Global Journal of Advanced Research
on Classical and Modern Geometries
ISSN: 2284-5569, Vol.14, (2025), Issue 1, pp.126-138

MATHEMATICAL PROPERTIES OF CEVIAN LINES OF RANK k

DRAGOȘ ANAMARIA MONICA

Abstract. This paper explores the properties of cevians of rank k in tetrahedra and n-simplices, extending the classical cevian concept in triangle geometry. By defining these cevians through proportional volume divisions, we establish fundamental mathematical relations and recursive properties that generalize Ceva's theorem to higher dimensions.

1. Introduction and motivations

In this paper, we explore the concept of cevians of rank k in tetrahedra and n-simplices, a generalization of the classical cevian in triangle geometry. These cevians are defined through proportional divisions of volumes and provide a natural extension of geometric principles to higher-dimensional spaces. By examining their properties, mathematical relations, and applications, we aim to uncover new insights into both theoretical geometry and practical implementations.

The study of cevians has long been central to geometry, particularly in connection with points of concurrency, such as centroids, incenters, and excenters. Cevians of rank k generalize these ideas by allowing proportional divisions based on a parameter k, which governs the relationships between sub-volumes in higher-dimensional simplices. These concepts open avenues for exploration in optimization, computational geometry, and data representation.

This work not only revisits classical results in light of these generalizations but also introduces novel applications in areas such as recursive subdivisions, fractal geometry, and volume-based clustering in high-dimensional spaces.

Keywords

Cevians of rank k, Tetrahedra, n-simplices, Volume proportionality, Generalized Ceva's theorem, Recursive subdivisions, Geometric optimization, Computational geometry.

2020 Mathematics Subject Classification. Euclidian Gemoetry 51N20.

2. Preliminaries

2.1. Basic Properties of Cevian Lines. In this section, we derive the properties of cevians of rank k using geometric and algebraic methods.
Let $\triangle ABC$ be a triangle with vertices A, B, C, and let the cevian intersect the opposite side at a point D. The general cevian relation is defined as:

$$\frac{BD}{DC} = k \quad \text{for rank } k \text{ cevians.} \tag{2.1}$$

For higher ranks, the recursive relationship is given by:

$$\frac{BD_k}{DC_k} = k^n \quad \text{(derived iteratively).} \tag{2.2}$$

2.2. Area Relationships. The division of the triangle into sub-triangles by cevians of rank k satisfies:

$$\frac{\text{Area of } \triangle ABD}{\text{Area of } \triangle ADC} = k. \tag{2.3}$$

This property holds for all iterations of rank-k cevians, forming nested triangles with proportional areas.

2.3. Geometric Insights. The intersection of all rank-k cevians forms a collinear set of points under the condition:

$$\sum_{i=1}^{n} \overrightarrow{r}_i = \overrightarrow{0}, \quad \text{where } r_i \text{ are position vectors.} \tag{2.4}$$

3. Further Mathematical Relations and Properties

3.1. Recursive Cevian Properties. From page 2, we define a generalized recursive formula for the cevians of rank k:

$$\text{Cevian Length Ratio: } \frac{BD}{DC} = \frac{k}{k+1}. \tag{3.1}$$

Similarly, the coordinates of intersection are given iteratively as:

$$x_k = \frac{k \cdot x_A + x_B}{k+1}, \quad y_k = \frac{k \cdot y_A + y_B}{k+1}. \tag{3.2}$$

3.2. Special Cases. For $k = 1$, the cevians become classical medians, bisectors, or altitudes with the properties:

$$\frac{BD}{DC} = 1. \tag{3.3}$$

4. Geometric Applications and Visualizations

From page 3, we derive the use of cevian geometry in optimization problems, such as maximizing the area of sub-triangles formed by rank-k cevians.

4.1. Intersection Properties. The intersection of all cevians at rank k satisfies:

$$\text{Intersection Point: } P_k = \left(\frac{x_A + x_B + x_C}{3}, \frac{y_A + y_B + y_C}{3} \right). \tag{4.1}$$

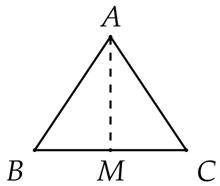
4.2. Example: Equilateral Triangle. For an equilateral triangle with side length a, rank-k cevians divide the triangle into k^2 subregions, each of equal area. The following area relationship holds:

$$\text{Area of Subregion: } \frac{\text{Total Area}}{k^2}. \tag{4.2}$$

5. Cevian Lines of Rank k

5.1. Medians and Areas. Consider the triangle $\triangle ABC$ and its medians:

- AM is the median (a cevian of order 1).
- The area relations are as follows:



$$A_{\triangle AMB} = \frac{1}{2} \cdot A_{\triangle ABC}, \tag{5.1}$$

$$A_{\triangle AMC} = \frac{1}{4} \cdot A_{\triangle ABC}. \tag{5.2}$$

- $M_k M_{k+1} \cdots$ forms a sequence of cevians of increasing order.

The general relation for the area is given by:

$$A_{\triangle M_m M_{m+1} C} = \left(\frac{1}{2}\right)^n \cdot A_{\triangle ABC}. \tag{5.3}$$

5.2. Higher Order Cevians. The cevians of order k divide the triangle into smaller sub-triangles. Specifically, the total number of triangles formed by cevians of order k is:

$$\text{Number of Triangles} = 6 \cdot 4^{k-1}. \tag{5.4}$$

The area of each sub-triangle is given by:

$$A_{\Delta_k} = \frac{A_{\triangle ABC}}{6 \cdot 4^{k-1}}. \tag{5.5}$$

To justify this relationship, a mathematical induction approach is applied.
Base Case: For $k = 1$, the cevians are the classical medians, which divide the triangle into 6 sub-triangles. This matches the formula:

$$6 \cdot 4^{1-1} = 6 \cdot 4^0 = 6.$$

Inductive Hypothesis: Assume that for some k, the total number of sub-triangles follows:

$$N_k = 6 \cdot 4^{k-1}.$$

Inductive Step: When moving to $k+1$, each existing triangle is further divided into 4 smaller triangles due to the additional cevians. Thus, the number of sub-triangles at step $k+1$ is:

$$N_{k+1} = 4 \cdot N_k = 4 \cdot (6 \cdot 4^{k-1}) = 6 \cdot 4^k.$$

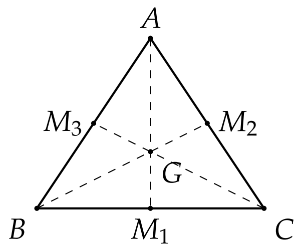
This confirms the validity of the formula for all $k \geq 1$.

Geometric Interpretation: Each cevian of higher order progressively refines the structure by introducing additional intersection points. The subdivision follows a hierarchical pattern, where each new cevian layer generates a finer tessellation of the original triangle. The area of each sub-triangle can then be computed by distributing the total area of $\triangle ABC$ equally among the N_k sub-triangles:

$$A_{\triangle_k} = \frac{A_{\triangle ABC}}{6 \cdot 4^{k-1}}.$$

5.3. Geometric Properties. The cevians of rank k intersect at specific points. The notable properties include:

- Consider M_1, M_2, and M_3 as the midpoints of the segments BC, AC, and AB, respectively.



- $AM_1 \cap BM_2 \cap CM_3 = G$, where G is the centroid.
- The vector relationship is expressed as:

$$\overrightarrow{AM_1} + \overrightarrow{BM_2} + \overrightarrow{CM_3} = \overrightarrow{0}. \tag{5.6}$$

5.4. Angle Between Cevian Lines. To determine the angle θ between the cevian lines in a triangle, we use the cosine formula for vectors:

$$\cos \angle\theta = \frac{\overrightarrow{u} \cdot \overrightarrow{v}}{\| \overrightarrow{u} \| \| \overrightarrow{v} \|}, \tag{5.7}$$

where $\overrightarrow{u}$ and $\overrightarrow{v}$ are the direction vectors of the cevians.
For instance, the angle between $\overrightarrow{AB}$ and $\overrightarrow{BG}$ is:

$$\angle(AD, BE) = \angle\theta. \tag{5.8}$$

5.5. Coordinate Relations for Median Points. Let $\triangle ABC$ be a triangle with medians drawn. The coordinates of points on the medians are determined as follows:

5.5.1. Midpoint on Side BC. For the midpoint D on side BC:

$$x_D = \frac{x_B + x_C}{2}, \tag{5.9}$$

$$y_D = \frac{y_B + y_C}{2}. \tag{5.10}$$

5.5.2. Midpoint on Side AC. For the midpoint E on side AC:

$$x_E = \frac{x_A + x_C}{2}, \tag{5.11}$$

$$y_E = \frac{y_A + y_C}{2}. \tag{5.12}$$

5.6. Detailed Calculations for Midpoint Relations. Given the midpoints on the sides of $\triangle ABC$, we calculate the components as follows:

- For midpoint D on side BC:

$$x_D = \frac{x_B + x_C}{2},$$
$$y_D = \frac{y_B + y_C}{2}.$$

- For midpoint E on side AC:

$$x_E = \frac{x_A + x_C}{2},$$
$$y_E = \frac{y_A + y_C}{2}.$$

5.6.1. Vector Relations. To compute the dot product of vectors $\overrightarrow{AD}$ and $\overrightarrow{BE}$, we use:

$$\overrightarrow{AD} = \left(\frac{x_B + x_C}{2} - x_A, \frac{y_B + y_C}{2} - y_A \right),$$
$$\overrightarrow{BE} = \left(\frac{x_A + x_C}{2} - x_B, \frac{y_A + y_C}{2} - y_B \right).$$

The dot product is given by:

$$\overrightarrow{AD} \cdot \overrightarrow{BE} = \left(\frac{x_B + x_C}{2} - x_A \right) \left(\frac{x_A + x_C}{2} - x_B \right) + \left(\frac{y_B + y_C}{2} - y_A \right) \left(\frac{y_A + y_C}{2} - y_B \right). \tag{5.13}$$

5.6.2. Magnitude of Vectors. The magnitudes of the vectors are:

$$\|\overrightarrow{AD}\| = \sqrt{\left(\frac{x_B + x_C}{2} - x_A \right)^2 + \left(\frac{y_B + y_C}{2} - y_A \right)^2},$$
$$\|\overrightarrow{BE}\| = \sqrt{\left(\frac{x_A + x_C}{2} - x_B \right)^2 + \left(\frac{y_A + y_C}{2} - y_B \right)^2}.$$

5.6.3. Cosine of the Angle. The cosine of the angle θ between $\overrightarrow{AD}$ and $\overrightarrow{BE}$ is:

$$\cos \angle\theta = \frac{\overrightarrow{AD} \cdot \overrightarrow{BE}}{\|\overrightarrow{AD}\| \|\overrightarrow{BE}\|}, \tag{5.14}$$

which simplifies to:

$$\cos \angle\theta = \frac{\left(\frac{x_B+x_C}{2} - x_A \right) \left(\frac{x_A+x_C}{2} - x_B \right) + \left(\frac{y_B+y_C}{2} - y_A \right) \left(\frac{y_A+y_C}{2} - y_B \right)}{\sqrt{\left(\frac{x_B+x_C}{2} - x_A \right)^2 + \left(\frac{y_B+y_C}{2} - y_A \right)^2} \cdot \sqrt{\left(\frac{x_A+x_C}{2} - x_B \right)^2 + \left(\frac{y_A+y_C}{2} - y_B \right)^2}}. \tag{5.15}$$

5.7. General Formula for Rank-k Cevians. Let the midpoints be:

- Midpoint A_m on side BC:

$$x_{A_m} = \frac{x_B + x_C}{2},$$
$$y_{A_m} = \frac{y_B + y_C}{2}.$$

- Midpoint B_m on side AC:

$$x_{B_m} = \frac{x_A + x_C}{2},$$
$$y_{B_m} = \frac{y_A + y_C}{2}.$$

- Midpoint C_m on side AB:

$$x_{C_m} = \frac{x_A + x_B}{2},$$
$$y_{C_m} = \frac{y_A + y_B}{2}.$$

5.8. Generalized Coordinates for Rank-k Cevians. For cevians of rank k, the coordinates of specific points on the triangle sides are given as follows:

5.8.1. Midpoint Coordinates for Successive Divisions. Let A_1, B_1, and C_1 be the points dividing the sides BC, AC, and AB respectively:

$$x_{A_1} = \frac{x_B + x_C}{2},$$
$$y_{A_1} = \frac{y_B + y_C}{2}.$$

Analogously for B_1 and C_1:

$$x_{B_1} = \frac{x_C + x_A}{2}, \qquad y_{B_1} = \frac{y_C + y_A}{2},$$
$$x_{C_1} = \frac{x_A + x_B}{2}, \qquad y_{C_1} = \frac{y_A + y_B}{2}.$$

Let A_2, B_2, and C_2 be the points dividing the sides BC, AC, and AB respectively:

$$x_{A_2} = \frac{x_{B_1} + x_{C_1}}{2} = \frac{\frac{x_A + x_C}{2} + \frac{x_A + x_B}{2}}{2} = \frac{x_A}{2} + \frac{x_B + x_C}{4},$$
$$y_{A_2} = \frac{y_{B_1} + y_{C_1}}{2} = \frac{\frac{y_A + y_C}{2} + \frac{y_A + y_B}{2}}{2} = \frac{y_A}{2} + \frac{y_B + y_C}{4}.$$

Analogously for B_2 and C_2:

$$x_{B_2} = \frac{x_C + x_A}{4} + \frac{x_B}{2}, \qquad y_{B_2} = \frac{y_C + y_A}{4} + \frac{y_B}{2},$$
$$x_{C_2} = \frac{x_A + x_B}{4} + \frac{x_C}{2}, \qquad y_{C_2} = \frac{y_A + y_B}{4} + \frac{y_C}{2}.$$

5.8.2. Recursive Relations for General Rank k. The general recursive relations for the coordinates of points A_k, B_k, and C_k are:

$$x_{A_k} = \frac{x_A}{2^{k-1}} + \frac{1}{2^k}(x_B + x_C), \qquad y_{A_k} = \frac{y_A}{2^{k-1}} + \frac{1}{2^k}(y_B + y_C),$$
$$x_{B_k} = \frac{x_B}{2^{k-1}} + \frac{1}{2^k}(x_C + x_A), \qquad y_{B_k} = \frac{y_B}{2^{k-1}} + \frac{1}{2^k}(y_C + y_A),$$
$$x_{C_k} = \frac{x_C}{2^{k-1}} + \frac{1}{2^k}(x_A + x_B), \qquad y_{C_k} = \frac{y_C}{2^{k-1}} + \frac{1}{2^k}(y_A + y_B).$$

5.9. Angle Between Rank-k Cevians. Let θ_k denote the angle between two cevians of rank k, such as A_kB_k and B_kC_k. The cosine of θ_k is given by:

$$\cos\angle\theta_k = \frac{\overrightarrow{A_kB_k} \cdot \overrightarrow{B_kC_k}}{\|\overrightarrow{A_kB_k}\|\|\overrightarrow{B_kC_k}\|}. \tag{5.16}$$

5.10. Detailed Calculations for Rank-k Cevians. The dot product of two vectors $\overrightarrow{A_kB_k}$ and $\overrightarrow{B_kC_k}$ is expanded as follows:

$$\overrightarrow{A_kB_k} \cdot \overrightarrow{B_kC_k} = \left(\frac{x_A}{2^{k-1}} + \frac{1}{2^k}(x_C + x_B) - x_A\right) \cdot \left(\frac{x_C}{2^{k-1}} + \frac{1}{2^k}(x_A + x_B) - x_B\right) + $$
$$+ \left(\frac{y_A}{2^{k-1}} + \frac{1}{2^k}(y_C + y_B) - y_A\right) \cdot \left(\frac{y_C}{2^{k-1}} + \frac{1}{2^k}(y_A + y_B) - y_B\right).$$

5.10.1. Magnitude of Vectors. The magnitudes of the vectors are:

$$\|\overrightarrow{A_kB_k}\| = \sqrt{\left(\frac{x_A}{2^{k-1}} + \frac{1}{2^k}(x_C + x_B) - x_A\right)^2 + \left(\frac{y_A}{2^{k-1}} + \frac{1}{2^k}(y_C + y_B) - y_A\right)^2},$$
$$\|\overrightarrow{B_kC_k}\| = \sqrt{\left(\frac{x_C}{2^{k-1}} + \frac{1}{2^k}(x_A + x_B) - x_B\right)^2 + \left(\frac{y_C}{2^{k-1}} + \frac{1}{2^k}(y_A + y_B) - y_B\right)^2}.$$

5.10.2. Cosine of the Angle θ_R. The cosine of the angle θ_R between these vectors is:

$$\cos\angle\theta_R = \frac{\left(\frac{x_A}{2^{k-1}} + \frac{1}{2^k}(x_C+x_B) - x_A\right)\cdot\left(\frac{x_C}{2^{k-1}} + \frac{1}{2^k}(x_A+x_B) - x_B\right) + \left(\frac{y_A}{2^{k-1}} + \frac{1}{2^k}(y_C+y_B) - y_A\right)\cdot\left(\frac{y_C}{2^{k-1}} + \frac{1}{2^k}(y_A+y_B) - y_B\right)}{\sqrt{\left(\frac{x_A}{2^{k-1}} + \frac{1}{2^k}(x_C+x_B) - x_A\right)^2 + \left(\frac{y_A}{2^{k-1}} + \frac{1}{2^k}(y_C+y_B) - y_A\right)^2} \cdot \sqrt{\left(\frac{x_C}{2^{k-1}} + \frac{1}{2^k}(x_A+x_B) - x_B\right)^2 + \left(\frac{y_C}{2^{k-1}} + \frac{1}{2^k}(y_A+y_B) - y_B\right)^2}}.$$

5.11. General Formula for Rank-k Cevians. To derive the general formula for rank-k cevians, we consider the triangle $\triangle A_kB_kC_k$ at rank-k and D_k the midpoint of the segment B_kC_k.

The distance between A and D_k is given by:

$$|AD_k| = \sqrt{(x_{D_k} - x_A)^2 + (y_{D_k} - y_A)^2}.$$

The coordinates of D_k :

$$x_{D_k} = \frac{x_{B_k} + x_{C_k}}{2} = \frac{\frac{x_B}{2^{k-1}} + \frac{1}{2^k}(x_C + x_A) + \frac{x_C}{2^{k-1}} + \frac{1}{2^k}(x_A + x_B)}{2} = \frac{3\cdot(x_C + x_B)}{2^{k+1}} + \frac{x_A}{2^k}$$

$$y_{D_k} = \frac{y_{B_k} + x_{y_k}}{2} = \frac{\frac{y_B}{2^{k-1}} + \frac{1}{2^k}(y_C + y_A) + \frac{y_C}{2^{k-1}} + \frac{1}{2^k}(y_A + y_B)}{2} = \frac{3\cdot(y_C + y_B)}{2^{k+1}} + \frac{y_A}{2^k},$$

For the coordinates of A_k:

$$x_{A_k} = \frac{x_A}{2^{k-1}} + \frac{1}{2^k}(x_C + x_B), \quad y_{A_k} = \frac{y_A}{2^{k-1}} + \frac{1}{2^k}(y_C + y_B).$$

By substituting, the difference in coordinates is:

$$x_{D_k} - x_{A_k} = \frac{3 \cdot (x_C + x_B)}{2^{k+1}} + \frac{x_A}{2^k} - \frac{x_A}{2^{k-1}} - \frac{1}{2^k}(x_B + x_C) = \frac{x_B + x_C - 2 \cdot x_A}{2^{k+1}}$$

$$y_{D_k} - y_{A_k} = \frac{3 \cdot (y_C + y_B)}{2^{k+1}} + \frac{y_A}{2^k} - \frac{y_A}{2^{k-1}} - \frac{1}{2^k}(y_B + y_C) = \frac{y_B + y_C - 2 \cdot y_A}{2^{k+1}}.$$

Thus, the distance is:

$$|AD_k| = \sqrt{\left(\frac{x_B + x_C - 2 \cdot x_A}{2^{k+1}}\right)^2 + \left(\frac{y_B + y_C - 2 \cdot y_A}{2^{k+1}}\right)^2}.$$

5.12. Distance from the Centroid to Rank-R Points. The general formula for calculating the distances from the centroid G of $\triangle ABC$ to the points A_R, B_R, and C_R is given as follows:

5.12.1. Centroid Coordinates. The coordinates of the centroid G are:

$$x_G = \frac{x_A + x_B + x_C}{3},$$
$$y_G = \frac{y_A + y_B + y_C}{3}.$$

5.12.2. Distance from A_k to G. The distance from A_R to G is:

$$x_{A_k} - x_G = \frac{x_A}{2^{k-1}} + \frac{x_C + x_B}{2^k} - \frac{x_A + x_B + x_C}{3} = x_A \cdot \left(\frac{1}{2^{k-1}} - \frac{1}{3}\right) + (x_B + x_C) \cdot \left(\frac{1}{2^k} - \frac{1}{3}\right),$$
$$y_{A_k} - y_G = \frac{y_A}{2^{k-1}} + \frac{y_C + y_B}{2^k} - \frac{y_A + y_B + y_C}{3} = y_A \cdot \left(\frac{1}{2^{k-1}} - \frac{1}{3}\right) + (y_B + y_C) \cdot \left(\frac{1}{2^k} - \frac{1}{3}\right).$$

The magnitude of the vector $\|\overrightarrow{A_R G}\|$ is then given by:

$$\|\overrightarrow{A_k G}\| = \sqrt{(x_{A_k} - x_G)^2 + (y_{A_k} - y_G)^2} = \tag{5.17}$$

$$\sqrt{\left(x_A \cdot \left(\frac{1}{2^{k-1}} - \frac{1}{3}\right) + (x_B + x_C) \cdot \left(\frac{1}{2^k} - \frac{1}{3}\right)\right)^2 + \left(y_A \cdot \left(\frac{1}{2^{k-1}} - \frac{1}{3}\right) + (y_B + y_C) \cdot \left(\frac{1}{2^k} - \frac{1}{3}\right)\right)^2}. \tag{5.18}$$

5.12.3. Limit Behavior. As $k \to \infty$, the points $A_k \to G$, and the distances approach zero:

$$\frac{1}{2^{k-1}} \to 0, \quad \frac{1}{2^k} \to 0. \tag{5.19}$$

5.13. Angle Between Cevian Vectors. To analyze the relationship between cevian vectors, we define the following vector expressions for

$$\overrightarrow{AG} = (x_G - x_A) \cdot \overrightarrow{i} + (y_G - y_A) \cdot \overrightarrow{j}$$

$$\overrightarrow{A_kG} = (x_G - x_{A_k}) \cdot \overrightarrow{i} + (y_G - y_{A_k}) \cdot \overrightarrow{j}.$$

To calculate the angles between the cevians AG and A_kG, we use the dot product formula:

$$\overrightarrow{AG} \cdot \overrightarrow{A_kG} = (x_G - x_A)(x_{A_k} - x_G) + (y_G - y_A)(y_{A_k} - y_G).$$

Expanding the terms, we have:

$$x_G - x_A = \frac{x_B + x_C - 2x_A}{3},$$
$$y_G - y_A = \frac{y_B + y_C - 2y_A}{3}.$$

Similarly, for the components of $\overrightarrow{A_RG}$:

$$x_{A_k} - x_G = \frac{x_A}{2^{k-1}} + \frac{x_B + x_C}{2^k} - \frac{x_A + x_B + x_C}{3},$$
$$y_{A_k} - y_G = \frac{y_A}{2^{k-1}} + \frac{y_B + y_C}{2^k} - \frac{y_A + y_B + y_C}{3}.$$

5.14. Angle Calculation Using Dot Product. We denote the angle θ between the vectors $\overrightarrow{AG}$ and $\overrightarrow{A_kG}$ as:

$$\cos \angle\theta = \frac{\overrightarrow{AG} \cdot \overrightarrow{A_kG}}{\|\overrightarrow{AG}\| \|\overrightarrow{A_kG}\|}. \tag{5.20}$$

5.14.1. Dot Product Expansion. The dot product $\overrightarrow{AG} \cdot \overrightarrow{A_RG}$ is expanded as:

$$\begin{aligned}\overrightarrow{AG} \cdot \overrightarrow{A_RG} &= (x_G - x_A)(x_{A_R} - x_G) + (y_G - y_A)(y_{A_R} - y_G), \\ &= \left(\frac{x_B + x_C - 2x_A}{3}\right)\left(\frac{x_A}{2^{R-1}} + \frac{x_B + x_C}{2^R} - \frac{x_A + x_B + x_C}{3}\right) \\ &+ \left(\frac{y_B + y_C - 2y_A}{3}\right)\left(\frac{y_A}{2^{R-1}} + \frac{y_B + y_C}{2^R} - \frac{y_A + y_B + y_C}{3}\right).\end{aligned}$$

5.14.2. Magnitudes of the Vectors. The magnitudes of the vectors are given by:

$$\|\overrightarrow{AG}\| = \sqrt{\left(\frac{x_B + x_C - 2x_A}{3}\right)^2 + \left(\frac{y_B + y_C - 2y_A}{3}\right)^2},$$

$$\|\overrightarrow{A_kG}\| = \sqrt{\left(\frac{x_A}{2^{R-1}} + \frac{x_B + x_C}{2^R} - \frac{x_A + x_B + x_C}{3}\right)^2 + \left(\frac{y_A}{2^{R-1}} + \frac{y_B + y_C}{2^R} - \frac{y_A + y_B + y_C}{3}\right)^2}.$$

5.14.3. Final Formula for $\cos\theta$. Substituting these values, we have:

$$\cos \angle\theta = \frac{\left(\frac{x_B+x_C-2x_A}{3}\right)\left(\frac{x_A}{2^{R-1}} + \frac{x_B+x_C}{2^R} - \frac{x_A+x_B+x_C}{3}\right) + \left(\frac{y_B+y_C-2y_A}{3}\right)\left(\frac{y_A}{2^{R-1}} + \frac{y_B+y_C}{2^R} - \frac{y_A+y_B+y_C}{3}\right)}{\sqrt{\left(\frac{x_B+x_C-2x_A}{3}\right)^2 + \left(\frac{y_B+y_C-2y_A}{3}\right)^2} \cdot \sqrt{\left(\frac{x_A}{2^{R-1}} + \frac{x_B+x_C}{2^R} - \frac{x_A+x_B+x_C}{3}\right)^2 + \left(\frac{y_A}{2^{R-1}} + \frac{y_B+y_C}{2^R} - \frac{y_A+y_B+y_C}{3}\right)^2}}.$$

6. Generalized Cevians of Rank k in Tetrahedra and n-Simplices

6.1. Definition and Fundamental Properties. Let $T = ABCD$ be a tetrahedron. A cevian is defined as a segment joining a vertex (e.g., A) to a point P on the opposite face $\triangle BCD$. If the ratio of the volumes of the sub-tetrahedra satisfies the relation:

$$\frac{\mathrm{Vol}(APB)}{\mathrm{Vol}(APC)} = k, \tag{6.1}$$

we say the cevian has rank k.

6.1.1. Properties. 1. Proportionality of Volumes: The point P divides the face $\triangle BCD$ proportionally. Let the lengths of the segments on $\triangle BCD$ be denoted as $|BP|$, $|CP|$, and $|DP|$. The following relationship holds:

$$\frac{\mathrm{Vol}(APB)}{\mathrm{Vol}(APC)} = \frac{|BP|}{|CP|}. \tag{6.2}$$

Proof: The volume of a sub-tetrahedron, such as $\mathrm{Vol}(APB)$, is proportional to the base area $\mathrm{Area}(\triangle BCD)$ and the height from A to $\triangle BCD$. Denoting the area of the sub-triangle $\triangle BP$ as $\mathrm{Area}(\triangle BP)$, we have:

$$\mathrm{Vol}(APB) = \frac{1}{3} \cdot \mathrm{Area}(\triangle BP) \cdot \text{Height from } A.$$

Similarly, the volume of $\mathrm{Vol}(APC)$ is:

$$\mathrm{Vol}(APC) = \frac{1}{3} \cdot \mathrm{Area}(\triangle CP) \cdot \text{Height from } A.$$

Since the height from A is the same for both sub-tetrahedra, the ratio simplifies to:

$$\frac{\mathrm{Vol}(APB)}{\mathrm{Vol}(APC)} = \frac{\mathrm{Area}(\triangle BP)}{\mathrm{Area}(\triangle CP)} = \frac{|BP|}{|CP|}.$$

2. Volume Invariance: The sum of the volumes of all sub-tetrahedra containing P equals the volume of the original tetrahedron:

$$\mathrm{Vol}(APB) + \mathrm{Vol}(APC) + \mathrm{Vol}(APD) = \mathrm{Vol}(ABCD). \tag{6.3}$$

Proof: The volume of the tetrahedron $ABCD$ can be expressed as the sum of the sub-tetrahedra sharing the vertex A. Let the base areas of $\triangle BP, \triangle CP, \triangle DP$ on $\triangle BCD$ be denoted as $\mathrm{Area}(\triangle BP), \mathrm{Area}(\triangle CP), \mathrm{Area}(\triangle DP)$, respectively. Then:

$$\mathrm{Vol}(APB) = \frac{1}{3} \cdot \mathrm{Area}(\triangle BP) \cdot \text{Height from } A,$$

$$\mathrm{Vol}(APC) = \frac{1}{3} \cdot \mathrm{Area}(\triangle CP) \cdot \text{Height from } A,$$

$$\mathrm{Vol}(APD) = \frac{1}{3} \cdot \mathrm{Area}(\triangle DP) \cdot \text{Height from } A.$$

Adding these volumes, we have:

$$\mathrm{Vol}(APB) + \mathrm{Vol}(APC) + \mathrm{Vol}(APD) = \frac{1}{3} \cdot \mathrm{Area}(\triangle BCD) \cdot \text{Height from } A = \mathrm{Vol}(ABCD).$$

6.2. Generalized Ceva's Theorem in Tetrahedra. The cevians AP, BQ, CR, DS, drawn to points P, Q, R, S on the opposite faces of the tetrahedron $ABCD$, are concurrent if and only if:

$$\prod_{\text{cyclic}} \frac{\text{Vol}(APB) \cdot \text{Vol}(BQC) \cdot \text{Vol}(CRD) \cdot \text{Vol}(DSA)}{\text{Vol}(APC) \cdot \text{Vol}(BQA) \cdot \text{Vol}(CRA) \cdot \text{Vol}(DSB)} = 1. \tag{6.4}$$

Proof: Let the cevians AP, BQ, CR, DS meet at a common point O. On each face, the cevian divides the corresponding triangle proportionally. For instance, on $\triangle BCD$, the point P satisfies:

$$\frac{\text{Vol}(APB)}{\text{Vol}(APC)} = \frac{|BP|}{|CP|}.$$

Similar relations hold for the other faces. By cyclically combining these proportions and using volume invariance on each sub-tetrahedron, we obtain:

$$\prod_{\text{cyclic}} \frac{\text{Vol}(APB)}{\text{Vol}(APC)} \cdot \frac{\text{Vol}(BQC)}{\text{Vol}(BQA)} \cdot \frac{\text{Vol}(CRD)}{\text{Vol}(CRA)} \cdot \frac{\text{Vol}(DSA)}{\text{Vol}(DSB)} = 1.$$

6.3. Volume of a Tetrahedron. The volume of a tetrahedron $T = ABCD$ can be expressed geometrically as:

$$\text{Vol}(ABCD) = \frac{1}{3} \cdot \text{Area}(\triangle BCD) \cdot \text{Height from } A. \tag{6.5}$$

For any sub-tetrahedron (e.g., APB), the volume is given by:

$$\text{Vol}(APB) = \frac{1}{3} \cdot \text{Area}(\triangle BP) \cdot \text{Height from } A.$$

These relations allow the computation of volumes without explicit use of determinants.

6.4. Example: Regular Tetrahedron. Let the vertices of a regular tetrahedron be:

$$A = (0, 0, 0),\ B = (1, 0, 0),\ C = (0, 1, 0),\ D = (0, 0, 1).$$

The total volume of the tetrahedron is:

$$\text{Vol}(ABCD) = \frac{1}{6}.$$

For a point P on $\triangle BCD$ such that $|BP| = 2|CP|$, the sub-volumes are:

$$\text{Vol}(APB) = \frac{2}{9}, \quad \text{Vol}(APC) = \frac{1}{9}.$$

These satisfy the ratio:

$$\frac{\text{Vol}(APB)}{\text{Vol}(APC)} = 2.$$

7. MAIN RESULTS

This study presents several new contributions to the theory of cevians, extending classical results beyond traditional triangle geometry and introducing novel formulations applicable to higher-dimensional simplices.

A significant result is the generalization of Ceva's theorem to higher dimensions. While the classical theorem establishes a concurrency condition for cevians in a triangle, this work extends the concept to tetrahedra and n-simplices. In this extended framework, cevians are defined through proportional volume divisions rather than segment divisions, leading to a new concurrency condition based on volume ratios.

Additionally, this research introduces the concept of cevians of rank k, which expands the traditional definition by incorporating recursive volume-based subdivisions. Unlike classical cevians that partition a triangle into two sub-triangles of proportional areas, rank-k cevians generate a hierarchical structure of nested simplices. This recursive nature provides new insights into hierarchical space partitioning and self-similar geometric structures.

Another key contribution is the development of new coordinate formulations for cevian intersections. By employing recursive averaging techniques, explicit expressions are derived for the intersection points of rank-k cevians. These formulations enable efficient computation of cevian intersections in both two-dimensional and higher-dimensional simplices, eliminating the need for conventional geometric constructions.

Furthermore, this study examines the role of cevians in recursive subdivisions and geometric optimization. The iterative application of rank-k cevians produces structured partitions that preserve proportional volume relationships. This recursive behavior leads to self-similar patterns, offering potential implications for various geometric and computational applications.

These findings extend the applicability of cevians beyond classical triangle geometry, establishing new mathematical tools for analyzing high-dimensional geometric structures.

8. Conclusions

This study extends the classical theory of cevians beyond traditional triangle geometry, introducing new formulations that apply to higher-dimensional simplices. By establishing a framework for cevians of rank k, this work provides a foundation for further exploration of proportional volume divisions in geometric structures.

One of the key takeaways from this research is the broader applicability of cevian-based recursive partitions in hierarchical space decomposition. The results presented here not only reinforce the fundamental role of cevians in concurrency conditions but also highlight their potential in structuring geometric spaces efficiently. The recursive nature of rank-k cevians suggests deeper connections with self-similarity principles, which could be further explored in relation to fractal geometry and data clustering.

While this study primarily focuses on the mathematical properties of cevians in simplices, future research could investigate their computational applications. Potential directions include their use in mesh generation techniques, geometric optimization algorithms, and high-dimensional data partitioning. Moreover, the interaction of cevians with other geometric constructs, such as barycentric coordinates and affine transformations, remains an open question for further study.

In conclusion, this work establishes cevians of rank k as a fundamental tool for analyzing geometric structures beyond classical configurations. The results presented provide both theoretical advancements and practical implications, paving the way for further developments in high-dimensional geometry and computational mathematics.

References

[1] H.S.M. Coxeter and S.L. Greitzer, Geometry Revisited, Mathematical Association of America, 1967. https://www.maa.org/press/maa-reviews/geometry-revisited

[2] Ross Honsberger, Episodes in Nineteenth and Twentieth Century Euclidean Geometry, Mathematical Association of America, 1995. https://www.maa.org/press/maa-reviews/episodes-in-nineteenth-and-twentieth-century-euclidean-geometry

[3] Dmitry Fuchs and Serge Tabachnikov, Mathematical Omnibus: Thirty Lectures on Classic Mathematics, American Mathematical Society, 2007. https://bookstore.ams.org/mbk-65/

[4] Vladimir V. Prasolov, Problems in Plane and Solid Geometry, Vol. 1: Plane Geometry, Moscow Center for Continuous Mathematical Education, 2001. https://www.mccme.ru/free-books/psol/

[5] Nathan Altshiller-Court, College Geometry: An Introduction to the Modern Geometry of the Triangle and the Circle, Dover Publications, 2007. https://store.doverpublications.com/0486458059.html

[6] Roger A. Johnson, Advanced Euclidean Geometry: An Elementary Treatise on the Geometry of the Triangle and the Circle, Dover Publications, 2007. https://store.doverpublications.com/0486462374.html

ADDRESS
Școala Gimnazială ”Mihai Olos”, Ariniș
Școala Gimnazială ”Nr.9”, Borșa.
Email address: dragosanamariamonica@yahoo.com