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A PLANE ANALYTIC GEOMETRY PROOF OF THE FORMULAS FOR THE INSCRIBED AND CIRCUMSCRIBED ELLIPSES OF THREE CONJUGATE ELLIPSES

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Abstract. Let OP_1, OP_2, OP_3 be three non-parallel segments in a plane ω. Let $\mathcal{E}_{P_1,P_2}$, $\mathcal{E}_{P_2,P_3}$ and $\mathcal{E}_{P_3,P_1}$ be the concentric ellipses having as conjugate semi-diameters the pairs (OP_1, OP_2), (OP_2, OP_3) and (OP_3, OP_1), respectively. We give here an alternative plane geometric proof for the formulas for the conjugate semi-diameters of the inscribed and circumscribed ellipses to $\mathcal{E}_{P_1,P_2}$, $\mathcal{E}_{P_2,P_3}$, $\mathcal{E}_{P_3,P_1}$.

1. Introduction and motivations

Let ω be a plane in the three-dimensional Euclidean space $\mathbb{E}^3$ and let $OP_1, OP_2, OP_3 \subset \omega$ be three non-parallel segments. We consider the three conjugate ellipses determined by OP_1, OP_2, OP_3, i.e., the concentric ellipses

$$\mathcal{E}_{P_1,P_2}, \quad \mathcal{E}_{P_2,P_3}, \quad \mathcal{E}_{P_3,P_1}$$

given by the pairs of conjugate semi-diameter (OP_1, OP_2), (OP_2, OP_3) and (OP_3, OP_1), respectively. It was shown (see [1, 2, 3, 11]) that there exist at most two distinct ellipses, with center O, which circumscribes $\mathcal{E}_{P_1,P_2}$, $\mathcal{E}_{P_2,P_3}$ and $\mathcal{E}_{P_3,P_1}$. These are the so-called Pohlke ellipse $\mathcal{E}_{\mathsf{P}}$ and the secondary Pohlke ellipse $\mathcal{E}_{\mathsf{S}}$. More precisely, writing

$$\overrightarrow{OP_3} = h\,\overrightarrow{OP_1} + k\,\overrightarrow{OP_2} \quad \text{with} \quad h, k \neq 0, \tag{1.1}$$

it is possible to see that:

i) the Pohlke ellipse $\mathcal{E}_{\mathsf{P}}$ is determined by the pair of conjugate semi-diameters given by the vectors (see [4, 6]):

$$\sqrt{\frac{1+h^2+k^2}{h^2+k^2}}\,\overrightarrow{OP_3} \quad \text{and} \quad \frac{-k\overrightarrow{OP_1} + h\overrightarrow{OP_2}}{\sqrt{h^2+k^2}}\,; \tag{1.2}$$

ii) setting $g = g(h,k)$ with

$$g(h,k) \stackrel{\text{def}}{=} \left[(h+k)^2 - 1)\right]\left[(h-k)^2 - 1)\right], \tag{1.3}$$

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the secondary Pohlke ellipse $\mathcal{E}_{\mathtt{S}}$ exists if and only if the coefficients h,k in (1.1) are such that $g>0$. See [5, 6, 9]. If $g>0$ a pair of conjugate semi-diameters is given by the vectors:

$$\sqrt{\frac{g+H^2+K^2}{g}}\left(\frac{H\overrightarrow{OP_1}+K\overrightarrow{OP_2}}{\sqrt{H^2+K^2}}\right) \quad \text{and} \quad \frac{-K\overrightarrow{OP_1}+H\overrightarrow{OP_2}}{\sqrt{H^2+K^2}}, \tag{1.4}$$

where $H=H(h,k)$ and $K=K(h,k)$ with

$$H(h,k)\overset{\text{def}}{=}h(h^2-k^2-1), \qquad K(h,k)\overset{\text{def}}{=}k(h^2-k^2+1).^{1} \tag{1.5}$$

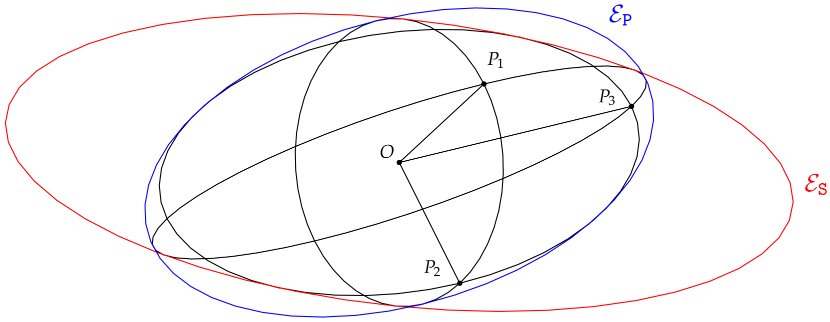

Figure 1. $\mathcal{E}_{\mathtt{P}}$ and $\mathcal{E}_{\mathtt{S}}$ with $P_1=(1.4,1.3), P_2=(1,-2), \overrightarrow{OP_3}=2.1\overrightarrow{OP_1}+0.9\overrightarrow{OP_2}$.

iii) When, instead of $g>0$, we assume

$$g<0 \quad \text{and} \quad g+H^2+K^2<0, \tag{1.6}$$

there exists a unique concentric ellipse $\mathcal{E}_{\mathtt{I}}$ inscribed in $\mathcal{E}_{P_1,P_2}$, $\mathcal{E}_{P_2,P_3}$, $\mathcal{E}_{P_3,P_1}$. A pair of conjugate semi-diameters is given (as for $\mathcal{E}_{\mathtt{S}}$) by the expressions in (1.4).[2] See [7, 8, 9].

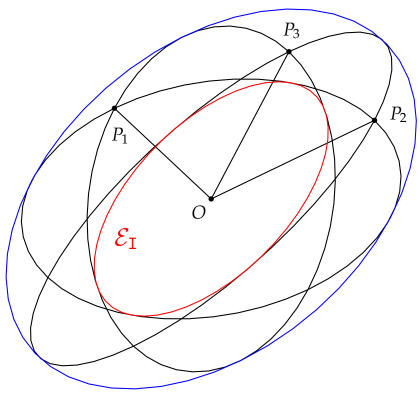

Figure 2. $\mathcal{E}_{\mathtt{P}}$ and $\mathcal{E}_{\mathtt{I}}$ with $P_1=(-1.6,1.5), P_2=(2.7,1.3), \overrightarrow{OP_3}=0.8\overrightarrow{OP_1}+0.95\overrightarrow{OP_2}$.

[1] It is immediate that $h,k\neq 0 \Rightarrow H^2+K^2>0$. Hence, the semi-diameters (1.4) are well defined. Noting the definition (1.3), we also have that $h,k\neq 0$ and $g(h,k)>0 \Rightarrow H,K\neq 0$.

[2] We can easily see that $g(h,k)<0 \Rightarrow h,k\neq 0$. From the identity (5.9) one can also deduce that $h,k\neq 0$ and $g+H^2+K^2\neq 0 \Rightarrow H,K\neq 0$.

2. Main results

Formulas (1.2), (1.4) are obtained in [5, 6, 7, 8] as a by-product of the construction of a specific parallel projection

$$\Pi : \mathbb{E}^3 \to \omega.$$

In case i) and ii) this parallel projection is applied to an appropriate sphere centered at O, say $\mathcal{S}$, and the circumscribed ellipse ($\mathcal{E}_{\mathtt{P}}$ or $\mathcal{E}_{\mathtt{S}}$) is obtained as the contour of $\Pi(\mathcal{S})$ in the plane ω. In case iii) the parallel projection is applied to a suitable one-sheeted hyperboloid of rotation, say $\mathcal{H}$, centered at O and with axis perpendicular to ω. The inscribed ellipse $\mathcal{E}_{\mathtt{I}}$ is the contour of $\Pi(\mathcal{H})$ in ω.

Here we propose direct proofs of formulas (1.2), (1.4) based exclusively upon arguments of plane analytic geometry. More precisely, we just need to apply the simple properties of the pairs of conjugate semi-diameters of the ellipse.

3. Preliminaries

Let $OU, OV \subset \omega$ be non-parallel segments.

Definition 3.1. We indicate with $\mathcal{E}_{U,V} \subset \omega$ the ellipse, centered at O, determined by the pair of conjugate semi-diameters (OU, OV). We also denote with $\overline{\mathcal{E}_{U,V}}$ the set of points $P \in \omega$ such that $OP \cap \mathcal{E}_{U,V} \subset \{P\}$.[3]

As it is known,[4] (OU', OV') is a pair of conjugate semi-diameters of $\mathcal{E}_{U,V}$ iff for some $\alpha \in [0, 2\pi)$ one has

$$\overrightarrow{OU'} = \cos\alpha\overrightarrow{OU} + \sin\alpha\overrightarrow{OV} \quad \text{and} \quad \overrightarrow{OV'} = \pm\big(-\sin\alpha\overrightarrow{OU} + \cos\alpha\overrightarrow{OV}\big). \tag{3.1}$$

In other words, fixed any pair of semi-diameters (OU, OV), we can say that

Claim 3.2. (OU', OV') is a pair of conjugate semi-diameters of $\mathcal{E}_{U,V}$ iff

$$\begin{pmatrix} \overrightarrow{OU'} \\ \overrightarrow{OV'} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & \pm 1 \end{pmatrix} \begin{pmatrix} \cos\alpha & \sin\alpha \\ -\sin\alpha & \cos\alpha \end{pmatrix} \begin{pmatrix} \overrightarrow{OU} \\ \overrightarrow{OV} \end{pmatrix}. \tag{3.2}$$

for some $\alpha \in [0, 2\pi)$.

Given $OS, OT \subset \omega$ non-parallel segments,

Definition 3.3. We say that $\mathcal{E}_{S,T}$ circumscribes $\mathcal{E}_{U,V}$ (equivalently, $\mathcal{E}_{U,V}$ is inscribed in $\mathcal{E}_{S,T}$) iff $\mathcal{E}_{U,V}$ and $\mathcal{E}_{S,T}$ are tangent and $\overline{\mathcal{E}_{U,V}} \subsetneqq \overline{\mathcal{E}_{S,T}}$.

It is then evident that:

Claim 3.4. $\mathcal{E}_{S,T}$ circumscribes $\mathcal{E}_{U,V}$ iff there exist a pair (OS', OT') of conjugate semi-diameters of $\mathcal{E}_{S,T}$ and a pair (OU', OV') of conjugate semi-diameters of $\mathcal{E}_{U,V}$ such that

$$\overrightarrow{OS'} = \lambda\,\overrightarrow{OU'} \quad \text{and} \quad \overrightarrow{OT'} = \pm\overrightarrow{OV'}, \tag{3.3}$$

with $\lambda > 1$. Conversely, $\mathcal{E}_{S,T}$ is inscribed in $\mathcal{E}_{U,V}$ iff (3.3) holds with $0 < \lambda < 1$.[5] In both cases the two ellipses are tangent at P with $\overrightarrow{OP} = \pm\overrightarrow{OT'}$.

[3] That is, $\mathcal{E}_{U,V}$ and the points inside $\mathcal{E}_{U,V}$.
[4] See, for instance, [10].
[5] We clearly have $\mathcal{E}_{S,T} = \mathcal{E}_{U,V}$ iff $\lambda = 1$.

We know define the following ellipses:

Definition 3.5. Given non-parallel segments $OU, OV \subset \omega$ and $u, v \neq 0$, we denote with $\mathcal{E}(U, V, u, v)$ the ellipse, centered at O, with conjugate semi-diameters given by the vectors

$$\sqrt{1+u^2+v^2}\left(\frac{u\overrightarrow{OU}+v\overrightarrow{OV}}{\sqrt{u^2+v^2}}\right) \quad \text{and} \quad \frac{-v\overrightarrow{OU}+u\overrightarrow{OV}}{\sqrt{u^2+v^2}}. \tag{3.4}$$

From Claim 3.2 and Claim 3.4, it is clear that:

Claim 3.6. $\mathcal{E}(U, V, u, v)$ circumscribes the concentric ellipse $\mathcal{E}_{U,V}$. Furthermore, the two ellipses are tangent at the point P with

$$\overrightarrow{OP} = \pm\frac{-v\overrightarrow{OU}+u\overrightarrow{OV}}{\sqrt{u^2+v^2}}. \tag{3.5}$$

Next we consider the functions $g = g(u, v)$, $H = H(u, v)$, $K = K(u, v)$ defined as in (1.3) and (1.5), noting that

$$u, v \neq 0 \;\Rightarrow\; H^2 + K^2 > 0. \tag{3.6}$$

Then, assuming

$$u, v \neq 0 \quad \text{and} \quad g\left(g + H^2 + K^2\right) > 0, \tag{3.7}$$

we can introduce a second type of ellipse:

Definition 3.7. Suppose (3.7) holds. We denote with $\widetilde{\mathcal{E}}(U, V, u, v)$ the ellipse, centered O, with conjugate semi-diameters given by the vectors

$$\sqrt{\frac{g+H^2+K^2}{g}}\left(\frac{H\overrightarrow{OU}+K\overrightarrow{OV}}{\sqrt{H^2+K^2}}\right) \quad \text{and} \quad \frac{-K\overrightarrow{OU}+H\overrightarrow{OV}}{\sqrt{H^2+K^2}}. \tag{3.8}$$

Since, by (3.6),

$$u, v \neq 0 \quad \text{and} \quad g > 0 \quad \Rightarrow \quad \frac{g+H^2+K^2}{g} > 1,$$

from Claim 3.2 and Claim 3.4 we have:

Claim 3.8. If $u, v \neq 0$ are such that $g > 0$, then $\widetilde{\mathcal{E}}(U, V, u, v)$ circumscribes $\mathcal{E}_{U,V}$ and the two ellipses are tangent at the point P with

$$\overrightarrow{OP} = \pm\frac{-K\overrightarrow{OU}+H\overrightarrow{OV}}{\sqrt{H^2+K^2}}. \tag{3.9}$$

On the contrary, if

$$g < 0 \quad \text{and} \quad g + H^2 + K^2 < 0, \tag{3.10}$$

we find

$$0 < \frac{g+H^2+K^2}{g} < 1. \tag{3.11}$$

Then Claim 3.2 together with the last part of Claim 3.4 give:

Claim 3.9. If $u, v \neq 0$ are such that $g < 0$ and $g + H^2 + K^2 < 0$, then $\widetilde{\mathcal{E}}(U, V, u, v)$ is inscribed in $\mathcal{E}_{U,V}$. The ellipses are tangent at the two points defined by (3.9).

4. Proof of formulas (1.2)

Let $OP_1, OP_2, OP_3 \subset \omega$ be three non-parallel segments. Writing

$$\overrightarrow{OP_3} = h\,\overrightarrow{OP_1} + k\,\overrightarrow{OP_2} \quad \text{with} \quad h,k \neq 0, \tag{4.1}$$

according to Def. 3.5 we can define the ellipse $\mathcal{E}(P_1,P_2,h,k)$. For Claim 3.6, $\mathcal{E}(P_1,P_2,h,k)$ circumscribes the concentric ellipse $\mathcal{E}_{P_1,P_2}$. We want to prove that:

Claim 4.1. $\mathcal{E}(P_1,P_2,h,k)$ circumscribes also $\mathcal{E}_{P_2,P_3}$ and $\mathcal{E}_{P_3,P_1}$.

From (4.1), we can write the equivalent expressions

$$\overrightarrow{OP_1} = -\frac{k}{h}\,\overrightarrow{OP_2} + \frac{1}{h}\,\overrightarrow{OP_3}, \tag{4.2}$$

$$\overrightarrow{OP_2} = \frac{1}{k}\,\overrightarrow{OP_3} - \frac{h}{k}\,\overrightarrow{OP_1}. \tag{4.3}$$

So we also define the ellipses:

- $\mathcal{E}(P_2,P_3,-\frac{k}{h},\frac{1}{h})$, which circumscribes the concentric ellipse $\mathcal{E}_{P_2,P_3}$
- $\mathcal{E}(P_3,P_1,\frac{1}{k},-\frac{h}{k})$, which circumscribes the concentric ellipse $\mathcal{E}_{P_3,P_1}$

by still applying Claim 3.6.

In conclusion, it will be sufficient to demonstrate that:

Claim 4.2. $\mathcal{E}(P_1,P_2,h,k) = \mathcal{E}(P_2,P_3,-\frac{k}{h},\frac{1}{h}) = \mathcal{E}(P_3,P_1,-\frac{1}{h},\frac{h}{k})$.

Proof. We will limit ourselves to proving the first equality, that is

$$\mathcal{E}(P_1,P_2,h,k) = \mathcal{E}(P_2,P_3,-\tfrac{k}{h},\tfrac{1}{h}). \tag{4.4}$$

The other can be proven in the same way.
By Def. 3.5, a pair of conjugate semi-diameters of $\mathcal{E}(P_1,P_2,h,k)$ is given by

$$\begin{aligned}\begin{pmatrix} \overrightarrow{OU} \\ \overrightarrow{OV} \end{pmatrix} &= \begin{pmatrix} \sqrt{1+h^2+k^2} & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \frac{h}{\sqrt{h^2+k^2}} & \frac{k}{\sqrt{h^2+k^2}} \\ -\frac{k}{\sqrt{h^2+k^2}} & \frac{h}{\sqrt{h^2+k^2}} \end{pmatrix} \begin{pmatrix} \overrightarrow{OP_1} \\ \overrightarrow{OP_2} \end{pmatrix} \\ &= \frac{1}{\sqrt{h^2+k^2}} \begin{pmatrix} L & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} h & k \\ -k & h \end{pmatrix} \begin{pmatrix} \overrightarrow{OP_1} \\ \overrightarrow{OP_2} \end{pmatrix} \\ &\stackrel{\text{def}}{=} A \begin{pmatrix} \overrightarrow{OP_1} \\ \overrightarrow{OP_2} \end{pmatrix},\end{aligned} \tag{4.5}$$

with $A = A(h,k)$ a 2×2 matrix and

$$L = L(h,k) \stackrel{\text{def}}{=} \sqrt{1+h^2+k^2}. \tag{4.6}$$

On the other hand, since

$$\begin{pmatrix} \overrightarrow{OP_2} \\ \overrightarrow{OP_3} \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ h & k \end{pmatrix} \begin{pmatrix} \overrightarrow{OP_1} \\ \overrightarrow{OP_2} \end{pmatrix}, \tag{4.7}$$

for $\mathcal{E}(P_2, P_3, -\frac{k}{h}, \frac{1}{h})$ Def. 3.5 gives the pair

$$\begin{aligned}
\begin{pmatrix} \overrightarrow{OU'} \\ \overrightarrow{OV'} \end{pmatrix} &= \begin{pmatrix} \sqrt{1+\frac{k^2}{h^2}+\frac{1}{h^2}} & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} -\frac{k}{h\sqrt{\frac{k^2}{h^2}+\frac{1}{h^2}}} & \frac{1}{h\sqrt{\frac{k^2}{h^2}+\frac{1}{h^2}}} \\ -\frac{1}{h\sqrt{\frac{k^2}{h^2}+\frac{1}{h^2}}} & -\frac{k}{h\sqrt{\frac{k^2}{h^2}+\frac{1}{h^2}}} \end{pmatrix} \begin{pmatrix} \overrightarrow{OP_2} \\ \overrightarrow{OP_3} \end{pmatrix} \\
&= -\frac{|h|/h}{\sqrt{1+k^2}} \begin{pmatrix} \frac{L}{|h|} & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} k & -1 \\ 1 & k \end{pmatrix} \begin{pmatrix} \overrightarrow{OP_2} \\ \overrightarrow{OP_3} \end{pmatrix} \\
&= -\frac{|h|/h}{\sqrt{1+k^2}} \begin{pmatrix} \frac{kL}{|h|} & -\frac{L}{|h|} \\ 1 & k \end{pmatrix} \begin{pmatrix} 0 & 1 \\ h & k \end{pmatrix} \begin{pmatrix} \overrightarrow{OP_1} \\ \overrightarrow{OP_2} \end{pmatrix} \\
&\stackrel{\text{def}}{=} B \begin{pmatrix} \overrightarrow{OP_1} \\ \overrightarrow{OP_2} \end{pmatrix},
\end{aligned} \tag{4.8}$$

with $B = B(h,k)$ a 2×2 matrix.
By Claim 3.2 we must prove that (3.2) holds for some $\alpha \in [0, 2\pi)$. Therefore, it will be sufficient to prove that BA^{-1}, i.e., the matrix

$$\frac{-\frac{|h|}{h}}{\sqrt{1+k^2}\sqrt{h^2+k^2}} \begin{pmatrix} \frac{kL}{|h|} & -\frac{L}{|h|} \\ 1 & k \end{pmatrix} \begin{pmatrix} 0 & 1 \\ h & k \end{pmatrix} \begin{pmatrix} h & -k \\ k & h \end{pmatrix} \begin{pmatrix} \frac{1}{L} & 0 \\ 0 & 1 \end{pmatrix}, \tag{4.9}$$

is equal to

$$\begin{pmatrix} 1 & 0 \\ 0 & \pm 1 \end{pmatrix} \begin{pmatrix} \cos\alpha & \sin\alpha \\ -\sin\alpha & \cos\alpha \end{pmatrix} \tag{4.10}$$

for a suitable choice of the sign $+, -$ and for some $\alpha \in [0, 2\pi)$.
A simple calculation shows that

$$\begin{aligned}
BA^{-1} &= \frac{1}{\sqrt{1+k^2}\sqrt{h^2+k^2}} \begin{pmatrix} h & -kL \\ -kL\frac{|h|}{h} & -h\frac{|h|}{h} \end{pmatrix} \\
&= \frac{1}{\sqrt{1+k^2}\sqrt{h^2+k^2}} \begin{pmatrix} 1 & 0 \\ 0 & -\frac{|h|}{h} \end{pmatrix} \begin{pmatrix} h & -kL \\ kL & h \end{pmatrix},
\end{aligned} \tag{4.11}$$

where, by (4.6),

$$\frac{1}{\sqrt{1+k^2}\sqrt{h^2+k^2}} \begin{pmatrix} h & -kL \\ kL & h \end{pmatrix} = \begin{pmatrix} \cos\alpha & \sin\alpha \\ -\sin\alpha & \cos\alpha \end{pmatrix} \tag{4.12}$$

with

$$\cos\alpha = \frac{h}{\sqrt{1+k^2}\sqrt{h^2+k^2}}, \quad \sin\alpha = \frac{-kL}{\sqrt{1+k^2}\sqrt{h^2+k^2}}. \tag{4.13}$$

Therefore BA^{-1} is of the form (4.10). □

5. Proof of formulas (1.4) in case *ii)*

We write again (4.1), but now we suppose $h, k \neq 0$ such that

$$g(h,k) > 0. \tag{5.1}$$

We can then define the ellipse $\widetilde{\mathcal{E}}(P_1,P_2,h,k)$, according to Def. 3.7. By Claim 3.8, we know that $\widetilde{\mathcal{E}}(P_1,P_2,h,k)$ circumscribes the concentric ellipse $\mathcal{E}_{P_1,P_2}$. As in the proof of formula (1.2), we want to show that:

Claim 5.1. $\widetilde{\mathcal{E}}(P_1,P_2,h,k)$ circumscribes also $\mathcal{E}_{P_2,P_3}$ and $\mathcal{E}_{P_3,P_1}$.

From definition (1.3), we easily get

$$h^4\, g\left(-\tfrac{k}{h},\tfrac{1}{h}\right) = k^4\, g\left(\tfrac{1}{k},-\tfrac{h}{k}\right) = g(h,k). \tag{5.2}$$

Thus we have also $g(-\frac{k}{h},\frac{1}{h}),\, g(\frac{1}{k},-\frac{h}{k}) > 0$. So, noting the equivalent expressions (4.2), (4.3) and taking into account Claim 3.8, we define the ellipses:

- $\widetilde{\mathcal{E}}(P_2,P_3,-\frac{k}{h},\frac{1}{h})$, which circumscribes the concentric ellipse $\mathcal{E}_{P_2,P_3}$
- $\widetilde{\mathcal{E}}(P_3,P_1,\frac{1}{k},-\frac{h}{k})$, which circumscribes the concentric ellipse $\mathcal{E}_{P_3,P_1}$

Therefore it will be sufficient to demonstrate that:

Claim 5.2. $\widetilde{\mathcal{E}}(P_1,P_2,h,k) = \widetilde{\mathcal{E}}(P_2,P_3,-\frac{k}{h},\frac{1}{h}) = \widetilde{\mathcal{E}}(P_3,P_1,\frac{1}{k},-\frac{h}{k})$.

Proof. As in the proof of Claim 4.2, we will prove only the first equality. That is, we will show that $\widetilde{\mathcal{E}}(P_1,P_2,h,k) = \widetilde{\mathcal{E}}(P_2,P_3,-\frac{k}{h},\frac{1}{h})$.
By Def. 3.7, a pair of conjugate semi-diameters of $\widetilde{\mathcal{E}}(P_1,P_2,h,k)$ is given by

$$\begin{aligned}\begin{pmatrix}\overrightarrow{OU}\\ \overrightarrow{OV}\end{pmatrix} &= \begin{pmatrix}\sqrt{\frac{g+H^2+K^2}{g}} & 0\\ 0 & 1\end{pmatrix}\begin{pmatrix}\frac{H}{\sqrt{H^2+K^2}} & \frac{K}{\sqrt{H^2+K^2}}\\ -\frac{K}{\sqrt{H^2+K^2}} & \frac{H}{\sqrt{H^2+K^2}}\end{pmatrix}\begin{pmatrix}\overrightarrow{OP_1}\\ \overrightarrow{OP_2}\end{pmatrix}\\ &= \frac{1}{\sqrt{H^2+K^2}}\begin{pmatrix}M & 0\\ 0 & 1\end{pmatrix}\begin{pmatrix}H & K\\ -K & H\end{pmatrix}\begin{pmatrix}\overrightarrow{OP_1}\\ \overrightarrow{OP_2}\end{pmatrix} \stackrel{\text{def}}{=} \widetilde{A}\begin{pmatrix}\overrightarrow{OP_1}\\ \overrightarrow{OP_2}\end{pmatrix},\end{aligned} \tag{5.3}$$

with $\widetilde{A} = \widetilde{A}(h,k)$ a 2×2 matrix and

$$M = M(g,H,K) \stackrel{\text{def}}{=} \sqrt{\frac{g+H^2+K^2}{g}}. \tag{5.4}$$

Taking into account (4.7), for $\widetilde{\mathcal{E}}(P_2,P_3,-\frac{k}{h},\frac{1}{h})$ Def. 3.7 gives the pair

$$\begin{aligned}\begin{pmatrix}\overrightarrow{OU'}\\ \overrightarrow{OV'}\end{pmatrix} &= \begin{pmatrix}\sqrt{\frac{g'+H'^2+K'^2}{g'}} & 0\\ 0 & 1\end{pmatrix}\begin{pmatrix}\frac{H'}{\sqrt{H'^2+K'^2}} & \frac{K'}{\sqrt{H'^2+K'^2}}\\ -\frac{K'}{\sqrt{H'^2+K'^2}} & \frac{H'}{\sqrt{H'^2+K'^2}}\end{pmatrix}\begin{pmatrix}\overrightarrow{OP_2}\\ \overrightarrow{OP_3}\end{pmatrix}\\ &= \frac{1}{\sqrt{H'^2+K'^2}}\begin{pmatrix}M' & 0\\ 0 & 1\end{pmatrix}\begin{pmatrix}H' & K'\\ -K' & H'\end{pmatrix}\begin{pmatrix}0 & 1\\ h & k\end{pmatrix}\begin{pmatrix}\overrightarrow{OP_1}\\ \overrightarrow{OP_2}\end{pmatrix}\\ &\stackrel{\text{def}}{=} \widetilde{B}\begin{pmatrix}\overrightarrow{OP_1}\\ \overrightarrow{OP_2}\end{pmatrix},\end{aligned} \tag{5.5}$$

with $\widetilde{B} = \widetilde{B}(h,k)$ a 2×2 matrix,

$$g' = g\left(-\tfrac{k}{h},\tfrac{1}{h}\right),\quad H' = H\left(-\tfrac{k}{h},\tfrac{1}{h}\right),\quad K' = K\left(-\tfrac{k}{h},\tfrac{1}{h}\right) \tag{5.6}$$

and $M' = M(g',H',K')$, with M according to the definition (5.4).

As above it is sufficient to prove that $\widetilde{B}\widetilde{A}^{-1}$ is of the form (4.10). To begin with, from definitions (1.3) and (1.5), and noting (5.2), it follows that:

$$g' = \frac{g(h,k)}{h^4}, \tag{5.7}$$

$$H' = \frac{K}{h^3}, \quad K' = \frac{N}{h^3} \quad \text{with} \quad N = N(h,k) \stackrel{\text{def}}{=} h^2 + k^2 - 1. \tag{5.8}$$

Furthermore, noting the identity

$$g + H^2 + K^2 \equiv (h^2 + k^2 - 1)\left[(h^2 - k^2)^2 - 1\right] \tag{5.9}$$

and definition (5.4), we easily get that

$$g' + H'^2 + K'^2 = \frac{g + H^2 + K^2}{h^6}, \tag{5.10}$$

$$M' = \frac{M}{|h|}. \tag{5.11}$$

Therefore, from (5.5), we obtain the expression

$$\begin{aligned} \widetilde{B} &= \frac{1}{\sqrt{\frac{K^2}{h^6} + \frac{N^2}{h^6}}} \begin{pmatrix} \frac{M}{|h|} & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \frac{K}{h^3} & \frac{N}{h^3} \\ -\frac{N}{h^3} & \frac{K}{h^3} \end{pmatrix} \begin{pmatrix} 0 & 1 \\ h & k \end{pmatrix} \\ &= \frac{\frac{|h|}{h}}{\sqrt{K^2 + N^2}} \begin{pmatrix} \frac{M}{|h|} & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} K & N \\ -N & K \end{pmatrix} \begin{pmatrix} 0 & 1 \\ h & k \end{pmatrix} \\ &= \frac{1}{\sqrt{K^2 + N^2}} \begin{pmatrix} 1 & 0 \\ 0 & -\frac{|h|}{h} \end{pmatrix} \begin{pmatrix} \frac{M}{h}K & \frac{M}{h}N \\ N & -K \end{pmatrix} \begin{pmatrix} 0 & 1 \\ h & k \end{pmatrix}. \end{aligned} \tag{5.12}$$

Moreover, we have

$$\begin{aligned} &\begin{pmatrix} 0 & 1 \\ h & k \end{pmatrix} \begin{pmatrix} H & -K \\ K & H \end{pmatrix} \begin{pmatrix} \frac{1}{M} & 0 \\ 0 & 1 \end{pmatrix} = \\ &= \begin{pmatrix} \frac{K}{M} & H \\ \frac{hH + kK}{M} & kH - hK \end{pmatrix} = \begin{pmatrix} \frac{K}{M} & H \\ \frac{(h^2 - k^2)N}{M} & -2hk \end{pmatrix}. \end{aligned} \tag{5.13}$$

Therefore, after some calculations, we find that $\widetilde{B}\widetilde{A}^{-1}$ can be expressed as

$$\frac{1}{\sqrt{K^2 + N^2}\sqrt{H^2 + K^2}} \begin{pmatrix} 1 & 0 \\ 0 & -\frac{|h|}{h} \end{pmatrix} \begin{pmatrix} \frac{K^2 + (h^2 - k^2)N^2}{h} & \frac{M}{h}(HK - 2hkN) \\ -\frac{(h^2 - k^2 - 1)KN}{M} & HN + 2hkK \end{pmatrix}. \tag{5.14}$$

To conclude, it is enough to observe that

$$\frac{K^2 + (h^2 - k^2)N^2}{h} = HN + 2hkK \stackrel{\text{def}}{=} \Phi, \tag{5.15}$$

$$\frac{M}{h}(HK - 2hkN) = \frac{(h^2 - k^2 - 1)KN}{M} \stackrel{\text{def}}{=} \Psi, \tag{5.16}$$

and that

$$\Phi^2 + \Psi^2 = (K^2 + N^2)(H^2 + K^2).^6 \tag{5.17}$$

This proves that $\widetilde{B}\widetilde{A}^{-1}$ is of the form (4.10) with

$$\cos\alpha = \frac{\Phi}{\sqrt{K^2+N^2}\sqrt{H^2+K^2}}, \quad \sin\alpha = \frac{\Psi}{\sqrt{K^2+N^2}\sqrt{H^2+K^2}}. \tag{5.18}$$

6. Proof of formulas (1.4) in case *iii*)

Assume that (1.6) is true, i.e., $g < 0$ and $g + H^2 + K^2 < 0$. Applying Claim 3.9 we know that $\widetilde{\mathcal{E}}(P_1, P_2, h, k)$ is inscribed in the concentric ellipse $\mathcal{E}_{P_1,P_2}$. We must prove that

Claim 6.1. $\widetilde{\mathcal{E}}(P_1, P_2, h, k)$ is inscribed also in $\mathcal{E}_{P_2,P_3}$ and $\mathcal{E}_{P_3,P_1}$.

Noting (5.2), (5.9) and (5.10), we find that the condition (1.6) still holds if we replace h, k with $-\frac{k}{h}, \frac{1}{h}$ or with $\frac{1}{k}, -\frac{h}{k}$. We can then define the ellipses $\widetilde{\mathcal{E}}(P_2, P_3, -\frac{k}{h}, \frac{1}{h})$ and $\widetilde{\mathcal{E}}(P_3, P_1, \frac{1}{k}, -\frac{h}{k})$ according to Def. 3.7. So, noting (4.2), (4.3) and applying Claim 3.9 again, we find that $\widetilde{\mathcal{E}}(P_2, P_3, -\frac{k}{h}, \frac{1}{h})$ is inscribed in $\mathcal{E}_{P_2,P_3}$ and that $\widetilde{\mathcal{E}}(P_3, P_1, \frac{1}{k}, -\frac{h}{k})$ is inscribed in $\mathcal{E}_{P_3,P_1}$. As in Section 4, it is therefore enough to prove that

$$\widetilde{\mathcal{E}}(P_1, P_2, h, k) = \widetilde{\mathcal{E}}(P_2, P_3, -\tfrac{k}{h}, \tfrac{1}{h}) = \widetilde{\mathcal{E}}(P_3, P_1, \tfrac{1}{k}, -\tfrac{h}{k}).$$

But the proof of these equalities is formally identical to the proof of Claim 5.2. □

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[6] It is not really necessary to check (5.17). In fact, from (5.3), (5.12) we know that $\det(\widetilde{A}) = M$ and $\det(\widetilde{B}) = -M|h|/h$. Therefore the matrix determined by the expression (5.14) must have determinant $-|h|/h$.