\ \mathrm{A}_3 = rs$ we obtain

$$a_1a_2a_3 = 4R\Delta = 4Rrs$$

By Heron's formula [1] $\Delta = \sqrt{s(s-a_1)(s-a_2)(s-a_3)}$ and $a_1a_2a_3 = 4Rrs$ we get

$$\sum a_1a_2 = s^2 + 4Rr + r^2.$$

We consider Euler's inequality $R \ge 2r$, then

$$\sum \cos^2\frac{A_k}{2} = \frac{2s}{a_1a_2a_3}\left(\sum a_1a_2 - s^2\right) = \frac{2s}{4Rrs}\left(s^2 + 4Rr + r^2 - s^2\right) = \frac{4R+r}{2R}$$
$$\le \frac{4R + \frac{1}{2}R}{2R} = \frac{9}{4}.$$
$$\sum \cos^2\frac{A_k}{2} = \frac{4R+r}{2R} = 2 + \frac{r}{2R} > 2,$$

Thus

$$2 < \cos^2\frac{A_1}{2} + \cos^2\frac{A_2}{2} + \cos^2\frac{A_3}{2} \le \frac{9}{4}.$$

Through the proof of (1), it is easy to obtain the proof of (2), so the proof of (2) will be omitted.
Lemma 2 In a triangle $A_1A_2A_3$, we get

$$0 < \sin^2 A_1 + \sin^2 A_2 + \sin^2 A_3 \leq \frac{9}{4} \tag{3}$$

$$\frac{3}{4} \leq \cos^2 A_1 + \cos^2 A_2 + \cos^2 A_3 < 3. \tag{4}$$

Proof By $\frac{a_1}{\sin A_1} = \frac{a_2}{\sin\ \mathrm{A}_2} = \frac{a_3}{\sin\ \mathrm{A}_3} = 2R$ and Gerretsen's inequality [1]$s^2 \leq 4R^2 + 4Rr + 3r^2$, we obtain

$$\sum \sin^2 A_k = \frac{1}{4R^2} \sum a_k^2 = \frac{1}{4R^2} \left[\left(\sum a_k \right)^2 - 2 \sum \mathrm{a}_1 \mathrm{a}_2 \right] = \frac{2s^2 - \sum a_1 a_2}{2R^2} = \frac{s^2 - 4Rr - r^2}{2R^2}$$
$$\leq \frac{4R^2 + 4Rr + 3r^2 - 4Rr - r^2}{2R^2} = \frac{2R^2 + r^2}{R^2} \leq \frac{2R^2 + \frac{1}{4}R^2}{R^2} = \frac{9}{4},$$
$$\sum \sin^2 A_k > 0,$$

Hence

$$0 < \sin^2 A_1 + \sin^2 A_2 + \sin^2 A_3 \leq \frac{9}{4}.$$

The proof of (4) will be omitted.

Lemma 3 In a triangle $A_1A_2A_3$, the following inequalities holds:

$$0 < \sin A_1 + \sin \mathrm{A}_2 + \sin \mathrm{A}_3 \leq \frac{3\sqrt{3}}{2} \tag{5}$$

$$1 < \cos A_1 + \cos \mathrm{A}_2 + \cos \mathrm{A}_3 \leq \frac{3}{2} \tag{6}$$

with equality if and only if $A_1 = A_2 = A_3 = \frac{\pi}{3}$.

3. Main results and proof

Proof of Theorem The left-hand side

$$\leq \frac{1}{6} \left(\cos \frac{\angle A_2A_1A_3}{2} + \cos \frac{\angle A_3A_1A_4}{2} + \cos \frac{\angle A_4A_1A_2}{2} + \cos \frac{\angle A_1A_2A_3}{2} + \cos \frac{\angle A_1A_2A_4}{2} \right.$$
$$+ \cos \frac{\angle A_3A_2A_4}{2} + \cos \frac{\angle A_1A_3A_2}{2} + \cos \frac{\angle A_1A_3A_4}{2} + \cos \frac{\angle A_2A_3A_4}{2} + \cos \frac{\angle A_1A_4A_2}{2} +$$
$$\left. \cos \frac{\angle A_1A_4A_3}{2} + \cos \frac{\angle A_2A_4A_3}{2} \right) + \frac{1}{6} \left(\sin \frac{\angle A_2PA_3}{2} + \sin \frac{\angle A_3PA_4}{2} + \sin \frac{\angle A_4PA_2}{2} + \right.$$
$$+ \sin \frac{\angle A_1PA_3}{2} + \sin \frac{\angle A_1PA_4}{2} + \sin \frac{\angle A_3PA_4}{2} + \sin \frac{\angle A_1PA_2}{2} + \sin \frac{\angle A_1PA_4}{2} +$$
$$\left. \sin \frac{\angle A_2PA_4}{2} + \sin \frac{\angle A_1PA_2}{2} + \sin \frac{\angle A_1PA_3}{2} + \sin \frac{\angle A_2PA_3}{2} \right)$$
$$= \frac{1}{6} \left(\cos \frac{\angle A_2A_1A_3}{2} + \cos \frac{\angle A_1A_2A_3}{2} + \cos \frac{\angle A_1A_3A_2}{2} \right) + \frac{1}{6} \left(\cos \frac{\angle A_3A_1A_4}{2} + \right.$$
$$\left. \cos \frac{\angle A_1A_3A_4}{2} + \cos \frac{\angle A_1A_4A_3}{2} \right) + \frac{1}{6} \left(\cos \frac{\angle A_2A_1A_4}{2} + \cos \frac{\angle A_1A_2A_4}{2} + \cos \frac{\angle A_1A_4A_2}{2} \right)$$

$$+\frac{1}{6}\left(\cos\frac{\angle A_3A_2A_4}{2}+\cos\frac{\angle A_2A_3A_4}{2}+\cos\frac{\angle A_2A_4A_3}{2}\right)+\frac{1}{6}\left(\sin\frac{\angle A_2PA_3}{2}+\right.$$
$$\left.+\sin\frac{\angle A_1PA_3}{2}+\sin\frac{\angle A_1PA_2}{2}\right)+\frac{1}{6}\left(\sin\frac{\angle A_1PA_3}{2}+\sin\frac{\angle A_1PA_4}{2}+\sin\frac{\angle A_3PA_4}{2}\right)+$$
$$\frac{1}{6}\left(\sin\frac{\angle A_1PA_2}{2}+\sin\frac{\angle A_1PA_4}{2}+\sin\frac{\angle A_2PA_4}{2}\right)+\frac{1}{6}\left(\sin\frac{\angle A_2PA_3}{2}+\sin\frac{\angle A_2PA_4}{2}+\right.$$
$$\left.\sin\frac{\angle A_3PA_4}{2}\right).$$

According to the properties of trihedral angles [1], we get

$$\begin{gathered}0<\angle A_2PA_3+\angle A_1PA_3+\angle A_1PA_2<2\pi\\0<\angle A_1PA_3+\angle A_1PA_4+\angle A_3PA_4<2\pi\\0<\angle A_1PA_2+\angle A_1PA_4+\angle A_2PA_4<2\pi\\0<\angle A_2PA_3+\angle A_2PA_4+\angle A_3PA_4<2\pi\end{gathered}$$

Or

$$\begin{gathered}0<\frac{1}{2}\angle A_2PA_3+\frac{1}{2}\angle A_1PA_3+\frac{1}{2}\angle A_1PA_2<\pi,\\0<\frac{1}{2}\angle A_1PA_3+\frac{1}{2}\angle A_1PA_4+\frac{1}{2}\angle A_3PA_4<\pi,\\0<\frac{1}{2}\angle A_1PA_2+\frac{1}{2}\angle A_1PA_4+\frac{1}{2}\angle A_2PA_4<\pi,\\0<\frac{1}{2}\angle A_2PA_3+\frac{1}{2}\angle A_2PA_4+\frac{1}{2}\angle A_3PA_4<\pi.\end{gathered}$$

By Lemma 3, we obtain

$$\frac{1}{3}\left(\sin\frac{\angle A_2PA_3}{2}+\sin\frac{\angle A_1PA_3}{2}+\sin\frac{\angle A_1PA_2}{2}\right)<\frac{1}{3}\times\frac{3\sqrt{3}}{2}=\frac{\sqrt{3}}{2},$$

similarly

$$\begin{gathered}\frac{1}{3}\left(\sin\frac{1}{2}\angle A_1PA_3+\sin\frac{1}{2}\angle A_1PA_4+\sin\frac{1}{2}\angle A_3PA_4\right)<\frac{\sqrt{3}}{2},\\\frac{1}{3}\left(\sin\frac{1}{2}\angle A_1PA_2+\sin\frac{1}{2}\angle A_1PA_4+\sin\frac{1}{2}\angle A_2PA_4\right)<\frac{\sqrt{3}}{2},\\\frac{1}{3}\left(\sin\frac{1}{2}\angle A_2PA_3+\sin\frac{1}{2}\angle A_2PA_4+\sin\frac{1}{2}\angle A_3PA_4\right)<\frac{\sqrt{3}}{2}.\end{gathered}$$

Thus

$$\frac{1}{6}\left(\sin\frac{\angle A_2PA_3}{2}+\sin\frac{\angle A_1PA_3}{2}+\sin\frac{\angle A_1PA_2}{2}\right)+\frac{1}{6}\left(\sin\frac{\angle A_1PA_3}{2}+\sin\frac{\angle A_1PA_4}{2}\right.$$
$$\left.+\sin\frac{\angle A_3PA_4}{2}\right)+\frac{1}{6}\left(\sin\frac{\angle A_1PA_2}{2}+\sin\frac{\angle A_1PA_4}{2}+\sin\frac{\angle A_2PA_4}{2}\right)+\frac{1}{6}\left(\sin\frac{\angle A_2PA_3}{2}\right.$$
$$\left.+\sin\frac{\angle A_2PA_4}{2}+\sin\frac{\angle A_3PA_4}{2}\right)<4\times\frac{\sqrt{3}}{4}=\sqrt{3}.$$

Since

$$\frac{\sum\cos^2\frac{A_k}{2}}{3}\geq\left(\frac{\sum\cos\frac{A_k}{2}}{3}\right)^2$$

By Lemma 1, $\sum\cos^2\frac{A_k}{2}\leq\frac{9}{4}$,
we have

$$\sum\cos\frac{A_k}{2}\leq 3\sqrt{\frac{1}{3}\sum\cos^2\frac{A_k}{2}}\leq 3\sqrt{\frac{1}{3}\times\frac{9}{4}}=\frac{3\sqrt{3}}{2},$$
$$\frac{1}{6}\left(\cos\frac{\angle A_2A_1A_3}{2}+\cos\frac{\angle A_1A_2A_3}{2}+\cos\frac{\angle A_1A_3A_2}{2}\right)+\frac{1}{6}\left(\cos\frac{\angle A_3A_1A_4}{2}+\cos\frac{\angle A_1A_3A_4}{2}\right.$$
$$\left.+\cos\frac{\angle A_1A_4A_3}{2}\right)+\frac{1}{6}\left(\cos\frac{\angle A_2A_1A_4}{2}+\cos\frac{\angle A_1A_2A_4}{2}+\cos\frac{\angle A_1A_4A_2}{2}\right)+\frac{1}{6}$$
$$\left(\cos\frac{\angle A_3A_2A_4}{2}+\cos\frac{\angle A_2A_3A_4}{2}+\cos\frac{\angle A_2A_4A_3}{2}\right)\leq 4\times\frac{1}{6}\times\frac{3\sqrt{3}}{2}=\sqrt{3}.$$

This completes the proof of Theorem.

Corollary 3.1.

$$\sqrt[6]{\sin\frac{\angle A_2A_1A_3}{2}\sin\frac{\angle A_3A_1A_4}{2}\sin\frac{\angle A_2A_1A_4}{2}\cos\frac{\angle A_2PA_3}{2}\cos\frac{\angle A_3PA_4}{2}\cos\frac{\angle A_2PA_4}{2}}+$$
$$\sqrt[6]{\sin\frac{\angle A_1A_2A_3}{2}\sin\frac{\angle A_1A_2A_4}{2}\sin\frac{\angle A_3A_2A_4}{2}\cos\frac{\angle A_1PA_3}{2}\cos\frac{\angle A_1PA_4}{2}\cos\frac{\angle A_3PA_4}{2}}+$$
$$\sqrt[6]{\sin\frac{\angle A_1A_3A_2}{2}\sin\frac{\angle A_1A_3A_4}{2}\sin\frac{\angle A_2A_3A_4}{2}\cos\frac{\angle A_1PA_2}{2}\cos\frac{\angle A_1PA_4}{2}\cos\frac{\angle A_2PA_4}{2}}+$$
$$\sqrt[6]{\sin\frac{\angle A_1A_4A_2}{2}\sin\frac{\angle A_1A_4A_3}{2}\sin\frac{\angle A_2A_4A_3}{2}\cos\frac{\angle A_1PA_2}{2}\cos\frac{\angle A_1PA_3}{2}\cos\frac{\angle A_2PA_3}{2}}$$

$<1+\sqrt{3}$.

3.1. **Proof Just like the proof method of Theorem, we can obtain.** The left-hand side

$$\le \frac{1}{6}\left(\sin\frac{\angle A_2A_1A_3}{2}+\sin\frac{\angle A_3A_1A_4}{2}+\sin\frac{\angle A_2A_1A_4}{2}+\cos\frac{\angle A_2PA_3}{2}+\cos\frac{\angle A_3PA_4}{2}\right.$$
$$\left.+\cos\frac{\angle A_2PA_4}{2}\right)+\frac{1}{6}\left(\sin\frac{\angle A_1A_2A_3}{2}+\sin\frac{\angle A_1A_2A_4}{2}+\sin\frac{\angle A_3A_2A_4}{2}+\cos\frac{\angle A_1PA_3}{2}+\right.$$
$$\left.\cos\frac{\angle A_1PA_4}{2}+\cos\frac{\angle A_3PA_4}{2}\right)+\frac{1}{6}\left(\sin\frac{\angle A_1A_3A_2}{2}+\sin\frac{\angle A_1A_3A_4}{2}+\sin\frac{\angle A_2A_3A_4}{2}+\right.$$
$$\left.+\cos\frac{\angle A_1PA_2}{2}+\cos\frac{\angle A_1PA_4}{2}+\cos\frac{\angle A_2PA_4}{2}\right)+\frac{1}{6}\left(\sin\frac{\angle A_1A_4A_2}{2}+\sin\frac{\angle A_1A_4A_3}{2}+\right.$$
$$\left.\sin\frac{\angle A_2A_4A_3}{2}+\cos\frac{\angle A_1PA_2}{2}+\cos\frac{\angle A_1PA_3}{2}+\cos\frac{\angle A_2PA_3}{2}\right)$$
$$=\frac{1}{6}\left(\cos\frac{\angle A_2PA_3}{2}+\cos\frac{\angle A_1PA_3}{2}+\cos\frac{\angle A_1PA_2}{2}\right)+\frac{1}{6}\left(\cos\frac{\angle A_3PA_4}{2}+\right.$$
$$\left.\cos\frac{\angle A_1PA_4}{2}+\cos\frac{\angle A_1PA_3}{2}\right)+\frac{1}{6}\left(\cos\frac{\angle A_2PA_4}{2}+\cos\frac{\angle A_1PA_4}{2}+\cos\frac{\angle A_1PA_2}{2}\right)$$
$$+\frac{1}{6}\left(\cos\frac{\angle A_3PA_4}{2}+\cos\frac{\angle A_2PA_4}{2}+\cos\frac{\angle A_2PA_3}{2}\right)+\frac{1}{6}\left(\sin\frac{\angle A_2A_1A_3}{2}+\right.$$
$$\left.+\sin\frac{\angle A_1A_2A_3}{2}+\sin\frac{\angle A_1A_3A_2}{2}\right)+\frac{1}{6}\left(\sin\frac{\angle A_1A_4A_3}{2}+\sin\frac{\angle A_1A_3A_4}{2}+\sin\frac{\angle A_3A_1A_4}{2}\right)+$$
$$\frac{1}{6}\left(\sin\frac{\angle A_1A_4A_2}{2}+\sin\frac{\angle A_1A_2A_4}{2}+\sin\frac{\angle A_2A_1A_4}{2}\right)+\frac{1}{6}\left(\sin\frac{\angle A_2A_4A_3}{2}+\sin\frac{\angle A_2A_3A_4}{2}+\right.$$
$$\left.\sin\frac{\angle A_3A_2A_4}{2}\right).$$

For each group of trihedral angles,such as $\angle A_2PA_3, \angle A_2PA_4, \angle A_3PA_4$, we can first obtain $\frac{1}{2}\angle A_2PA_3, \frac{1}{2}\angle A_2PA_4, \frac{1}{2}\angle A_3PA_4$, and $0 < \frac{1}{2}\angle A_2PA_3 + \frac{1}{2}\angle A_2PA_4 + \frac{1}{2}\angle A_3PA_4 < \pi$. We can approximate $\frac{1}{2}\angle A_2PA_3, \frac{1}{2}\angle A_2PA_4$, and $\frac{1}{2}\angle A_3PA_4$ as the three interior angles of a triangle. According to Lemma 3, for triangle $A_1A_2A_3, 1 < \cos A_1 + \cos \mathrm{A}_2 + \cos \mathrm{A}_3 \le \frac{3}{2}$ with equality if and only if $A_1 = A_2 = A_3 = \frac{\pi}{3}$. So $1 < \cos\frac{\angle A_2PA_3}{2} + \cos\frac{\angle A_2PA_4}{2} + \cos\frac{\angle A_3PA_4}{2} < \frac{3}{2}$. Similarly

$$1 < \cos\frac{\angle A_1PA_3}{2}+\cos\frac{\angle A_1PA_4}{2}+\cos\frac{\angle A_3PA_4}{2} < \frac{3}{2},$$
$$1 < \cos\frac{\angle A_1PA_2}{2}+\cos\frac{\angle A_1PA_4}{2}+\cos\frac{\angle A_2PA_4}{2} < \frac{3}{2},$$

$$1 < \cos\frac{\angle A_1PA_2}{2}+\cos\frac{\angle A_1PA_3}{2}+\cos\frac{\angle A_2PA_3}{2} < \frac{3}{2}$$

Using Lemma 3, we have

$$\frac{1}{6}\left(\cos\frac{\angle A_2PA_3}{2}+\cos\frac{\angle A_1PA_3}{2}+\cos\frac{\angle A_1PA_2}{2}\right)+\frac{1}{6}\left(\cos\frac{\angle A_3PA_4}{2}+\cos\frac{\angle A_1PA_4}{2}\right.$$
$$\left.+\cos\frac{\angle A_1PA_3}{2}\right)+\frac{1}{6}\left(\cos\frac{\angle A_2PA_4}{2}+\cos\frac{\angle A_1PA_4}{2}+\cos\frac{\angle A_1PA_2}{2}\right)+\frac{1}{6}\left(\cos\frac{\angle A_3PA_4}{2}\right.$$
$$\left.+\cos\frac{\angle A_2PA_4}{2}+\cos\frac{\angle A_2PA_3}{2}\right)+\frac{1}{6}\left(\sin\frac{\angle A_2A_1A_3}{2}+\sin\frac{\angle A_1A_2A_3}{2}+\sin\frac{\angle A_1A_3A_2}{2}\right)+$$
$$\frac{1}{6}\left(\sin\frac{\angle A_1A_4A_3}{2}+\sin\frac{\angle A_1A_3A_4}{2}+\sin\frac{\angle A_3A_1A_4}{2}\right)+\frac{1}{6}\left(\sin\frac{\angle A_1A_4A_2}{2}+\sin\frac{\angle A_1A_2A_4}{2}\right.$$
$$\left.+\sin\frac{\angle A_2A_1A_4}{2}\right)+\frac{1}{6}\left(\sin\frac{\angle A_2A_4A_3}{2}+\sin\frac{\angle A_2A_3A_4}{2}+\sin\frac{\angle A_3A_2A_4}{2}\right)$$
$$<4\times\frac{1}{6}\times\frac{3\sqrt{3}}{2}+4\times\frac{1}{6}\times\frac{3}{2}=1+\sqrt{3}.$$

So we complete the proof of Corollary 3.1.
Let P be a point in the interior of a triangle $A_1A_2A_3$, then

Corollary 3.2.

$$\sqrt{\cos\frac{\angle A_2A_1A_3}{2}\sin\frac{\angle A_2PA_3}{2}}+\sqrt{\cos\frac{\angle A_1A_2A_3}{2}\sin\frac{\angle A_1PA_3}{2}}+\sqrt{\cos\frac{\angle A_1A_3A_2}{2}\sin\frac{\angle A_1PA_2}{2}}$$
$$\leq\frac{3\sqrt{3}}{2}$$

with equality if and only if the triangle $A_1A_2A_3$ is equilateral and P is its center.

Proof.

$$\sqrt{\cos\frac{\angle A_2A_1A_3}{2}\sin\frac{\angle A_2PA_3}{2}}+\sqrt{\cos\frac{\angle A_1A_2A_3}{2}\sin\frac{\angle A_1PA_3}{2}}+\sqrt{\cos\frac{\angle A_1A_3A_2}{2}\sin\frac{\angle A_1PA_2}{2}}$$
$$\leq\frac{1}{2}\left(\cos\frac{1}{2}\angle A_2A_1A_3+\sin\frac{1}{2}\angle A_2PA_3\right)+\frac{1}{2}\left(\cos\frac{1}{2}\angle A_1A_2A_3+\sin\frac{1}{2}\angle A_1PA_3\right)+$$
$$\frac{1}{2}\left(\cos\frac{1}{2}\angle A_1A_3A_2+\sin\frac{1}{2}\angle A_1PA_2\right)$$
$$=\frac{1}{2}\sum\cos\frac{\angle A_k}{2}+\frac{1}{2}\sum\sin\frac{\angle A_1PA_2}{2}\leq\frac{1}{2}\times\frac{3\sqrt{3}}{2}+\frac{1}{2}\times3\sqrt{\frac{1}{3}\sum\sin^2\frac{\angle A_1PA_2}{2}}$$
$$=\frac{3\sqrt{3}}{4}+\frac{3}{2}\sqrt{\frac{1}{3}\times\frac{9}{4}}=\frac{3\sqrt{3}}{2}.$$

With equality if and only if $\cos\frac{\angle A_2A_1A_3}{2}=\sin\frac{\angle A_2PA_3}{2}, \cos\frac{\angle A_1A_2A_3}{2}=\sin\frac{1}{2}\angle A_1PA_3, \cos\frac{\angle A_1A_3A_2}{2}=\sin\frac{\angle A_1PA_2}{2}$ or the triangle $A_1A_2A_3$ is an equilateral and P is its center. □

Corollary 3.3.

$$\cos\frac{\angle A_2A_1A_3}{2}\sin\frac{\angle A_2PA_3}{2}+\cos\frac{\angle A_1A_2A_3}{2}\sin\frac{\angle A_1PA_3}{2}+\cos\frac{\angle A_2A_3A_1}{2}\sin\frac{\angle A_2PA_1}{2}$$
$$\leq\frac{9}{4}.$$

with equality if and only if the triangle $A_1A_2A_3$ is equilateral and P is its center.

Corollary 3.4.

$$\cos\frac{\angle A_2A_1A_3}{2}\cos\frac{\angle A_2PA_3}{2}+\cos\frac{\angle A_1A_2A_3}{2}\cos\frac{\angle A_1PA_3}{2}+\cos\frac{\angle A_2A_3A_1}{2}\cos\frac{\angle A_2PA_1}{2} \leq \frac{3\sqrt{3}}{4}.$$

$$\sin\frac{\angle A_2A_1A_3}{2}\sin\frac{\angle A_2PA_3}{2}+\sin\frac{\angle A_1A_2A_3}{2}\sin\frac{\angle A_1PA_3}{2}+\sin\frac{\angle A_2A_3A_1}{2}\sin\frac{\angle A_2PA_1}{2} \leq \frac{3\sqrt{3}}{4}.$$

4. Concluding remarks

Above, we discussed two trigonometric inequalities regarding an interior point of a tetrahedron [2, 3, 4, 5]. Next, I would like to ask : Can the Theorem in this article be extended to n-dimensional simplex? If possible, what form will it take?

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