Global Journal of Advanced Research
on Classical and Modern Geometries
ISSN: 2284-5569, Vol.14, (2025), Issue 1, pp.26-43

SUPER TWISTED PRODUCTS

TONG WU[1], YONG WANG[2,*], XUE WANG[2]

Abstract. In the paper,we define the W_2-curvature tensor on super Riemannian manifolds. And we compute the curvature tensor, the Ricci tensor and the W_2-curvature tensor on super twisted product spaces. Furthermore, we investigate the W_2-curvature flat super twisted product manifolds. Finally, we get a result that a mixed Ricci-flat super twisted product semi-Riemannian manifold can be expressed as a super warped product semi-Riemannian manifold.

1. Introduction and motivations

The concept of warped products was first introduced by Bishop and ONeil (see [2]) to construct examples of Riemannian manifolds with negative curvature. Singly warped products have a natural generalization. The (singly) twisted product $B \times_h F$ of two pseudo-Riemannian manifolds (B, g_B) and (F, g_F) with a smooth function $h : B \times F \to (0, \infty)$ is the product manifold $B \times F$ with the metric tensor $g = g_B \oplus h^2 g_F$. Here, (B, g_B) is called the base manifold, (F, g_F) is called as the fiber manifold and h is called as the warping function. In Riemannian geometry, warped product manifolds and their generic forms have been used to construct new examples with interesting curvature properties since then. In [5], F. Dobarro and E. Dozo had studied from the viewpoint of partial differential equations and variational methods, the problem of showing when a Riemannian metric of constant scalar curvature can be produced on a product manifolds by a warped product construction. In [6], Ehrlich, Jung and Kim got explicit solutions to warping function to have a constant scalar curvature for generalized Robertson-Walker space-times. In [1], explicit solutions were also obtained for the warping function to make the space-time as Einstein when the fiber is also Einstein. It is shown that a mixed Ricci-flat twisted product semi-Riemannian manifold can be expressed as a warped product semi-Riemannian manifold in [10].
Pokhariyal and Mishra first defined the W_2-curvature tensor and they studied its physical and geometrical properties in [11]. In [13] and [14], Sular and Özgur studied warped product manifolds with a semi-symmetric metric connection and a semi-symmetric non-metric connection, they computed curvature of semi-symmetric metric connection and semi-symmetric non-metric connection and considered Einstein warped product manifolds with a semi-symmetric metric connection and a semi-symmetric non-metric connection. In

2010 Mathematics Subject Classification. 53C40; 53C42.
Key words and phrases. The W_2-curvature tensor; the curvature tensor; Ricci tensor; super twisted product space; mixed Ricci-flat.

[15], Wang studied the Einstein multiply warped products with a semi-symmetric metric connection and the multiply warped products with a semi-symmetric metric connection with constant scalar curvature.
On the other hand, in [3], the definition of super warped product spaces was given. Einstein warped products were studied in [7]. In [8], several new super warped product spaces were given and the authors also studied the Einstein equations with cosmological constant in these new super warped product spaces. In [16], Wang studied super warped product spaces with a semi-symmetric metric connection. In [12], Shenawy. S and Ünal. B studyed the W_2-curvature tensor on (singly) warped product manifolds as well as on generalized Robertson-Walker and standard static space-time and investigated W_2-curvature flat warped product manifolds. In this paper, we define the W_2-curvature tensor on super twisted products. Our motivation is to study super twisted products and explore the Ricci tensor and the W_2-curvature tensor on super twisted product manifolds. This paper is organized as follows. In Section 2, we state some definitions of super manifolds and super Riemannian metrics. We also define the W_2-curvature tensor on super Riemannian manifolds. In Section 3, we compute the curvature tensor, the Ricci tensor on super twisted product spaces. Further, we give the W_2-curvature tensor of the Levi-civita connection on super twisted product spaces. In Section 4, we investigate W_2-curvature flat super twisted product manifolds. Finally, we get a result that a mixed Ricci-flat super twisted product semi-Riemannian manifold can be expressed as a super warped product semi-Riemannian manifold.

2. Preliminaries

In this section, we give some definitions about Riemannian supergeometry.

Definition 2.1. (Definition 1 in [3]) A locally $\mathbb{Z}_2$-ringed space is a pair $S := (|S|, \mathcal{O}_S)$ where $|S|$ is a second-countable Hausdorff space, and a $\mathcal{O}_S$ is a sheaf of $\mathbb{Z}_2$-graded $\mathbb{Z}_2$-commutative associative unital $\mathbb{R}$-algebras, such that the stalks $\mathcal{O}_{S,p}$, $p \in |S|$ are local rings.

In this context, $\mathbb{Z}_2$-commutative means that any two sections $s, t \in \mathcal{O}_S(|U|)$, $|U| \subset |S|$ open, of homogeneous degree $|s| \in \mathbb{Z}_2$ and $|t| \in \mathbb{Z}_2$ commute up to the sign rule $st = (-1)^{|s||t|} ts$. $\mathbb{Z}_2$-ring space $U^{m|n} := (U, C^\infty_{U^m} \otimes \wedge \mathbb{R}^n)$, is called standard superdomain where $C^\infty_{U^m}$ is the sheaf of smooth functions on U and $\wedge \mathbb{R}^n$ is the exterior algebra of $\mathbb{R}^n$. We can employ (natural) coordinates $x^I := (x^a, \xi^A)$ on any $\mathbb{Z}_2$-domain, where x^a form a coordinate system on U and the ξ^A are formal coordinates.

Definition 2.2. (Notation and preliminary concepts in [4]) A supermanifold of dimension $m|n$ is a super ringed space $M = (|M|, \mathcal{O}_M)$ that is locally isomorphic to $\mathbb{R}^{m|n}$ and $|M|$ is a second countable and Hausdorff topological space.

The tangent sheaf $\mathcal{T}M$ of a $\mathbb{Z}_2$-manifold M is defined as the sheaf of derivations of sections of the structure sheaf, i.e., $\mathcal{T}M(|U|) := \mathrm{Der}(\mathcal{O}_M(|U|))$, for arbitrary open set $|U| \subset |M|$. Naturally, this is a sheaf of locally free $\mathcal{O}_M$-modules. Global sections of the tangent sheaf are referred to as vector fields. We denote the $\mathcal{O}_M(|M|)$-module of vector fields as $\mathrm{Vect}(M)$. The dual of the tangent sheaf is the cotangent sheaf, which we denote as $\mathcal{T}^*M$. This is also a sheaf of locally free $\mathcal{O}_M$-modules. Global section of the cotangent sheaf we will refer to as one-forms and we denote the $\mathcal{O}_M(|M|)$-module of one-forms as $\Omega^1(M)$.

Definition 2.3. (Definition 4 in [3]) A Riemannian metric on a $\mathbb{Z}_2$-manifold M is a $\mathbb{Z}_2$-homogen
-eous, $\mathbb{Z}_2$-symmetric, non-degenerate, $\mathcal{O}_M$-linear morphisms of sheaves $\langle -,-\rangle_g : \mathcal{T}M \otimes \mathcal{T}M \to \mathcal{O}_M$. A $\mathbb{Z}_2$-manifold equipped with a Riemannian metric is referred to as a Riemannian $\mathbb{Z}_2$-manifold.

We will insist that the Riemannian metric is homogeneous with respect to the $\mathbb{Z}_2$-degree, and we will denote the degree of the metric as $|g| \in \mathbb{Z}_2$. Explicitly, a Riemannian metric has the following properties:
(1)$|\langle X,Y\rangle_g| = |X| + |Y| + |g|$,
(2)$\langle X,Y\rangle_g = (-1)^{|X||Y|}\langle Y,X\rangle_g$,
(3) If $\langle X,Y\rangle_g = 0$ for all $Y \in Vect(M)$, then $X = 0$,
(4) $\langle fX+Y,Z\rangle_g = f\langle X,Z\rangle_g + \langle Y,Z\rangle_g$,
for arbitrary (homogeneous) $X,Y,Z \in \mathrm{Vect}(M)$ and $f \in C^\infty(M)$. We will say that a Riemannian metric is even if and only if it has degree zero. Similarly, we will say that a Riemannian metric is odd if and only if it has degree one. Any Riemannian metric we consider will be either even or odd as we will only be considering homogeneous metrics.

Definition 2.4. (Definition 9 in [3]) An affine connection on a $\mathbb{Z}_2$-manifold is a $\mathbb{Z}_2$-degree preserving map

$$\nabla : \mathrm{Vect}(M) \times \mathrm{Vect}(M) \to \mathrm{Vect}(M); \quad (X,Y) \mapsto \nabla_X Y,$$

which satisfies the followings
1) Bi-linearity

$$\nabla_X(Y+Z) = \nabla_X Y + \nabla_X Z; \quad \nabla_{X+Y}Z = \nabla_X Z + \nabla_Y Z,$$

2)$C^\infty(M)$-linearrity in the first argument

$$\nabla_{fX}Y = f\nabla_X Y,$$

3)The Leibniz rule

$$\nabla_X(fY) = X(f)Y + (-1)^{|X||f|}f\nabla_X Y,$$

for all homogeneous $X,Y,Z \in \mathrm{Vect}(M)$ and $f \in C^\infty(M)$.

Definition 2.5. (Definition 10 in [3]) The torsion tensor of an affine connection
$T_\nabla : \quad \mathrm{Vect}(M) \otimes_{C^\infty(M)} \mathrm{Vect}(M) \to \mathrm{Vect}(M)$ is defined as

$$T_\nabla(X,Y) := \nabla_X Y - (-1)^{|X||Y|}\nabla_Y X - [X,Y],$$

for any (homogeneous) $X,Y \in \mathrm{Vect}(M)$. An affine connection is said to be symmetric if the torsion vanishes.

Definition 2.6. (Definition 11 in [3]) An affine connection on a Riemannian $\mathbb{Z}_2$-manifold (M,g) is said to be metric compatible if and only if

$$X\langle Y,Z\rangle_g = \langle \nabla_X Y, Z\rangle_g + (-1)^{|X||Y|}\langle Y, \nabla_X Z\rangle_g,$$

for any $X,Y,Z \in \mathrm{Vect}(M)$.

Theorem 2.1. (Theorem 1 in [3]) There is a unique symmetric (torsionless) and metric compatible affine connection ∇^L on a Riemannian $\mathbb{Z}_2$-manifold (M, g) which satisfies the Koszul formula

$$\begin{aligned} 2\left\langle \nabla^L_X Y, Z \right\rangle_g &= X\langle Y, Z\rangle_g + \langle [X,Y], Z\rangle_g \\ &+ (-1)^{|X|(|Y|+|Z|)}(Y\langle Z, X\rangle_g - \langle [Y,Z], X\rangle_g) \\ &- (-1)^{|Z|(|X|+|Y|)}(Z\langle X, Y\rangle_g - \langle [Z,X], Y\rangle_g), \end{aligned} \tag{2.1}$$

for all homogeneous $X, Y, Z \in \mathsf{Vect}(M)$.

Definition 2.7. (Definition 13 in [3]) The Riemannian curvature tensor of an affine connection

$$R_\nabla : \quad \mathsf{Vect}(M) \otimes_{C^\infty(M)} \mathsf{Vect}(M) \otimes_{C^\infty(M)} \mathsf{Vect}(M) \to \mathsf{Vect}(M)$$

is defined as

$$R_\nabla(X, Y)Z = \nabla_X \nabla_Y - (-1)^{|X||Y|}\nabla_Y \nabla_X - \nabla_{[X,Y]}Z, \tag{2.2}$$

for all X, Y and $Z \in \mathsf{Vect}(M)$.

Directly from the definition it is clear that

$$R_\nabla(X, Y)Z = -(-1)^{|X||Y|}R_\nabla(Y, X)Z, \tag{2.3}$$

for all X, Y and $Z \in \mathsf{Vect}(M)$.

Definition 2.8. (Definition 14 in [3]) The Ricci curvature tensor of an affine connection is the symmetric rank-2 covariant tensor defined as

$$Ric_\nabla(X, Y) := (-1)^{|\partial_{x^I}|(|\partial_{x^I}|+|X|+|Y|)}\frac{1}{2}\left[R_\nabla(\partial_{x^I}, X)Y + (-1)^{|X||Y|}R_\nabla(\partial_{x^I}, Y)X\right]^I, \tag{2.4}$$

where $X, Y \in \mathsf{Vect}(M)$ and $[\ \ ]^I$ denotes the coefficient of ∂_{x^I} and ∂_{x^I} is the natural frame of $\mathcal{T}M$.

Definition 2.9. (Definition 16 in [3]) Let $f \in C^\infty(M)$ be an arbitrary function on a Riemannian $\mathbb{Z}_2$-manifold (M, g). The gradient of f is the unique vector field $\mathrm{grad}_g f$ such that

$$X(f) = (-1)^{|f||g|}\left\langle X, \mathrm{grad}_g f \right\rangle_g, \tag{2.5}$$

for all $X \in \mathsf{Vect}(M)$.

Definition 2.10. (Definition 17 in [3]) Let (M, g) be a Riemannian $\mathbb{Z}_2$-manifold and let ∇^L be the associated Levi-Civita connection. The covariant divergence is the map Div_L : $\mathsf{Vect}(M) \to C^\infty(M)$, given by

$$\mathrm{Div}_L(X) = (-1)^{|\partial_{x^I}|(|\partial_{x^I}|+|X|)}(\nabla_{\partial_{x^I}} X)^I, \tag{2.6}$$

for any arbitrary $X \in \mathsf{Vect}(M)$.

Definition 2.11. (Definition 18 in [3]) Let (M, g) be a Riemannian $\mathbb{Z}_2$-manifold and let ∇^L be the associated Levi-Civita connection. The connection Laplacian (acting on functions) is the differential operator of $\mathbb{Z}_2$-degree $|g|$ defined as

$$\triangle_g(f) = \mathrm{Div}_L(\mathrm{grad}_g f), \tag{2.7}$$

for any and all $f \in C^\infty(M)$.

Definition 2.12. Let (M, g) be a Riemannian $\mathbb{Z}_2$-manifold and ∇ be the Levi-Civita connection associated to the Riemannian metric g^M. Let $D \subseteq TM$ be a super distribution and $D^\perp \subseteq TM$ is the orthogonal distribution to D, then $g^M = g^D + g^{D^\perp}$. Let $\pi^D : TM \to D$, $\pi^{D^\perp} : TM \to D^\perp$ be the projections. For $X, Y \in \Gamma(D)$, we define $\nabla^D_X Y = \pi^D(\nabla_X Y)$, then we have the fundamental form of submanifold on Riemannian $\mathbb{Z}_2$-manifold (M, g)

$$\nabla_X Y = \nabla^D_X Y + B(X, Y), \tag{2.8}$$

$$\nabla_X \xi = -(-1)^{|X||\xi|} A_\xi X + L^\perp_X \xi, \tag{2.9}$$

where $B(X, Y) = \pi^{D^\perp} \nabla_X Y$, $L^\perp_X \xi = \pi^{D^\perp} \nabla_X \xi$ and $\pi^D \nabla_X \xi = -(-1)^{|X||\xi|} A_\xi X$ for any homogenous $\xi \in \Gamma(D^\perp)$ and $X, Y \in \Gamma(D)$.

Then

$$B(fX, Y) = fB(X, Y), \quad B(X, fY) = (-1)^{|f||X|} B(X, Y)$$
$$B(X, Y) = (-1)^{|X||Y|} B(Y, X) + \pi^{D^\perp}[X, Y];$$
$$A_{f\xi} X = f A_\xi X, \quad A_\xi f X = (-1)^{|f||\xi|} A_\xi X, \quad g^{D^\perp}(B(X, Y), \xi) = (-1)^{|X|(|Y|+|\xi|)} g^D(Y, A_\xi X). \tag{2.10}$$

When D is a submanifold of M, we also have similar formula.

Definition 2.13. Let $(M^{m,n}, g)$ be a Riemannian $\mathbb{Z}_2$-manifold, the W_2-curvature tensor is also given by

$$W_2(X, Y, Z, T) = g(K(X, Y)T, Z), \tag{2.11}$$

for any homogenous $X, Y, Z, T \in \mathrm{Vect}(M)$ and where

$$K(X, Y)T := R(X, Y)T - \frac{1}{(m - n - 1)}[X \cdot Ric(Y, T) - (-1)^{|Y||T|} Ric(X, T)Y]. \tag{2.12}$$

3. The W_2-curvature tensor on super twisted products

Let $(M = M_1 \times_\mu M_2, g_\mu = \pi_1^* g_1 + \pi_1^*(\mu)\pi_2^* g_2)$ be the super twisted product with $|g| = |g_1| = |g_2| = 0$ and $|\mu| = 0$ and $\mu \in C^\infty(M)$ and its body $\varepsilon(\mu) > 0$. For simplicity, we assume that $\mu = h^2$ with $|h| = 0$. Let $\nabla^{L,\mu}$ be the Levi-Civita connection on (M, g_μ) and ∇^{L,M_1} (resp. ∇^{L,M_2}) be the Levi-Civita connection on (M_1, g_1) (resp. (M_2, g_2)).

Lemma 3.1. For $X, Y, Z \in \mathrm{Vect}(M_1)$ and $U, W, V \in \mathrm{Vect}(M_2)$, we have

$$(1) \nabla^{L,\mu}_X Y = \nabla^{L,M_1}_X Y,$$

$$(2) \nabla^{L,\mu}_X U = \frac{X(h)}{h} U,$$

$$(3) \nabla^{L,\mu}_U X = (-1)^{|U||X|} \frac{X(h)}{h} U,$$

$$(4) \nabla^{L,\mu}_U W = \frac{U(h)}{h} W + (-1)^{|U||W|} \frac{W(h)}{h} U - (-1)^{|V|(|U|+|W|)} \frac{g_2(U, W)}{h} \mathrm{grad}_{g_2} h$$
$$- h g_2(U, W) \mathrm{grad}_{g_1} h + \nabla^{L,M_2}_U W. \tag{3.1}$$

Proof. (1)By (2.1) and $[X, V] = 0$, we have $g_\mu(\nabla^{L,\mu}_X Y, Z) = g_1(\nabla^{L,M_1}_X Y, Z)$ and $g_\mu(\nabla^{L,\mu}_X Y, V) = 0$, so (1) holds.
(2)Similarly, we have $g_\mu(\nabla^{L,\mu}_X U, Y) = 0$ and $2g_\mu(\nabla^{L,\mu}_X U, V) = \frac{X(\mu)}{\mu} g_\mu(U, V)$, so (2) holds by $\mu = h^2$.
(3)By (2) and $\nabla^{L,\mu}$ having no torsion, we have (3).
(4)By (2.1) and (2.5), we have

$$\begin{aligned} 2g_\mu(\nabla^{L,\mu}_U W, X) &= -(-1)^{|X|(|U|+|W|)} X(\mu) g_2(U, W) \\ &= -(-1)^{|X|(|U|+|W|)} g_1(X, \mathrm{grad}_{g_1}(\mu)) g_2(U, W) \\ &= -g_\mu(g_2(U, W)\mathrm{grad}_{g_1}(\mu), X). \end{aligned} \tag{3.2}$$

And

$$\begin{aligned} &2g_\mu(\nabla^{L,\mu}_U W, V) \\ &= U(h^2)g_2(W, V) + (-1)^{|U||W|} W(h^2) g_2(U, V) - (-1)^{|V|(|U|+|W|)} V(h^2) g_2(U, W) \\ &+ 2g_\mu(\nabla^{L,M_2}_U W, V), \end{aligned} \tag{3.3}$$

then we have

$$P^{M_2}\nabla^{L,\mu}_U W = \frac{U(h)}{h} W + (-1)^{|U||W|} \frac{W(h)}{h} U - (-1)^{|V|(|U|+|W|)} \frac{g_2(U, W)}{h} \mathrm{grad}_{g_2} h, \tag{3.4}$$

so (4) holds. □

Let $R^{L,\mu}$ denotes the curvature tensor of the Levi-Civita connection on $(M^{m,n}, g_\mu)$ and let R^{L,M_1} (resp. R^{L,M_2}) be the curvature tensor of the Levi-Civita connection on (M_1, g_1) (resp. (M_2, g_2)). Let $H^h_{M_1}(X, Y) := XY(h) - \nabla^{L,M_1}_X Y(h)$, then $H^h_{M_1}(fX, Y) = f H^h_{M_1}(X, Y)$ and $H^h_{M_1}(X, fY) = (-1)^{|f||X|} f H^h_{M_1}(X, Y)$, where $H^h_{M_1}$ is a $(0, 2)$ tensor.

Proposition 3.1. For $X, Y, Z \in \mathrm{Vect}(M_1)$ and $U, V, W \in \mathrm{Vect}(M_2)$, we have

(1)$R^{L,\mu}(X, Y)Z = R^{L,M_1}(X, Y)Z$,

(2)$R^{L,\mu}(V, X)Y = -(-1)^{|V|(|X|+|Y|)} \frac{H^h_{M_1}(X, Y)}{h} V$,

(3)$R^{L,\mu}(X, Y)V = 0$,

(4)$R^{L,\mu}(V, W)X = (-1)^{|W||X|} V\left(\frac{X(h)}{h}\right) W - (-1)^{|V|(|W|+|X|)} W\left(\frac{X(h)}{h}\right) V$,

(5)$R^{L,\mu}(X, V)W = (-1)^{(|X|+|V|)|W|} W\left(\frac{X(h)}{h}\right) V - (-1)^{|X|(|V|+|W|)+|V||W|} g_2(W, V)$
$\mathrm{grad}_{g_2} \frac{X(h)}{h} - (-1)^{|X|(|V|+|W|)} \frac{g_\mu(V, W)}{h} \nabla^{L,M_1}_X (\mathrm{grad}_{g_1} h)$,

(6)$R^{L,\mu}(V, W)U = R^{L,M_2}(V, W)U + (-1)^{|U||W|} g_\mu(V, U) \mathrm{grad}_{g_1} \frac{W(h)}{h} - (-1)^{(|U|+|W|)|V|}$
$g_\mu(W, U)\mathrm{grad}_{g_1} \frac{V(h)}{h} - (-1)^{|V|(|W|+|U|)} \frac{(\mathrm{grad}_{g_1} h)(h)}{h^2} g_2(W, U)V + (-1)^{|W||U|} \frac{(\mathrm{grad}_{g_1} h)(h)}{h^2}$
$g_2(V, U)W. \qquad (3.5)$

Proof. (1)By Lemma 3.1 and (2.2), we can get (1).
(2)By Lemma 3.1 and the Leibniz rule, we have

$$\nabla_V^{L,\mu}\nabla_X^{L,\mu}Y=(-1)^{|V|(|X|+|Y|)}\frac{\nabla_X^{L,M_1}Y(h)}{h}V,\quad -\nabla_X^{L,\mu}\nabla_V^{L,\mu}Y=-(-1)^{|V|(|X|+|Y|)}\frac{XY(h)}{h}V,\tag{3.6}$$

by (2.2) and $[V,X]=0$ and the definition of $H^h_{M_1}(X,Y)$, we get (2).
(3) By Lemma 3.1, we have $\nabla_X^{L,\mu}\nabla_Y^{L,\mu}V=\frac{XY(h)}{h}V$ and by (2.2) and the definition of $[X,Y]$, we get (3).
(4)By Lemma 3.1, we have

$$\nabla_V^{L,\mu}\nabla_W^{L,\mu}X=(-1)^{|W||X|}\left[V\left(\frac{X(h)}{h}\right)W+(-1)^{|V||X|}\frac{X(h)}{h}\nabla_V^{L,\mu}W\right],\tag{3.7}$$

$$-\nabla_W^{L,\mu}\nabla_V^{L,\mu}X=-(-1)^{|V||X|}\left[W\left(\frac{X(h)}{h}\right)V+(-1)^{|W||X|}\frac{X(h)}{h}\nabla_W^{L,\mu}V\right],\tag{3.8}$$

$$-\nabla_{[V,W]}^{L,\mu}X=(-1)^{(|V|+|W|)|X|}\frac{X(h)}{h}[V,W],\tag{3.9}$$

then by (2.2) and $\nabla^{L,\mu}$ having no torsion, we get (4).
(5)For $W_1\in \mathrm{Vect}(M_2)$, by (4) and (4.12) in [9], we have

$$\begin{aligned}&g_\mu(R^{L,\mu}(X,V)W,W_1)\\&=(-1)^{(|X|+|V|)(|W|+|W_1|)}g_\mu(R^{L,\mu}(W,W_1)X,V)\\&=(-1)^{(|X|+|V|)(|W|+|W_1|)}g_\mu((-1)^{|X||W_1|}W\left(\frac{X(h)}{h}\right)W_1-(-1)^{(|X|+|W_1|)|W|}W_1\left(\frac{X(h)}{h}\right)W,V)\\&=(-1)^{(|X|+|V|)|W|+|V||W_1|}W\left(\frac{X(h)}{h}\right)g_\mu(W_1,V)-(-1)^{(|X|+|W|)|W_1|+|V|(|W|+|W_1|)}W_1\left(\frac{X(h)}{h}\right)\\&g_\mu(W,V),\end{aligned}\tag{3.10}$$

by (2.5), we have

$$W_1\left(\frac{X(h)}{h}\right)=(-1)^{|W_1||X|}g_2(\mathrm{grad}_{g_2}\frac{X(h)}{h},W_1),\tag{3.11}$$

then,

$$\begin{aligned}P^{M_2}R^{L,\mu}(X,V)W=(-1)^{|X|(|V|+|W|)}W\left(\frac{X(h)}{h}\right)V-(-1)^{|X|(|V|+|W|)+|V||W|}g_2(W,V)\\ \mathrm{grad}_{g_2}\frac{X(h)}{h}.\end{aligned}\tag{3.12}$$

By Proposition 9 in [3] and (2.2), we have

$$\begin{aligned}g_\mu(R^{L,\mu}(X,V)W,Y)=-(-1)^{|W||Y|}g_\mu(R^{L,\mu}(X,V)Y,W)\;=-(-1)^{(|W|+|V|)|Y|}\frac{H^h_{M_1}(X,Y)}{h}\\ g_\mu(V,W).\end{aligned}\tag{3.13}$$

By the definition of $\mathbf{grad}_{g_1}(h)$ and ∇^{L,M_1} preserving the metric, we can get

$$g_1(\nabla_X^{L,M_1}(\mathbf{grad}_{g_1}h),Y)=(-1)^{|Y||g_1|}H_{M_1}^h(X,Y)=H_{M_1}^h(X,Y),\tag{3.14}$$

so

$$g_\mu(R^{L,\mu}(X,V)W,Y)=-(-1)^{|X|(|V|+|W|)}g_\mu(\frac{g_\mu(V,W)}{h}\nabla_X^{L,M_1}(\mathbf{grad}_{g_1}h),Y).\tag{3.15}$$

By (3.7) and (3.10), we get (5).
(6) By (2.5), we have

$$\begin{aligned}g_\mu(R^{L,\mu}(V,W)U,X)&=-(-1)^{|X||U|}g_\mu(R^{L,\mu}(V,W)X,U)\\&=-(-1)^{|X||U|}g_\mu((-1)^{|X||W|}V\left(\frac{X(h)}{h}\right)W-(-1)^{|V|(|X|+|W|)}\\&W\left(\frac{X(h)}{h}\right)V,U)\\&=(-1)^{|X||U|+|V|(|X|+|W|)}g_\mu(W\left(\frac{X(h)}{h}\right)V,U)-(-1)^{|X|(|U|+|W|)}\\&g_\mu(V\left(\frac{X(h)}{h}\right)W,U).\end{aligned}\tag{3.16}$$

By $XW=(-1)^{|X||W|}WX$ and $XV=(-1)^{|X||V|}VX$, we have

$$P^{M_1}R^{L,\mu}(V,W)U=(-1)^{|U||W|}g_\mu(V,U)\mathrm{grad}_{g_2}\frac{W(h)}{h}-(-1)^{(|U|+|W|)|V|}g_\mu(W,U)\mathrm{grad}_{g_2}\frac{V(h)}{h}.\tag{3.17}$$

By Definition 2.7 and Definition 2.12, we have

$$R^D(X,Y)Z=(R(X,Y)Z)^\top-(-1)^{|Y||Z|}A_{B(X,Z)}Y+(-1)^{|X|(|Y|+|Z|)}A_{B(Y,Z)}X,$$

where R^D (resp. R) denotes the curvature tensor of ∇^D (resp. ∇) and $R^\top$ denotes tangential component of R.
Then by $g(B(X,Y),\xi)=(-1)^{(|Y|+|\xi|)|X|}g(Y,A_\xi X)$, we have

$$g(A_\xi X,Y)=(-1)^{|Y|(|X|+|\xi|)}g(Y,A_\xi X)=(-1)^{|\xi|(|X|+|Y|)}g(B(X,Y),\xi),\tag{3.18}$$

so

$$\begin{aligned}&g(R^D(X,Y)Z,W)\\&=g(R(X,Y)Z),W)-(-1)^{|Y||Z|}g(A_{B(X,Z)}Y,W)+(-1)^{|X|(|Y|+|Z|)}g(A_{B(Y,Z)}X,W)\\&=g(R(X,Y)Z),W)-(-1)^{|X|(|Y|+|W|)+|W||Z|}g(B(Y,W),B(X,Z))+(-1)^{|W|(|Y|+|Z|)}\\&g(B(X,W),B(Y,Z)).\end{aligned}\tag{3.19}$$

By $B(U,W)=-hg_2(U,W)\mathrm{grad}_{g_1}h=-\frac{g_\mu(U,W)}{h}\mathrm{grad}_{g_1}h$, we have

$$\begin{aligned}&g_\mu(R^{M_2}(V,W)U,W_1)\\&=g_\mu(R^{L,\mu}(X,Y)Z),W_1)-(-1)^{|V|(|W|+|W_1|)+|W_1||U|}g_\mu(B(W,W_1),B(V,U))\\&+(-1)^{|W_1|(|W|+|U|)}g_\mu(B(V,W_1),B(W,U))\end{aligned}$$

$$= g_\mu(R^{L,\mu}(X,Y)Z),W_1) + (-1)^{|V|(|W|+|U|)}\frac{|\mathrm{grad}_{g_1}h|_g^2}{h^2}g_\mu(W,U)g_\mu(V,W_1)$$
$$-(-1)^{|W||U|}\frac{|\mathrm{grad}_{g_1}h|_g^2}{h^2}g_\mu(V,U)g_\mu(W,W_1),$$
$$= g_\mu(R^{L,\mu}(X,Y)Z),W_1) + (-1)^{|V|(|W|+|U|)}\frac{(\mathrm{grad}_{g_1}h)(h)}{h^2}g_\mu(W,U)g_\mu(V,W_1)$$
$$-(-1)^{|W||U|}\frac{(\mathrm{grad}_{g_1}h)(h)}{h^2}g_\mu(V,U)g_\mu(W,W_1), \tag{3.20}$$

then, we get (6). □

In the following, we compute the Ricci tensor of manifold $M^{m,n}$. Let M_1 (resp. M_2) have the (p,m_1) (resp. (q,m_2)) dimension, where $n_1 = p - m_1$, $n_2 = q - m_2$ and $m - n = n_1 + n_2$. Let $\partial_{x^I} = \{\partial_{x^a}, \partial_{\xi^A}\}$ (resp. $\partial_{y^J} = \{\partial_{y^b}, \partial_{\eta^B}\}$) denote the natural tangent frames on M_1 (resp. M_2). Let $\mathrm{Ric}^{L,\mu}$ (resp. Ric^{L,M_1}, Ric^{L,M_2}) denote the Ricci tensor of (M, g_μ) (resp. (M_1, g_1), (M_2, g_2)). Then by (2.4), (2.7) and (3.5), we have

Proposition 3.2. The following equalities holds

$$(1)\mathrm{Ric}^{L,\mu}(\partial_{x^L},\partial_{x^K}) = \mathrm{Ric}^{L,M_1}(\partial_{x^L},\partial_{x^K}) - \frac{(q-m_2)}{h}H^h_{M_1}(\partial_{x^L},\partial_{x^K}),$$
$$(2)\mathrm{Ric}^{L,\mu}(\partial_{x^L},\partial_{y^J}) = -(q-m_2-1)(-1)^{|\partial_{x_L}||\partial_{y^J}|}\partial_{y^J}\left(\frac{\partial_{x^L}(h)}{h}\right),$$
$$(3)\mathrm{Ric}^{L,\mu}(\partial_{y^J},\partial_{x^L}) = -(q-m_2-1)\partial_{y^J}\left(\frac{\partial_{x^L}(h)}{h}\right),$$
$$(4)\mathrm{Ric}^{L,\mu}(\partial_{y^L},\partial_{y^J}) = \mathrm{Ric}^{L,M_2}(\partial_{y^L},\partial_{y^J}) - g_\mu(\partial_{y^L},\partial_{y^J})\cdot[\frac{\triangle^L_{g_1}(h)}{h} + (q-m_2-1)\frac{(\mathrm{grad}_{g_1}h)(h)}{h^2}]. \tag{3.21}$$

Proof. (1)By Definition 2.8 and Proposition 3.1, we have

$$\mathrm{Ric}^{L,\mu}(\partial_{x^L},\partial_{x^K})$$
$$= \sum_I (-1)^{|\partial_{x^I}|(|\partial_{x^I}|+|\partial_{x^L}|+|\partial_{x^K}|)}\frac{1}{2}[R^{L,\mu}(\partial_{x^I},\partial_{x^L})\partial_{x^K} + (-1)^{|\partial_{x^L}||\partial_{x^K}|}R^{L,\mu}(\partial_{x^I},\partial_{x^K})\partial_{x^L}]^I$$
$$+\sum_J (-1)^{|\partial_{y^J}|(|\partial_{y^J}|+|\partial_{x^L}|+|\partial_{x^K}|)}\frac{1}{2}[R^{L,\mu}(\partial_{y^J},\partial_{x^L})\partial_{x^K} + (-1)^{|\partial_{x^L}||\partial_{x^K}|}R^{L,\mu}(\partial_{y^J},\partial_{x^K})\partial_{x^L}]^J$$
$$= \mathrm{Ric}^{L,M_1}(\partial_{x^L},\partial_{x^K}) + \sum_J (-1)^{|\partial_{y^J}|(|\partial_{y^J}|+|\partial_{x^L}|+|\partial_{x^K}|)}\frac{1}{2}\Big[-(-1)^{|\partial_{y^J}|(|\partial_{x^L}|+|\partial_{x^K}|)}\frac{H^h_{M_1}(\partial_{x^L},\partial_{x^K})}{h}$$
$$\partial_{y^J} - (-1)^{|\partial_{x^L}||\partial_{x^K}|}(-1)^{|\partial_{y^J}|(|\partial_{x^L}|+|\partial_{x^K}|)}\frac{H^h_{M_1}(\partial_{x^K},\partial_{x^L})}{h}\partial_{y^J}\Big]^J$$
$$= \mathrm{Ric}^{L,M_1}(\partial_{x^L},\partial_{x^K}) - \sum_J (-1)^{|\partial_{y^J}||\partial_{y^J}|}\frac{1}{2}\Big[\frac{H^h_{M_1}(\partial_{x^L},\partial_{x^K})}{h} + (-1)^{|\partial_{x^L}||\partial_{x^K}|}\frac{H^h_{M_1}(\partial_{x^K},\partial_{x^L})}{h}\Big], \tag{3.22}$$

by $(-1)^{|\partial_{y^J}||\partial_{y^J}|}=\sum_{j=1}^{q}(-1)^0+\sum_{k=1}^{m_2}(-1)^1=q-m_2$, we have

$$\mathrm{Ric}^{L,\mu}(\partial_{x^L},\partial_{x^K})=\mathrm{Ric}^{L,M_1}(\partial_{x^L},\partial_{x^K})-\frac{(q-m_2)}{2h}\left[\frac{H^h_{M_1}(\partial_{x^L},\partial_{x^K})}{h}+(-1)^{|\partial_{x^L}||\partial_{x^K}|}\frac{H^h_{M_1}(\partial_{x^K},\partial_{x^L})}{h}\right], \tag{3.23}$$

by ∇^{L,M_1} having no torsion, we have

$$\begin{aligned}
&\frac{H^h_{M_1}(\partial_{x^L},\partial_{x^K})}{h}-(-1)^{|\partial_{x^L}||\partial_{x^K}|}\frac{H^h_{M_1}(\partial_{x^K},\partial_{x^L})}{h}\\
&=\partial_{x^L}(\partial_{x^K}(h))-\nabla^{L,M_1}_{\partial_{x^L}}\partial_{x^K}(h)-(-1)^{|\partial_{x^L}||\partial_{x^K}|}\partial_{x^K}(\partial_{x^L}(h))+(-1)^{|\partial_{x^L}||\partial_{x^K}|}\nabla^{L,M_1}_{\partial_{x^K}}\partial_{x^L}(h)\\
&=-[-[\partial_{x^L},\partial_{x^K}](h)+\nabla^{L,M_1}_{\partial_{x^L}}\partial_{x^K}(h)-(-1)^{|\partial_{x^L}||\partial_{x^K}|}\partial_{x^K}(\partial_{x^L}(h))]\\
&=-T^{L,M_1}(\partial_{x^L},\partial_{x^K})(h)\\
&=0,
\end{aligned} \tag{3.24}$$

so (1) holds.

(2)By Definition 2.8, we get

$$\begin{aligned}
&\mathrm{Ric}^{L,\mu}(\partial_{x^L},\partial_{y^J})\\
&=\sum_I(-1)^{|\partial_{x^I}|(|\partial_{x^I}|+|\partial_{x^L}|+|\partial_{y^J}|)}\frac{1}{2}[R^{L,\mu}(\partial_{x^I},\partial_{x^L})\partial_{y^J}+(-1)^{|\partial_{x^L}||\partial_{y^J}|}R^{L,\mu}(\partial_{x^I},\partial_{y^J})\partial_{x^L}]^I\\
&+\sum_K(-1)^{|\partial_{y^K}|(|\partial_{y^K}|+|\partial_{x^L}|+|\partial_{y^J}|)}\frac{1}{2}[R^{L,\mu}(\partial_{y^K},\partial_{x^L})\partial_{x^K}+(-1)^{|\partial_{x^L}||\partial_{y^J}|}R^{L,\mu}(\partial_{y^K},\partial_{y^J})\partial_{x^L}]^K\\
&=\sum_K(-1)^{|\partial_{y^K}|(|\partial_{y^K}|+|\partial_{x^L}|+|\partial_{y^J}|)}\frac{1}{2}[R^{L,\mu}(\partial_{y^K},\partial_{x^L})\partial_{x^K}+(-1)^{|\partial_{x^L}||\partial_{y^J}|}R^{L,\mu}(\partial_{y^K},\partial_{y^J})\partial_{x^L}]^K\\
&=\sum_K(-1)^{|\partial_{y^K}|(|\partial_{y^K}|+|\partial_{x^L}|+|\partial_{y^J}|)}\frac{1}{2}[-(-1)^{|\partial_{y^K}||\partial_{x^L}|}R^{L,\mu}(\partial_{x^L},\partial_{y^K})\partial_{y^J}+(-1)^{|\partial_{x^L}||\partial_{y^J}|}\\
&R^{L,\mu}(\partial_{y^K},\partial_{y^J})\partial_{x^L}]^K,
\end{aligned} \tag{3.25}$$

by Propsition 3.1, we get

$$\begin{aligned}
&R^{L,\mu}(\partial_{x^L},\partial_{y^K})\partial_{y^J}\\
&=(-1)^{(|\partial_{y^K}|+|\partial_{x^L}|)|\partial_{y^J}|}\partial_{y^J}\left(\frac{\partial_{x^L}(h)}{h}\right)\partial_{y^K}-(-1)^{|\partial_{x^L}|(|\partial_{y^K}|+|\partial_{y^J}|)+|\partial_{y^K}||\partial_{y^J}|}g_2(\partial_{y^J},\partial_{y^K})\mathrm{grad}_{g_2}\frac{\partial_{x^L}(h)}{h}.
\end{aligned} \tag{3.26}$$

By [3], we have

$$\mathrm{grad}_g f=\sum^I(-1)^{|f||g|+|\partial_{y^J}|(|f|+|g|)}\frac{\partial f}{\partial_{y^J}}g^{JI}\partial_{y^I}, \tag{3.27}$$

and

$$g^{\alpha K}=(-1)^{|\partial_{y^K}|^2+|\partial_{y^\alpha}|^2+|g|^2+|\partial_{y^K}||\partial_{y^\alpha}|}g^{K\alpha}, \tag{3.28}$$

then

$$
\begin{aligned}
&[(-1)^{|\partial_{x^L}|(|\partial_{y^K}|+|\partial_{y^J}|)+|\partial_{y^K}||\partial_{y^J}|}g_2(\partial_{y^J},\partial_{y^K})\mathrm{grad}_{g_2}(\partial_{x^L}(lnh))]^{\partial_{y^K}}\\
&=(-1)^{|\partial_{x^L}|(|\partial_{y^K}|+|\partial_{y^J}|)+|\partial_{y^K}||\partial_{y^J}|}g_2(\partial_{y^J},\partial_{y^K})\sum_\alpha\partial_{x^L}(\partial_{y^\alpha}(lnh))g_2^{\alpha K}\\
&=(-1)^{|\partial_{x^L}|(|\partial_{y^K}|+|\partial_{y^J}|)+|\partial_{y^K}||\partial_{y^J}|}(-1)^{(|\partial_{y^K}|+|\partial_{y^J}|)(|\partial_{y^\alpha}|+|\partial_{x^L}|)}\partial_{x^L}(\partial_{y^\alpha}(lnh))g_2(\partial_{y^J},\partial_{y^K})g_2^{\alpha K}\\
&=\sum_\alpha(-1)^{|\partial_{y^\alpha}|+|\partial_{y^K}|+|\partial_{y^\alpha}||\partial_{y^K}|}(-1)^{(|\partial_{y^J}|+|\partial_{y^K}|)|\partial_{x^L}|+|\partial_{y^J}||\partial_{y^K}|}(-1)^{(|\partial_{y^J}|+|\partial_{y^K}|)(|\partial_{x^L}|+|\partial_{y^\alpha}|)}\\
&\partial_{x^L}(\partial_{y^\alpha}(lnh))g_2(\partial_{y^J},\partial_{y^K})g_2^{K\alpha}\\
&=\sum_\alpha(-1)^{|\partial_{y^K}|^2+|\partial_{y^\alpha}|}(-1)^{(|\partial_{y^\alpha}|+|\partial_{y^K}|)|\partial_{y^J}|}\partial_{x^L}(\partial_{y^\alpha}(lnh))g_2(\partial_{y^J},\partial_{y^K})g_2^{K\alpha},
\end{aligned}
\tag{3.29}
$$

so

$$
\begin{aligned}
&\sum_K(-1)^{(|\partial_{y^K}|+|\partial_{y^J}|+|\partial_{x^L}|)|\partial_{y^K}|}(-1)^{|\partial_{y^K}||\partial_{x^L}|}[(-1)^{|\partial_{x^L}|(|\partial_{y^K}|+|\partial_{y^J}|)+|\partial_{y^K}||\partial_{y^J}|}g_2(\partial_{y^J},\partial_{y^K})\\
&\mathrm{grad}_{g_2}(\partial_{x_L}(lnh))]^{\partial_{y^K}}\\
&=\sum_{K,\alpha}(-1)^{|\partial_{y^\alpha}|+|\partial_{y^J}||\partial_{y^\alpha}|}\partial_{x^L}(\partial_{y^\alpha}(lnh))g_2(\partial_{y^J},\partial_{y^K})g_2^{K\alpha}\\
&=\sum_\alpha(-1)^{|\partial_{y^\alpha}|+|\partial_{y^J}||\partial_{y^\alpha}|}\partial_{x^L}(\partial_{y^\alpha}(lnh))\delta_J^\alpha\\
&=(-1)^{|\partial_{y^J}|+|\partial_{y^J}|^2}\partial_{x^L}(\partial_{y^J}(lnh))\\
&=\partial_{x^L}(\partial_{y^J}(lnh))\\
&=(-1)^{|\partial_{y^J}||\partial_{x^L}|}\partial_{y^J}(\partial_{x^L}(lnh))\\
&=(-1)^{|\partial_{y^J}||\partial_{x^L}|}\partial_{y^J}\left(\frac{\partial_{x^L}(h)}{h}\right).
\end{aligned}
\tag{3.30}
$$

And

$$
\begin{aligned}
&\frac{1}{2}\sum_K(-1)^{|\partial_{y^K}|(|\partial_{y^K}|+|\partial_{x^L}|+|\partial_{y^J}|)}[-(-1)^{|\partial_{y^K}||\partial_{x^L}|}R^{L,\mu}(\partial_{x^L},\partial_{y^K})\partial_{y^J}]^{\partial_{y^K}}\\
&=-\frac{1}{2}(q-m_2-1)(-1)^{|\partial_{x_L}||\partial_{y^J}|}\partial_{y^J}\left(\frac{\partial_{x^L}(h)}{h}\right),
\end{aligned}
\tag{3.31}
$$

similarly,

$$
\begin{aligned}
&\frac{1}{2}\sum_K(-1)^{|\partial_{y^K}|(|\partial_{y^K}|+|\partial_{x^L}|+|\partial_{y^J}|)}[(-1)^{|\partial_{x^L}||\partial_{y^J}|}R^{L,\mu}(\partial_{y^K},\partial_{y^J})\partial_{x^L}]^K\\
&=-\frac{1}{2}(q-m_2-1)(-1)^{|\partial_{x_L}||\partial_{y^J}|}\partial_{y^J}\left(\frac{\partial_{x^L}(h)}{h}\right),
\end{aligned}
\tag{3.32}
$$

so (2) holds.
(3)By $\mathrm{Ric}^{L,\mu}(\partial_{y^J},\partial_{x^L})=(-1)^{|\partial_{x^L}||\partial_{y^J}|}\mathrm{Ric}^{L,\mu}(\partial_{x^L},\partial_{y^J})$, (3) holds.
(4)By Definition 2.8 and Proposition 3.1, we have

$$
\mathrm{Ric}^{L,\mu}(\partial_{y^L},\partial_{y^J})
$$

$$
\begin{aligned}
&= \sum_I (-1)^{|\partial_{x^I}|(|\partial_{x^I}|+|\partial_{y^L}|+|\partial_{y^J}|)} \frac{1}{2}[R^{L,\mu}(\partial_{x^I},\partial_{y^L})\partial_{y^J} + (-1)^{|\partial_{y^L}||\partial_{y^J}|} R^{L,\mu}(\partial_{x^I},\partial_{y^J})\partial_{y^L}]^I \\
&+ \sum_K (-1)^{|\partial_{y^K}|(|\partial_{y^K}|+|\partial_{y^L}|+|\partial_{y^J}|)} \frac{1}{2}[R^{L,\mu}(\partial_{y^K},\partial_{y^L})\partial_{y^J} + (-1)^{|\partial_{y^L}||\partial_{y^J}|} R^{L,\mu}(\partial_{y^K},\partial_{y^J})\partial_{y^L}]^K \\
&= \Delta_1 + \Delta_2, \qquad (3.33)
\end{aligned}
$$

where

$$
\begin{aligned}
\Delta_1 &:= \sum_I (-1)^{|\partial_{x^I}|(|\partial_{x^I}|+|\partial_{y^L}|+|\partial_{y^J}|)} \frac{1}{2}[R^{L,\mu}(\partial_{x^I},\partial_{y^L})\partial_{y^J} + (-1)^{|\partial_{y^L}||\partial_{y^J}|} R^{L,\mu}(\partial_{x^I},\partial_{y^J})\partial_{y^L}]^I \\
\Delta_2 &:= \sum_K (-1)^{|\partial_{y^K}|(|\partial_{y^K}|+|\partial_{y^L}|+|\partial_{y^J}|)} \frac{1}{2}[R^{L,\mu}(\partial_{y^K},\partial_{y^L})\partial_{y^J} + (-1)^{|\partial_{y^L}||\partial_{y^J}|} R^{L,\mu}(\partial_{y^K},\partial_{y^J})\partial_{y^L}]^K. \qquad (3.34)
\end{aligned}
$$

by Propsition 3.5, we have

$$
\begin{aligned}
&\sum_I (-1)^{|\partial_{x^I}|(|\partial_{x^I}|+|\partial_{y^L}|+|\partial_{y^J}|)} [R^{L,\mu}(\partial_{x^I},\partial_{y^L})\partial_{y^J}]^I \\
&= -\sum_I (-1)^{|\partial_{x^I}|(|\partial_{x^I}|+|g|)} \frac{g_\mu(\partial_{y^L},\partial_{y^J})}{h} \left[\frac{\nabla^{L,M_1}_{\partial_{x^I}}(\mathrm{grad}_{g_1} h)}{h}\right]^I \\
&= -\frac{g_\mu(\partial_{y^L},\partial_{y^J})}{h} \sum_I (-1)^{|\partial_{x^I}|(|\partial_{x^I}|+|\mathrm{grad}_{g_1} h|)} \left[\frac{\nabla^{L,M_1}_{\partial_{x^I}}(\mathrm{grad}_{g_1} h)}{h}\right]^I \\
&= -\frac{g_\mu(\partial_{y^L},\partial_{y^J})}{h} Div_{\nabla^{L,M_1}} \mathrm{grad}_{g_1} h \\
&= -\frac{g_\mu(\partial_{y^L},\partial_{y^J})}{h} \triangle^L_{g_1}(h), \qquad (3.35)
\end{aligned}
$$

then, we get

$$
\begin{aligned}
\Delta_1 &= \frac{1}{2}[-g_\mu(\partial_{y^L},\partial_{y^J})\frac{\triangle^L_{g_1}(h)}{h} - (-1)^{|\partial_{y^L}||\partial_{y^J}|} g_\mu(\partial_{y^L},\partial_{y^J})\frac{\triangle^L_{g_1}(h)}{h}], \\
&= -\frac{g_\mu(\partial_{y^L},\partial_{y^J})}{h}\triangle^L_{g_1}(h), \qquad (3.36)
\end{aligned}
$$

$$
\begin{aligned}
\Delta_2 &= \mathrm{Ric}^{L,M_2}(\partial_{y^L},\partial_{y^J}) + \sum_K (-1)^{|\partial_{y^K}|(|\partial_{y^K}|+|\partial_{y^L}|+|\partial_{y^J}|)} \frac{1}{2}\Big\{ (-1)^{|\partial_{y^L}||\partial_{y^J}|} g_\mu(\partial_{y^K},\partial_{y^J}) \mathrm{grad}_{g_2} \frac{\partial_{y^L}(h)}{h} \\
&- (-1)^{(|\partial_{y^L}|+|\partial_{y^J}|)|\partial_{y^K}|} g_\mu(\partial_{y^L},\partial_{y^J}) \mathrm{grad}_{g_2} \frac{\partial_{y^K}(h)}{h} - (-1)^{(|\partial_{y^L}|+|\partial_{y^J}|)|\partial_{y^K}|} \\
&- (-1)^{|\partial_{y^K}|(|\partial_{y^L}|+|\partial_{y^J}|)} \frac{(\mathrm{grad}_{g_1} h)(h)}{h^2} g_\mu(\partial_{y^L},\partial_{y^J})\partial_{y^K} + (-1)^{|\partial_{y^L}||\partial_{y^J}|} \frac{(\mathrm{grad}_{g_1} h)(h)}{h^2} \\
& g_\mu(\partial_{y^K},\partial_{y^J})\partial_{y^L} + (-1)^{|\partial_{y^L}||\partial_{y^J}|} \Big[(-1)^{|\partial_{y^L}||\partial_{y^J}|} g_\mu(\partial_{y^K},\partial_{y^L}) \mathrm{grad}_{g_2} \frac{\partial_{y^J}(h)}{h} \\
&- (-1)^{(|\partial_{y^L}|+|\partial_{y^J}|)|\partial_{y^K}|} g_\mu(\partial_{y^K},\partial_{y^L}) (\mathrm{grad}_{g_2} \frac{\partial_{y^K}(h)}{h} - (-1)^{(|\partial_{y^L}|+|\partial_{y^J}|)|\partial_{y^K}|} \frac{(\mathrm{grad}_{g_1} h)(h)}{h^2}
\end{aligned}
$$

$$g_\mu(\partial_{y^J},\partial_{y^L})\partial_{y^K}+(-1)^{|\partial_{y^L}||\partial_{y^J}|}\frac{(\mathrm{grad}_{g_1}h)(h)}{h^2}g_\mu(\partial_{y^K},\partial_{y^L})\partial_{y^J}\Big]\Big]\Big\}$$

$$=Ric^{L,M_2}(\partial_{y^L},\partial_{y^J})-g_\mu(\partial_{y^L},\partial_{y^J})(q-m_2-1)\frac{(\mathrm{grad}_{g_1}h)(h)}{h^2}, \tag{3.37}$$

then,

$$\Delta_1+\Delta_2=\mathrm{Ric}^{L,M_2}(\partial_{y^L},\partial_{y^J})-g_\mu(\partial_{y^L},\partial_{y^J})[\frac{\triangle^L_{g_1}(h)}{h}+(q-m_2-1)\frac{(\mathrm{grad}_{g_1}h)(h)}{h^2}], \tag{3.38}$$

so (4) holds.

□

Theorem 3.1. Let $M=M_1\times M_2$ be a singly twisted product manifold with the metric tensor $g=g_1\oplus h^2g_2$. If $X,Y,Z\in Vect(M_1)$, $U,V,Q\in Vect(M_2)$, then

$$\begin{aligned}
&(1)K^{L,\mu}(X,Y)Z=K^{L,M_1}(X,Y)Z+\frac{n_2}{(m-n-1)(n_1-1)}[X\cdot Ric^{L,M_1}(Y,Z)-(-1)^{|Y||Z|}\\
&\qquad Ric^{L,M_1}(X,Z)Y]+\frac{q-m_2}{(m-n-1)h}[X\cdot H^h_{M_1}(Y,Z)-(-1)^{|Y||Z|}H^h_{M_1}(X,Z)Y]\\
&(2)K^{L,\mu}(X,Y)Q=-\frac{q-m_2-1}{m-n-1}\left[(-1)^{|Y||Q|}Q\left(\frac{X(h)}{h}\right)Y-X\cdot Q\left(\frac{Y(h)}{h}\right)\right]\\
&(3)K^{L,\mu}(U,V)X=(-1)^{|V||X|}U\left(\frac{X(h)}{h}\right)V+(-1)^{|U|(|V|+|X|)}V\left(\frac{X(h)}{h}\right)U\\
&\qquad+\frac{q-m_2-1}{m-n-1}\left[(-1)^{|V||X|}U\cdot X\left(\frac{X(h)}{h}\right)-(-1)^{|X|(|V|+|U|)}X\left(\frac{U(h)}{h}\right)V\right]\\
&(4)K^{L,\mu}(X,V)Y=\frac{1}{m-n-1}(-1)^{|V||Y|}[(m-n-q+m_2-1)H^h_{M_1}(X,Y)+Ric^{L,M_1}(X,Y)]V\\
&\qquad+\frac{q-m_2-1}{m-n-1}X\cdot Y\frac{V(h)}{h}\\
&(5)K^{L,\mu}(X,U)V=(-1)^{|X|(|V|+|U|)}V\left(\frac{X(h)}{h}\right)U-(-1)^{|X|(|V|+|U|)+|U||V|}g_2(U,V)[h\triangle^L_{g_1}(h)\\
&\qquad+(q-m_2-1)\mathrm{grad}_{g_1}(h)(h)]+\frac{q-m_2-1}{m-n-1}(-1)^{|V||U|}V\frac{X(h)}{h}U\\
&(6)K^{L,\mu}(U,V)Q=K^{L,M_2}(U,V)Q+\frac{n_1}{(m-n-1)(n_2-1)}[U\cdot Ric^{L,M_2}(V,Q)-(-1)^{|V||Q|}\\
&\qquad Ric^{L,M_2}(U,Q)V]+(-1)^{|V||Q|}g_\mu(U,Q)\mathrm{grad}_{g_2}\frac{V(h)}{h}-(-1)^{|U|(|V|+|Q|)}\\
&\qquad g_\mu(V,Q)\mathrm{grad}_{g_2}\frac{U(h)}{h}-(-1)^{|U|(|V|+|Q|)}\frac{\mathrm{grad}_{g_1}(h)(h)}{h^2}g_2(V,Q)U\\
&\qquad+(-1)^{|V||Q|}\frac{\mathrm{grad}_{g_1}(h)(h)}{h^2}g_2(U,Q)V+\frac{1}{m-n-1}[U\cdot g_2(V,Q)(h\triangle^L_{g_1}(h)\\
&\qquad+(q-m_2-1)\mathrm{grad}_{g_1}(h)(h))]-\frac{1}{m-n-1}[(-1)^{|V||Q|}g_2(U,Q)(h\triangle^L_{g_1}(h)\\
&\qquad+(q-m_2-1)\mathrm{grad}_{g_1}(h)(h))V].
\end{aligned} \tag{3.39}$$

Proof. (1)By Definition 2.13 and Propsition 3.2, we have

$$\begin{aligned}
K^{L,\mu}(X,Y)Z &= R^{L,\mu}(X,Y)Z - \frac{1}{(m-n-1)}[X\cdot Ric^{L,\mu}(Y,Z) - (-1)^{|Y||Z|}Ric^{L,\mu}(X,Z)Y] \\
&= R^{L,M_1}(X,Y)Z - \frac{1}{(m-n-1)}X\cdot[Ric^{L,M_1}(Y,Z) - \frac{q-m_2}{h}H^h_{M_1}(Y,Z)] \\
&+ \frac{1}{(m-n-1)}(-1)^{|Y||Z|}[Ric^{L,M_1}(X,Z) - \frac{q-m_2}{h}H^h_{M_1}(X,Z)]Y \\
&= K^{L,M_1}(X,Y)Z + \frac{n_2}{(m-n-1)(n_1-1)}[X\cdot Ric^{L,M_1}(Y,Z) - (-1)^{|Y||Z|} \\
& Ric^{L,M_1}(X,Z)Y] + \frac{q-m_2}{(m-n-1)h}[X\cdot H^h_{M_1}(Y,Z) - (-1)^{|Y||Z|}H^h_{M_1}(X,Z)Y],
\end{aligned} \tag{3.40}$$

then (1) holds.

(2)By Definition 2.13 and Propsition 3.2, we have

$$\begin{aligned}
& K^{L,\mu}(X,Y)Q \\
&= R^{L,\mu}(X,Y)Q - \frac{1}{(m-n-1)}[X\cdot Ric^{L,\mu}(Y,Q) - (-1)^{|Y||Q|}Ric^{L,\mu}(X,Q)Y] \\
&= -\frac{1}{(m-n-1)}\left[X\cdot -\frac{q-m_2-1}{2}Q\left(\frac{Y(h)}{h}\right) + (-1)^{|Y||Q|}\frac{q-m_2-1}{2}Q\left(\frac{X(h)}{h}\right)Y\right] \\
&= -\frac{q-m_2-1}{2(m-n-1)}\left[(-1)^{|Y||Z|}Q\left(\frac{X(h)}{h}\right)Y - X\cdot Q\left(\frac{Y(h)}{h}\right)\right],
\end{aligned} \tag{3.41}$$

so (2) holds.

(3)By Definition 2.13 and Propsition 3.2, we have

$$\begin{aligned}
& K^{L,\mu}(U,V)X \\
&= R^{L,\mu}(U,V)X - \frac{1}{(m-n-1)}[U\cdot Ric^{L,\mu}(V,X) - (-1)^{|V||X|}Ric^{L,\mu}(U,X)V] \\
&- \frac{1}{(m-n-1)}\left[U\cdot -(q-m_2-1)X\left(\frac{V(h)}{h}\right) + (-1)^{(|V||X|}(q-m_2-1)X\left(\frac{U(h)}{h}\right)V\right] \\
&= (-1)^{|V||X|}U\left(\frac{X(h)}{h}\right)V + (-1)^{|U|(|V|+|X|)}V\left(\frac{X(h)}{h}\right)U \\
&- \frac{q-m_2-1}{2(m-n-1)h}\left[U\cdot X\left(\frac{V(h)}{h}\right) - (-1)^{(|V||X|}X\left(\frac{U(h)}{h}\right)V\right],
\end{aligned} \tag{3.42}$$

so (3) holds.

(4)By Definition 2.13 and Propsition 3.2, we have

$$\begin{aligned}
K^{L,\mu}(X,V)Y &= R^{L,\mu}(X,V)Y - \frac{1}{(m-n-1)}[X\cdot Ric^{L,\mu}(V,Y) - (-1)^{|V||Y|}Ric^{L,\mu}(X,Y)V] \\
&= (-1)^{|X||V|}R^{L,\mu}(V,X)Y - \frac{1}{m-n-1}[X\cdot -(q-m_2-1)Y\left(\frac{V(h)}{h}\right) \\
&- (-1)^{(|V||Y|}(Ric^{L,M_1}(X,Y) - \frac{q-m_2}{h}H^h_{M_1}(X,Y))V]
\end{aligned}$$

$$= \frac{1}{m-n-1}(-1)^{|V||Y|}[(n-q+m_2-1)H^h_{M_1}(X,Y) + Ric^{L,M_1}(X,Y)]V$$
$$+ \frac{q-m_2-1}{m-n-1} X\cdot Y\left(\frac{V(h)}{h}\right), \tag{3.43}$$

so (4) holds.
(5)By Definition 2.13 and Propsition 3.2, we have

$$K^{L,\mu}(X,U)V = R^{L,\mu}(X,U)V - \frac{1}{m-n-1}[X\cdot Ric^{L,\mu}(U,V) - (-1)^{|U||V|}Ric^{L,\mu}(X,V)U]$$
$$= (-1)^{|X|(|U|+|V|)}V\left(\frac{X(h)}{h}\right)U - (-1)^{|X|(|U|+|V|)+|V||U|}g_2(V,U)\mathrm{grad}_{g_2}\frac{X(h)}{h}$$
$$- (-1)^{|X|(|U|+|V|)}hg_2(U,V)\nabla^{L,M_1}_X(\mathrm{grad}_{g_2}h) - \frac{1}{m-n-1}\Bigg\{X\cdot\Bigg[Ric^{L,M_2}(U,V)$$
$$- g_\mu(U,V)\left(\frac{\triangle^L_{g_1}(h)}{h} + (q-m_2-1)\frac{(\mathrm{grad}_{g_1}h)(h)}{h^2}\right)\Bigg] + (-1)^{|U||V|}\frac{(q-m_2-1)}{2}$$
$$V\left(\frac{X(h)}{h}\right)U\Bigg\}$$
$$= (-1)^{|X|(|U|+|V|)}V\left(\frac{X(h)}{h}\right)U - (-1)^{|X|(|U|+|V|)+|V||U|}g_2(V,U)\mathrm{grad}_{g_2}\frac{X(h)}{h}$$
$$- (-1)^{|X|(|U|+|V|)}hg_2(U,V)\nabla^{L,M_1}_X(\mathrm{grad}_{g_2}h) - \frac{1}{m-n-1}X\cdot Ric^{L,M_2}(U,V)$$
$$+ \frac{1}{m-n-1}X\cdot g_2(U,V)[h\triangle^L_{g_1}(h) + (-1)^{|U||V|}\frac{(q-m_2-1)}{2}V\left(\frac{X(h)}{h}\right)U, \tag{3.44}$$

so (5) holds.
(6)By Definition 2.13 and Propsition 3.2, we have

$$K^{L,\mu}(U,V)Q$$
$$= R^{L,\mu}(U,V)Q - \frac{1}{m-n-1}[U\cdot Ric^{L,\mu}(V,Q) - (-1)^{|V||Q|}Ric^{L,\mu}(U,Q)V]$$
$$= R^{L,M_2}(U,V)Q + (-1)^{|Q||V|)}g_\mu(U,Q)\mathrm{grad}_{g_2}\frac{V(h)}{h}$$
$$- (-1)^{|U|(|Q|+|V|)}g_\mu(V,Q)\mathrm{grad}_{g_2}\frac{U(h)}{h} - (-1)^{|U|(|Q|+|V|)}\frac{(\mathrm{grad}_{g_1}h)(h)}{h^2}g_2(V,Q)U$$
$$+ (-1)^{|V||Q|}\frac{(\mathrm{grad}_{g_1}h)(h)}{h^2}g_2(U,Q)V$$
$$- \frac{1}{m-n-1}\Bigg\{U\cdot\Bigg[Ric^{L,M_2}(V,Q) - g_\mu(V,Q)\left(\frac{\triangle^L_{g_1}(h)}{h} + (q-m_2-1)\frac{(\mathrm{grad}_{g_1}h)(h)}{h^2}\right)\Bigg]$$
$$- (-1)^{|V||Q|}\Bigg[Ric^{L,M_2}(U,Q) - g_\mu(U,Q)\left(\frac{\triangle^L_{g_1}(h)}{h} + (q-m_2-1)\frac{(\mathrm{grad}_{g_1}h)(h)}{h^2}\right)\Bigg]V\Bigg\}$$
$$= K^{L,M_2}(U,V)Q + \frac{n_1}{(m-n-1)(n_2-1)}[U\cdot Ric^{L,M_2}(V,Q) - (-1)^{|V||Q|}Ric^{L,M_2}(U,Q)V]$$

$$+(-1)^{|Q||V|)}g_\mu(U,Q)\text{grad}_{g_2}\frac{V(h)}{h}-(-1)^{|U|(|Q|+|V|)}g_\mu(V,Q)\text{grad}_{g_2}\frac{U(h)}{h}$$
$$-(-1)^{|U|(|Q|+|V|)}\frac{(\text{grad}_{g_1}h)(h)}{h^2}g_2(V,Q)U+(-1)^{|V||Q|}\frac{(\text{grad}_{g_1}h)(h)}{h^2}g_2(U,Q)V$$
$$+\frac{1}{m-n-1}U\cdot\left[g_2(V,Q)(h\triangle^L_{g_1}(h))+(q-m_2-1)(\text{grad}_{g_1}h)(h)\right]$$
$$-\frac{1}{m-n-1}\left[(-1)^{|V||Q|}g_2(U,Q)(h\triangle^L_{g_1}(h))+(q-m_2-1)(\text{grad}_{g_1}h)(h)\right]V, \tag{3.45}$$

so (6) holds.

□

4. Mixed Ricci flat super twisted products

Definition 4.1. Let $M=M_1\times_\mu M_2$ be a super twisted product of (M_1,g_1) and (M_2,g_2) with twisting function h, then $M=M_1\times_\mu M_2$ is called mixed Ricci-flat if $Ric(X,V)=0$ for all X $\in Vect(M_1)$ and $V\in Vect(M_2)$.

Theorem 4.1. Let $M=M_1\times_\mu M_2$ be a super twisted product of (M_1,g_1) and (M_2,g_2) with twisting function h and $q-m_2-1\neq 0$. Then, $Ric(X,V)=0$ for all X $\in Vect(M_1)$ and $V\in Vect(M_2)$ if and only if $M=M_1\times_\mu M_2$ can be expressed as a super warped product, $M=M_1\times_\mu M_2$ of (M_1,g_1) and $(M_2,\widehat{g_2})$ with a warping function $\widehat{\Phi}$, where $\widehat{g_2}$ is a conformal metric tensor to g_2.

Proof. First, we prove the sufficiency of the theorem. By Propsiton 3.2 and $\frac{X(h)}{h}=X(lnh)$, we know

$$\text{Ric}^{L,\mu}(X,V)=-(q-m_2-1)(-1)^{|X||V|}VX(lnh)=0, \tag{4.1}$$

then by $q-m_2-1\neq 0$, $VX(lnh)=0$ and $XV(lnh)=0$, $XV(lnh)=0$ implies that $V(lnh)$ only depends on the points of M_2, and similarly, $VX(lnh)=0$ implies that $X(lnh)$ only depends on the points of M_1. Thus h can be expressed as a sum of two functions Φ and Ψ which are defined on M_1 and M_2, respectively, that is, $lnh(s,t)=\phi(s)+\psi(t)$ for any $(s,t)\in M_1\times M_2$. Hence $h=e^\phi e^\psi$, that is, $h=\Phi(s)\Psi(t)$, where $\Phi=e^\phi$ and $\Psi=e^\psi$ for any $(s,t)\in M_1\times M_2$. Thus we can write $g=g_1\oplus\Phi^2\widehat{g_2}$, where $\widehat{g_2}=\Psi^2 g_2$, that is, a super twisted product $M_1\times_\mu M_2$ can be expressed as a super warped product $M_1\times_\mu M_2$, where the metric tensor of M_2 is $\widehat{g_2}$ given above.
By Proposition 3.2, we find that it's obvious about the necessity. □

Theorem 4.2. Let $M=M_1\times_\mu M_2$ be a super twisted product of (M_1,g_1) and (M_2,g_2) with twisting function h. If M is a W_2-curvature flat super twisted product, then $M=M_1\times_\mu M_2$ can be expressed as a super warped product.

Proof. By Theorem 3.1, we know

$$K^{L,\mu}(X,Y)Q=-\frac{q-m_2-1}{m-n-1}\left[(-1)^{|Y||Q|}Q\frac{X(h)}{h}Y-X\cdot Q\frac{Y(h)}{h}\right]=0. \tag{4.2}$$

If $q-m_2-1\neq 0$, let $Q=\partial_{y^K},\ X=\partial_{x^I},\ Y=\partial_{x^J}$, when $I\neq J$, we have

$$(-1)^{|\partial_{x^J}||\partial_{y^K}|}\partial_{y^K}[\partial_{x^I}(lnh)]\partial_{x^J}-\partial_{x^I}\cdot\partial_{y^K}[\partial_{x^J}(lnh)]=0, \tag{4.3}$$

Because a pair (I, J) is arbitrary, then $I \neq J$ implies that $\partial_{y^K}[\partial_{x^I}(lnh)] = 0$. So

$$lnh = \phi(x) + \psi(y), \tag{4.4}$$

for any $(x, y) \in M_1 \times M_2$, then $h = e^{\phi(x)}e^{\psi(y)}$.
If $q - m_2 - 1 = 0$, then by Theorem 3.1, we know

$$K^{L,\mu}(U, V)X = (-1)^{|V||X|}UX(lnh)V + (-1)^{|U|(|V|+|X|)}VX(lnh)U = 0, \tag{4.5}$$

similarly, let $U = \partial_{y^P},\ V = \partial_{y^Q},\ X = \partial_{x^I}$, when $P \neq Q$, we have

$$(-1)^{|\partial_{x^I}||\partial_{y^Q}|}\partial_{y^P}[\partial_{x^I}(lnh)]\partial_{y^Q} + (-1)^{|\partial_{y^P}|(|\partial_{y^Q}|+|\partial_{x^I}|)}\partial_{y^Q}[\partial_{x^I}(lnh)]\partial_{y^P} = 0. \tag{4.6}$$

Similar to (4.4), we can get

$$lnh = \omega(x) + \nu(y), \tag{4.7}$$

for any $(x, y) \in M_1 \times M_2$, then $h = e^{\omega(x)}e^{\nu(y)}$, therefore we can get Theorem 4.2. □

Corollary 4.1. Let $M = M_1 \times_\mu M_2$ be a super twisted product of (M_1, g_1) and (M_2, g_2) with twisting function h. Then, M is a super warped product if and only if $K^{L,\mu}(X, Y)Q = 0$ for $X, Y \in Vect(M_1), Q \in Vect(M_2)$.

Acknowledgments. The work was supported in part by NSFC (No.11771070). The authors thank the referee for his (or her) careful reading and helpful comments.

References

[1] Alías, L. Romero, A. Sánchez, M. Spacelike hypersurfaces of constant mean curvature and Calabi-Bernstein type problems. Tohoku Math. J., 49 (1997): 337-345.
[2] Bishop, R. O'Neill, B. Manifolds of negative curvature. Trans. Am. Math. Soc., 145 (1969): 1-49.
[3] Bruce, A. Grabowski, J. Riemannian structures on $\mathbb{Z}_2^n$-manifolds. Mathematics, 8 (2020): 1469.
[4] Bruce, A. Grabowski, J. Odd Connections on Supermanifolds: Existence and relation with Affine Connections. J. Phys. A., 53(45) (2020): 455203, 24 pp.
[5] Dobarro, F. Dozo, E. Scalar curvature and warped products of Riemannian manifolds. Trans. Am. Math. Soc., 303 (1987): 161-168.
[6] Ehrlich, P. Jung, Y. Kim, S. Constant scalar curvatures on warped product manifolds. Tsukuba J. Math., 20(1) (1996): 239-265.
[7] Gebarowski, A. On Einstein warped products. Tensor (N.S.), 52(3) (1993): 204–207.
[8] Gholami, F. Darabi, Y. Mohammadi, M. Varsaie, S. Roshande, M. Einstein equations with cosmological constant in super Space-Time. arXiv: 2108.11437, 2021.
[9] Goertsches, O. Riemannian supergeometry. Math. Z., 260(3) (2008): 557-593.
[10] Fernández-López, E. García-Río, E. Kupeli, D. ünal, B. A curvature condition for a twisted productto be a warped product. Manuscripta Math., 106(2) (2001): 213-217.
[11] Pokhariyal, G. Mishra, R. Curvature tensors and their relativistic significance. Yokohama Math. J., 18 (1970): 105-108.
[12] Shenawy, S. Ünal, B. The W_2-curvature tensor on warped product manifolds and applications. Int. Geom. Methods M. Phys., (2016): 1650099.
[13] Sular, S. Özgür, C. Warped products with a semi-symmetric metric connection. Taiwanese J. Math., 15(4) (2011): 1701–1719.
[14] Sular, S. Özgür, C. Warped Products with a semi-symmetric non-metric connection. Arab. J. Sci. Eng., 36(3) (2011): 461-473.
[15] Wang, Y. Multiply warped products with a semi-symmetric metric connection. Abstr. Appl. Anal., (2014): Art. ID 742371, 12 pp.
[16] Wang, Y. Super warped products with a semi-symmetric metric connection. arXiv: 2112.00887, 2021.

[1]Department of Mathematics
Northeastern University
Shenyang, 110819, China.

[2]School of Mathematics and Statistics
Northeast Normal University
Changchun, 130024, China.
Email address: wut977@nenu.edu.cn
Email address: wangy581@nenu.edu.cn
Email address: 1203907183@qq.com