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SOME REMARKS ON χ_H-QUOTIENT

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ABSTRACT. In this paper we will continue our investigation over the theory of χ_H-quotient for Riemannian manifolds. The χ_H-quotient was introduced by us in paper [5]. This paper is dedicated to find some new inequalities in this respect.

1. INTRODUCTION

Few invariants are known in Finsler-Riemannian geometry. In this respect there was many research papers in present days.
As we know B_x^n represents in Finsler geometry the unit ball in a Finsler space centered at $p \in M$, where (M, F) is a Finsler manifold, i.e.

$$B_x^n = \{y \in \mathbb{R}^n | \, |y| = F(x,y) < 1\}$$

and B^n represents the unit ball in the Euclidean space centered at origin:

$$B^n = \{x \in \mathbb{R}^n | \, |x| < 1\}$$

where $|x| = \sqrt{\delta_{ij}x^i x^j}$.
Also, in Finsler geometry we know that $\sigma_F(x)$ is given by:

$$\sigma_F(x) = \frac{volume(B^n)}{volume(B_x^n)}$$

Let now recall one classical definition

Definition 1.1. *The distorsion $\tau = \tau(x,y)$ for a Finsler space is defined by:*

$$\tau(x,y) = ln\left(\frac{\sqrt{det(g_{ij})}}{\sigma_F(x)}\right)$$

Acording also to Shen, when $F = \sqrt{g_{ij}(x)y^i y^j}$, is Riemannian, then $\sigma_F(x) = \sqrt{det(g_{ij}(x))}$.

Definition 1.2. *([6]) A pseudo-Riemannian metric of metric signature (p,q) on a smooth manifold M of dimension $n = p+q$, is a smooth symmetric differentiable 2-form g on M, such that, at each point $x \in M$, g_x is non-degenerate on T_xM with the signature (p,q). We call (M,g) a pseudo-Riemannian manifold.*

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For a coordinate chart $(U, x^1, ..., x^n)$, the Christoffel symbols Γ^k_{ij} of the Levi-Civita connection are related to the components of the metric g in the following way:

$$\Gamma^k_{ij} = \frac{1}{2}\left(\frac{\partial g_{li}}{\partial x^j} + \frac{\partial g_{jl}}{\partial x^i} - \frac{\partial g_{ij}}{\partial x^l}\right)$$

If $f : M \to \mathbb{R}$ is a smooth function, then the second covariant derivative of the function f is given by:

$$\nabla^2_g f = \left(\frac{\partial^2 f}{\partial x^i \partial x^j} - \Gamma^k_{ij}\frac{\partial f}{\partial x^k}\right) dx^i \otimes dx^j$$

is called the Hessian of the function f.
In work [2], the authors have used the following notations, that we will also use in this paper:

$$f_{,i} = \frac{\partial f}{\partial x^i}, f_{,ij} = \frac{\partial^2 f}{\partial x^i \partial x^j} - \Gamma^k_{ij} f_{,m}$$

$$f_{,ijk} = \frac{\partial f_{,ij}}{\partial x^k} - \Gamma^l_{ki} f_{,lj} - \Gamma^l_{kj} f_{,li}.$$

Definition 1.3. *([5]) For a pseudo-Riemannian manifold (M, g), we will denote by*

$$\chi_H = \frac{\|Hf(x)\|_{HS}}{\|Hf_1(x_1)\|_{HS}} \tag{1.1}$$

the Hessian χ-quotient for two smooth function $f, f_1 : M \to \mathbb{R}$. Here $\|\cdot\|_{HS}$ represent the Frobenius (Hilbert-Schmidt) norm of a Hessian matrix attached to the pseudo-Riemannian manifold. Here the point x represent the critical point for the first Hessian of the smooth function f and the point x_1 represent the critical point of the Hessian of the second function f_1.

Some properties of the χ-quotient was also presented in paper [5].
Some important results regarding the "size" of a matrix were established in a series of papers recently. In this respect, please see [8], [1]
As we know, the two norm for a matrix A is given by

$$\|A\|_2 = max_{\|x\|=1}\|Ax\| = max_{x\neq 0}\frac{\|Ax\|}{\|x\|}$$

The Frobenius (or the Hilbert-Schmidt norm) of a matrix $A = (A_{ij})$ is defined as follows:

$$\|A\|_{HS} = \sqrt{\sum_{i=1}^{n}\sum_{j=1}^{n} A^2_{ij}}.$$

The operator norm of a matrix $A = (A_{ij})$ is given by

$$\|A\|_{op} = max_{\|x\|=1}\|Ax\|$$

. Next, we will be focused on Hilbert-Scwartz norm (or Frobenius norm) $\|\cdot\|_{HS}$ because we will use it to established some new main results of this paper.

$$\|A \cdot B\|_{HS} \leq \|A\|_{HS} \cdot \|B\|_{HS} \tag{1.2}$$

$$\|A \cdot B\|_{op} \leq \|A\|_{op} \cdot \|B\|_{op} \tag{1.3}$$

$$\|A\|_{HS} \leq \sqrt{n}\,\|A\|_{op} \tag{1.4}$$

In the previous inequality, the equality take place when $A = I_n$.

$$|det(A)| \leq \|A\|_{HS}^{n} \tag{1.5}$$

Here, A denotes any positive definite symmetric matrix, r is the rank of A, and $\sigma_{min}(A)$ denotes the minimum singular value of A.
Some interesting results from paper [4], [1] will be presented next. The commutator of two matrices according to [4] is given in the following conjecture:

Theorem 1.1. *Suppose $n \in \mathbb{N}$ is arbitrary. Let X and Y be two nxn matrices. Then:*

$$|[A,B]| \leq 2\,\|A\|_{HS}^{2} \cdot \|B\|_{HS}^{2} \tag{1.6}$$

2. Main Results

Starting from (1.6), we are ready now to give our first result for this paper:

Theorem 2.1. *Suppose $n \in \mathbb{N}$ is arbitrary. Let $Hf(x)$ and $Hf_1^{-1}(x_1)$ be the Hessian matrices attached to the functions $f, f_1 : M \to \mathbb{R}$. Then the following inequalities holds:*

$$\left|\left[Hf(x), Hf_1^{-1}(x_1)\right]\right| \leq \sqrt{2}\chi_H \leq \|Hf(x)\|_{HS}^{2} + \left\|Hf_1^{-1}(x_1)\right\|_{HS}^{2} \tag{2.1}$$

Proof. Starting from (1.6), we will proof the first part of the above inequality. Replacing in (1.6), the matrix A and B, with $Hf(x)$ and $Hf_1^{-1}(x_1)$, we get:

$$\left|\left[Hf(x), Hf_1^{-1}(x_1)\right]\right| \leq \sqrt{2}\,\|Hf(x)\|_{HS}\left\|Hf_1^{-1}(x_1)\right\|_{HS}$$

. But, $\chi_H = \|Hf(x)\|_{HS}\left\|Hf_1^{-1}(x_1)\right\|_{HS}$, so the first part of the theorem is proved. □

For the proof of the second part of the theorem, we can observe that this part follows directly from the means inequality.
One interesting inequality is presented in Crux Math. from 2002:

$$\frac{1}{2a} + \frac{1}{2b} \geq \frac{2}{1+ab} \tag{2.2}$$

for every two nonnegative numbers $a, b > 0$. Using this inequality, we are ready now for the following theorem:

Theorem 2.2.

$$\frac{1}{2\,\|Hf(x)\|_{HS}} + \frac{1}{2\left\|Hf_1^{-1}(x_1)\right\|_{HS}} \geq \frac{2}{1+\chi_H} \tag{2.3}$$

The mathematical means are very important in the theory of inequalities. Let us remember some of them for two strictly positive numbers a and b:
Arithmetic mean:

$$m_a = \frac{a+b}{2}$$

Geometric mean:

$$m_a = \sqrt{ab}$$

Harmonic mean:

$$m_a = \frac{2ab}{a+b} = \frac{2}{\frac{1}{a}+\frac{1}{b}}.$$

We are ready to give now the following Lemma:

Lemma 2.1. *For two strictly positive numbers a and b, the following inequality holds:*

$$9m_g \leq 5m_a + 4m_h \tag{2.4}$$

Proof. Using the link between means:

$$m_h = \frac{m_g^2}{m_a}$$

we have:
We start with the following true identity

$$4(m_a - m_g)^2 + m_a(m_a - m_g) \geq 0$$

$$4(m_a^2 - 2m_a m_g + m_g^2) + m_a^2 - m_a m_g \geq 0$$

$$5m_a^2 - 9m_a m_g + 4m_g^2 \geq 0$$

$$5m_a^2 - 5m_a m_g \geq 4m_a m_g - 4m_g^2$$

$$5(m_a - m_g) \geq 4\left(m_g - \frac{m_g^2}{m_a}\right)$$

Using now

$$m_h = \frac{m_g^2}{m_a}$$

Finally we get:

$$5m_a - 5m_g \geq 4m_g - 4m_h$$

and from this we get the desired inequality

$$9m_g \leq 5m_a + 4m_h$$

□

Using the above Lemma, we are ready now to formulate our next result:

Theorem 2.3. *Suppose $n \in \mathbb{N}$ is arbitrary. Let $Hf(x)$ and $Hf_1^{-1}(x_1)$ be the Hessian matrices attached to the functions $f, f_1 : M \to \mathbb{R}$. Then the following inequalities holds:*

$$\frac{5}{2}\left(\|Hf(x)\|_{HS} + \left\|Hf_1^{-1}(x_1)\right\|_{HS}\right) + \frac{8}{\|Hf(x)\|_{HS} + \left\|Hf_1^{-1}(x_1)\right\|_{HS}} \geq 9\sqrt{\chi_H} \tag{2.5}$$

The proof of this theorem is directly applying the previous Lemma. Finally, let us consider the following polynomial $P(t) = t^2 - ln(t) \geq 0$ for $t > 1$. The following inequality holds:

$$P((x+y)^2) \geq P(\sqrt{2xy})$$

because:

$$(x+y)^2 - 2ln(x+y) \geq 2xy - ln(\sqrt{2xy})$$

for every $x, y > 1$. The above inequality could be rewritten using the mathematical means as follows:

$$P(4m_a^2) \geq P(\sqrt{2}m_g) \tag{2.6}$$

We can now formulate now our last result

Theorem 2.4. *Suppose $n \in \mathbb{N}$ is arbitrary. Let $Hf(x)$ and $Hf_1^{-1}(x_1)$ be the Hessian matrices attached to the functions $f, f_1 : M \to \mathbb{R}$. Then the following inequalities holds:*

$$\left(\|Hf(x)\|_{HS} + \left\|Hf_1^{-1}(x_1)\right\|_{HS}\right)^4 - ln\left(\|Hf(x)\|_{HS} + \left\|Hf_1^{-1}(x_1)\right\|_{HS}\right) \geq 2\chi_H^2 - ln(\sqrt{2}\chi_H). \tag{2.7}$$

The proof is direct using the previous inequality.

3. Conclusion

In this paper, we have investigated some interesting inequalities of the χ_H quotient. Finnaly, let us observe that also a lot of inequalities can be formulated using the χ_H quotient, so further investigations in this respect could be done using not only classical inequalities but also derived inequalities from classical ones.

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