Global Journal of Advanced Research
on Classical and Modern Geometries
ISSN: 2284-5569, Vol. 14, (2025), Issue 1, pp. 1-25

# A NEW APPROACH OF THE FERMAT-TORRICELLI CONFIGURATION

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ABSTRACT. In this paper, we aim to approach the Fermat-Torricelli configuration in a new way. Given a triangle $ABC$, instead of the six equilateral triangles constructed (inside or outside) on its sides, we consider three equilateral triangles $AA_1A_2$, $BB_1B_2$, $CC_1C_2$ with the same orientation as the given triangle and such that $A_1$, $A_2 \in BC$, $B_1$, $B_2 \in CA$, and $C_1$, $C_2 \in AB$ (Fig. 1). This new configuration is denoted $\mathcal{F}$. It is shown that $\mathcal{F}$ includes the classic Fermat-Torricelli configuration, and new concepts are introduced and studied.

## 1. INTRODUCTION

Consider an arbitrary triangle $ABC$ with sidelenghts $a, b, c$. Let $A_+$ and $A_-$ be the vertices of the equilateral triangles built on the $BC$ outside and inside the triangle $ABC$, respectively; similarly define points $B_+, B_-$ and $C_+, C_-$.
Denote by $\mathcal{T}_+^a$ and $\mathcal{T}_-^a$ the circumcircles of the equilateral triangles $A_+BC$ and $A_-BC$, respectively, and similarly we define $\mathcal{T}_+^b$ and $\mathcal{T}_-^b$, $\mathcal{T}_+^c$ and $\mathcal{T}_-^c$. These six circles are called the *Torricelli circles* of triangle $ABC$. We also note with $N_+^a, N_-^a, N_+^b, N_-^b, N_+^c, N_-^c$ the circumcenters of the six Torricelli circles. The triangles $N_+^aN_+^bN_+^c$ and $N_-^aN_-^bN_-^c$ are called the *outer and inner Napoleon triangles* of triangle $ABC$, respectively.
It is known that *Fermat points* (or *Torricelli points* or *isogonic points*) are defined by $F_+ := AA_+ \cap BB_+ \cap CC_+ = \mathcal{T}_+^a \cap \mathcal{T}_+^b \cap \mathcal{T}_+^c$ and $F_- := AA_- \cap BB_- \cap CC_- = \mathcal{T}_-^a \cap \mathcal{T}_-^b \cap \mathcal{T}_-^c$ and *Napoleon points* by $N_+ := AN_+^a \cap BN_+^b \cap CN_+^c$, and $N_- := AN_-^a \cap BN_-^b \cap CN_-^c$.
The *classic Fermat-Torricelli configuration* starts with points $A$, $B$, $C$, $A_+$, $B_+$, $C_+$, $A_-$, $B_-$, $C_-$ (Fig. 1) and then, by introducing new elements, a construction with wonderful properties is developed. On this subject there is a vast specialized literature - treatises, textbooks and papers: [6], [1], [7], [4], [10], [8] etc.

Now, we propose a new approach to the Fermat-Torricelli configuration. We construct the equilateral triangle $AA_1A_2$ such that $A_1, A_2 \in BC$ and having the same orientation as the given triangle. In fact, the equilateral triangle $AA_1A_2$ has the vertex $A$ and the altitude $h_a$ in common with the triangle $ABC$. Similarly, the equilateral triangles $BB_1B_2$

2010 *Mathematics Subject Classification.* 51M15; 51N20.
*Key words and phrases.* Torricelli circles; Fermat points; Napoleon points; Napoleon triangles; Euler line; orthocentroidal circle; radical axis; similar triangles; perspective triangles; configuration $\mathcal{F}$.

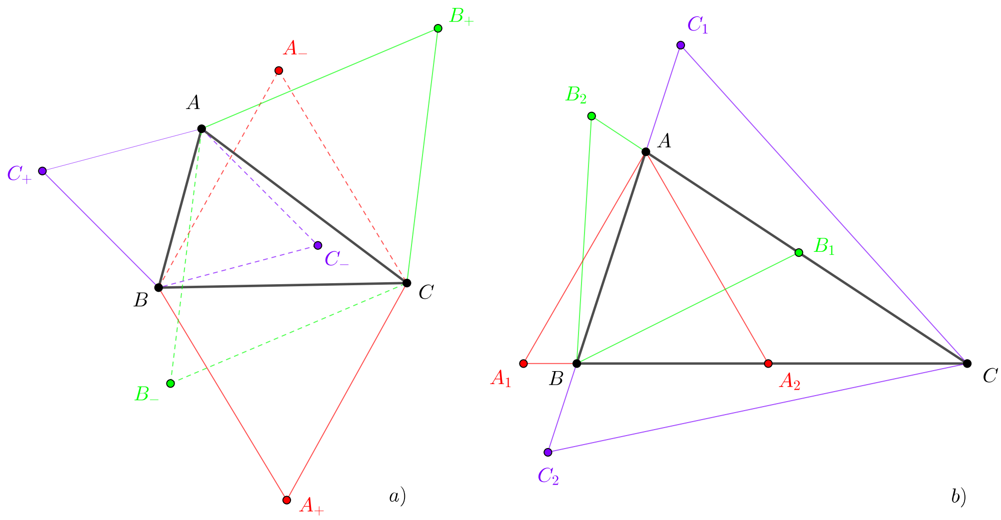


Fig. 1

and $CC_1C_2$ are constructed. Denote $\mathcal{F}$ the configuration that starts with the points $A$, $B$, $C$, $A_1$, $A_2$, $B_1$, $B_2$, $C_1$, $C_2$ (Fig. 1). We will see that $\mathcal{F}$ includes the classic Fermat-Torricelli configuration and that important concepts of triangle geometry are naturally involved in this framework or new ones can be introduced.

## 2. Basics of the configuration $\mathcal{F}$

We begin the study of the configuration $\mathcal{F}$ by defining in its terms the main objects of the classical configuration: Fermat points, Torricelli circles, Napoleon triangles, and Napoleon points.
The first objects of configuration $\mathcal{F}$ that we introduce are Torricelli circles and Fermat points.

**Proposition 2.1.** *We have:*
*(i) the quadrilaterals* $BC_2CB_1, CC_1AA_2, AB_2A_1B$ *are cyclic and the circles circumscribed to them are precisely* $\mathcal{T}_+^a, \mathcal{T}_+^b, \mathcal{T}_+^c$, *respectively;*
*(ii) the quadrilaterals* $BCC_1B_2, CAA_1C_2, ABA_2B_1$ *are cyclic and the circles circumscribed to them are precisely* $\mathcal{T}_-^a, \mathcal{T}_-^b, \mathcal{T}_-^c$, *respectively (Fig. 2).*

*Proof.* (i) Since the triangles $BB_1B_2, CC_1C_2$ are equilateral, it results that $\angle BB_1B_2 = \angle BC_2C = \frac{\pi}{3}$. From $\angle BB_1B_2 = \angle BC_2C$, it follows that $BC_2CB_1$ is a cyclic quadrilateral. On the other hand, since $\angle BC_2C = \frac{\pi}{3}$, we deduce that the equilateral triangle built on $BC$ outside the given triangle has the vertex $A_+$ on the circumcircle of quadrilateral $BC_2CB_1$. So, the circumcircle of $BC_2CB_1$ is $\mathcal{T}_+^a$. The remaining statements are proven in the same way.

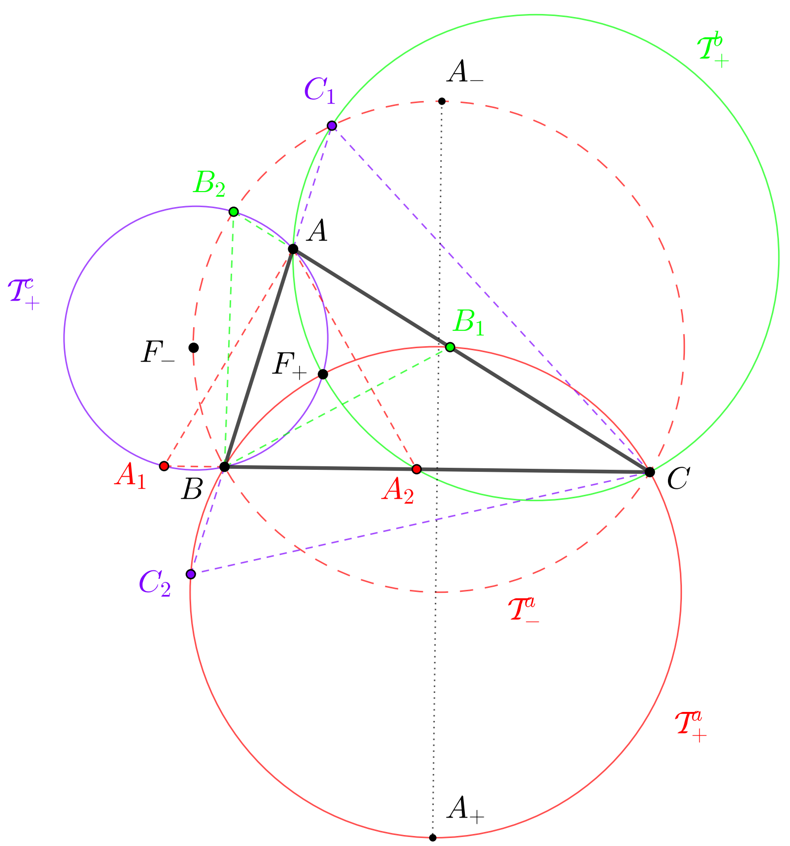


Fig. 2

(ii) Since $\angle BB_2C = \angle BC_1C = \dfrac{\pi}{3}$, it follows that $BCC_1B_2$ is a cyclic quadrilateral. Obviously, the equilateral triangle $BCA_-$ is inscribed in the circumcircle of this quadrilateral. As a result, the circumcircle of the quadrilateral $BCC_1B_2$ is $\mathcal{T}^a_-$. Etc. $\square$

We agree to note $(ABC)$ the circumcircle of the triangle $ABC$, $(ABCD)$ the circumcircle of the cyclic quadrilateral $ABCD$, etc.

**Corollary 2.1.** *(i)* $(BC_2CB_1) \cap (CC_1AA_2) \cap (AB_2A_1B) = F_+$*;*
*(ii)* $(BCC_1B_2) \cap (CAA_1C_2) \cap (ABA_2B_1) = F_-$ *(Fig. 2).*

*Proof.* (i) By Proposition 2.1, we have: $(BC_2CB_1) \cap (CC_1AA_2) \cap (AB_2A_1B) = \mathcal{T}^a_+ \cap \mathcal{T}^b_+ \cap \mathcal{T}^c_+$. This and the fact that $\mathcal{T}^a_+ \cap \mathcal{T}^b_+ \cap \mathcal{T}^c_+ = F_+$ lead to the statement (i).
(ii) It is demonstrated similarly. $\square$

**Corollary 2.2.** *(i) The circumcenters of the quadrilaterals* $BC_2CB_1$, $CC_1AA_2$, $AB_2A_1B$ *are the points* $N^a_+$, $N^b_+$, $N^c_+$, *respectively.*
*(ii) The circumcenters of the quadrilaterals* $BCC_1B_2$, $CAA_1C_2$, $ABA_2B_1$ *are the points* $N^a_-$, $N^b_-$, $N^c_-$, *respectively (Fig. 3).*

In order to highlight the basic triangles $AA_1A_2$, $BB_1B_2$, $CC_1C_2$, we reformulate the statements of the previous corollary.

**Corollary 2.3.** *(i) $N_+^a$ is the intersection of the perpendicular bisectors of the sides $BB_1$ and $CC_2$ of the triangles $BB_1B_2$ and $CC_1C_2$;*
*(ii) $N_-^a$ is the intersection of the perpendicular bisectors of the sides $BB_2$ and $CC_1$ of the same triangles.*
*The other vertices of Napoleon triangles are similarly defined.*

So, Fermat points, Napoleon points, and Napoleon triangles are defined in the configuration $\mathcal{F}$ as easily as in the classic one.

**Remark 2.1.** *From the above, it follows that the configuration $\mathcal{F}$ coincides with the classic Fermat-Torricelli configuration and represents only a new way of approaching it. We will see that the choice of triangles $AA_1A_2$, $BB_1B_2$, $CC_1C_2$ as fundamental elements of $\mathcal{F}$ will allow us to highlight an important number of new and interesting properties of this configuration.*

For the next sentence, we recall some known results.

**Lemma 2.1.** *(i) The lengths of the sides of triangles $AA_1A_2$, $BB_1B_2$, $CC_1C_2$ are given by*

$$l_a = \frac{4\Delta}{\sqrt{3}a}, \quad l_b = \frac{4\Delta}{\sqrt{3}b}, \quad l_c = \frac{4\Delta}{\sqrt{3}c}.$$

*(ii) We have:*

$$l_+^2 = \frac{1}{2}\left(a^2+b^2+c^2+4\sqrt{3}\Delta\right), \quad l_-^2 = \frac{1}{2}\left(a^2+b^2+c^2-4\sqrt{3}\Delta\right)$$

([7, p. 220]*), where $l_+$ and $l_-$ denote the common lengths of the segments $AA_+, BB_+, CC_+$ and $AA_-, BB_-, CC_-$ respectively, and $\Delta$ is the area of triangle $ABC$.*

**Lemma 2.2.** *Let $P$ be the trace of $AF_1$ on the side $BC$ (Fig. 4). Then*
*(i) $AP = \dfrac{4\Delta l_+}{4\Delta+\sqrt{3}a^2}$,*
*(ii) $BP = \dfrac{4\Delta+\sqrt{3}\left(c^2+a^2-b^2\right)}{2\left(4\Delta+\sqrt{3}a^2\right)}a$, and $CP = \dfrac{4\Delta+\sqrt{3}\left(a^2+b^2-c^2\right)}{2\left(4\Delta+\sqrt{3}a^2\right)}a$,*
*and similar formulas with respect to vertices $B$ and $C$ (*[2, p. 11]*).*

**Proposition 2.2.** *(i) The cevians $BB_2, CC_1$ intersect on the Fermat cevian $AF_+$.*
*(ii) The cevians $BB_1$, $CC_2$ intersect on the Fermat cevian $AF_-$.*
*Similar properties relative to Fermat cevians $BF_+$, $BF_-$, and $CF_+$, $CF_-$ (Fig. 4).*

*Proof.* (i) We have to show that

$$\frac{\overline{PB}}{\overline{PC}} \cdot \frac{\overline{B_2C}}{\overline{B_2A}} \cdot \frac{\overline{C_1A}}{\overline{C_1B}} = -1.$$

But, $BP$, $CP$ are given by Lemma 2.2, and

$$AB_2 = \frac{1}{2}l_b - c\cos A = \frac{2\Delta}{\sqrt{3}b} - \frac{b^2+c^2-a^2}{2b} = \frac{4\Delta-\sqrt{3}\left(b^2+c^2-a^2\right)}{2\sqrt{3}b},$$

$$B_2C = b + AB_2 = \frac{4\Delta+\sqrt{3}\left(a^2+b^2-c^2\right)}{2\sqrt{3}b},$$

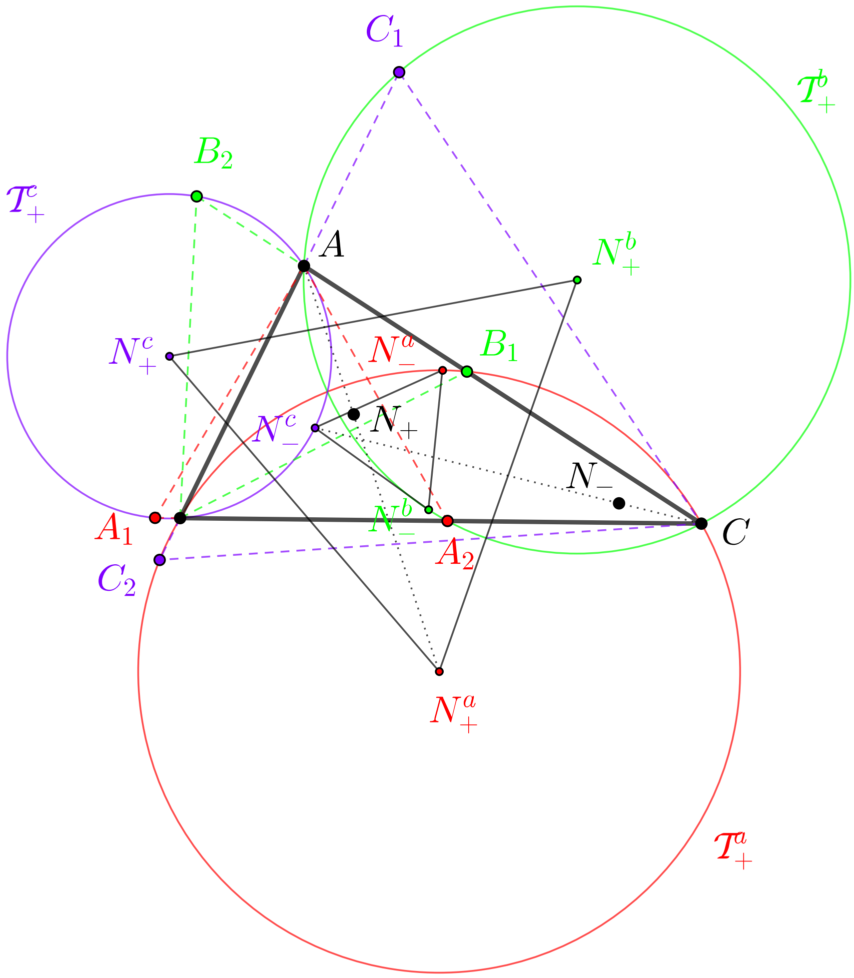


Fig. 3

$$C_1A = \frac{1}{2}l_c - b\cos A = \frac{2\Delta}{\sqrt{3}c} - \frac{b^2+c^2-a^2}{2c} = \frac{4\Delta - \sqrt{3}\left(b^2+c^2-a^2\right)}{2\sqrt{3}c},$$

$$C_1B = c + C_1A = \frac{4\Delta + \sqrt{3}\left(c^2+a^2-b^2\right)}{2\sqrt{3}c}.$$

Substituting the found expressions and performing the calculations, we immediately find that the above equality is verified. □

**Remark 2.2.** *Based on Proposition 2.2, we can define the points $F_+$ and $F_-$ as follows:*

$$F_+ := AX_1 \cap BY_1 \cap CZ_1, \quad F_- := AX_2 \cap BY_2 \cap CZ_2,$$

*where $X_1 := BB_2 \cap CC_1, X_2 := BB_1 \cap CC_2$ etc. Thus, if the configuration $\mathcal{F}$ is constructed on the basis of the equilateral triangles $AA_1A_2, BB_1B_2, CC_1C_2$, this definition of points $F_+$ and $F_-$*

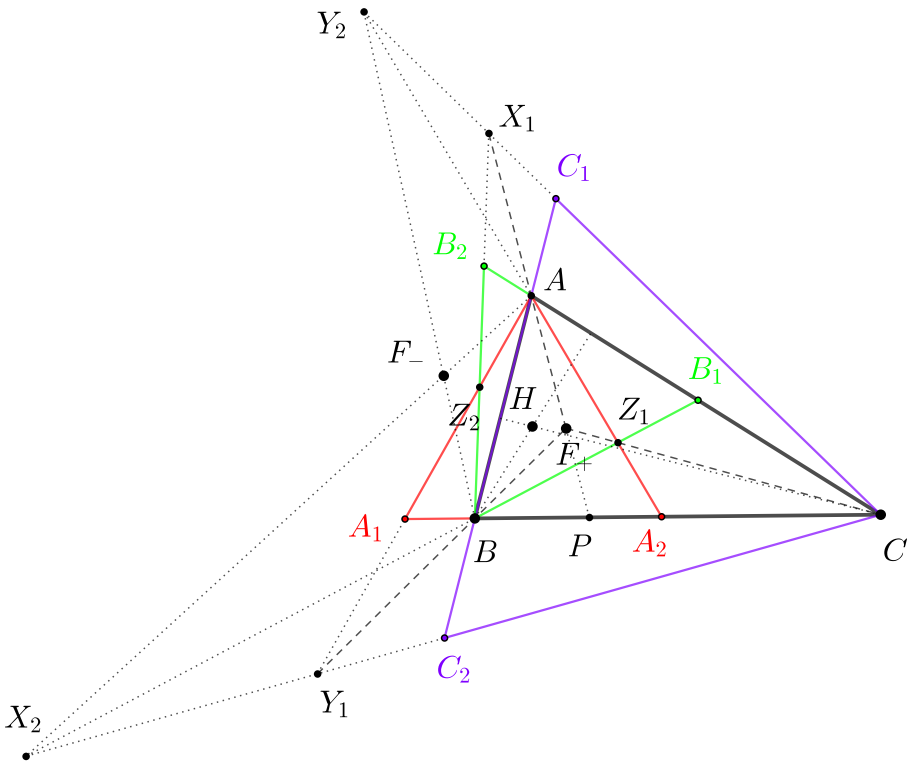


Fig. 4

*corresponds to the usual one in the classical configuration ($F_+ := AA_+ \cap BB_+ \cap CC_+$, $F_- := AA_- \cap BB_- \cap CC_-$).*

Denote by $O_a, O_b, O_c$ the centers of the equilateral triangles $AA_1A_2$, $BB_1B_2$, $CC_1C_2$, respectively.

**Proposition 2.3.** *We have: (i) $N_+^a = B_2O_b \cap C_1O_c$, $N_+^b = C_2O_c \cap A_1O_a$, $N_+^c = A_2O_a \cap B_1O_b$; (ii) $N_-^a = B_1O_b \cap C_2O_c$, $N_-^b = C_1O_c \cap A_2O_a$, $N_-^c = A_1O_a \cap B_2O_b$ (Fig. 5).*

*Proof.* (i) Consider the cyclic quadrilateral $BCC_1B_2$. By Proposition 2.1, the circle $(BCC_1B_2)$ is Torricelli circle $\mathcal{T}_-^a$. Since $\widehat{B_2} = \widehat{C_1} = \dfrac{\pi}{3}$, then the interior bisectors of these angles intersect in the middle of the minor arc $\overset{\frown}{BC}$ of the circle $\mathcal{T}_-^a$, i.e. in the point $N_+^a$. Obviously, the interior bisectors of $\widehat{B_2}$ and $\widehat{C_1}$ are $B_2O_b$ and $C_1O_c$, respectively. Hence, $B_2O_b \cap C_1O_c = N_+^a$.
(ii) Consider the cyclic quadrilateral $BB_1CC_2$ and we do the same.
The proof is complete. □

**Remark 2.3.** *The preceding proposition provides a way of defining Napoleon triangles different from the above and the classical one.*

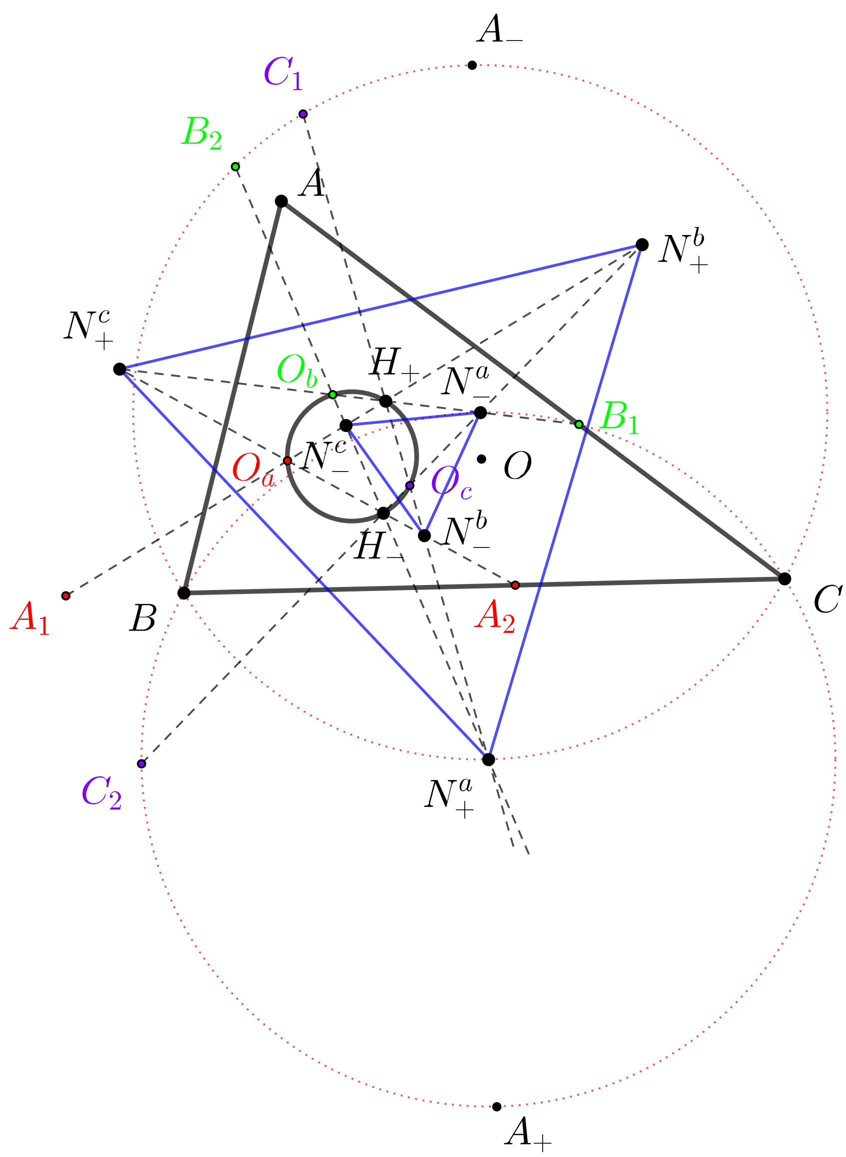


Fig. 5

It is absolutely necessary to make precise the position of points $A_1$, $A_2$, $B_1$, $B_2$, $C_1$, $C_2$ on the sides of the triangle $ABC$. This is equivalent to the position of triangles $AC_1B_2$, $BA_1C_2$, $CB_1A_2$ in relation to $ABC$ (Fig. 6).

The form of the given triangle is decisive. If $C < \frac{\pi}{3}$, then $\widehat{DAC} > \frac{\pi}{6}$, and the point $A_2$ lies between $D$ and $C$. Moreover, we also have that $\widehat{EBC} > \frac{\pi}{6}$, hence the point $B_1$ lies between $E$ and $C$. Consequently, the triangle $CB_1A_2$ overlaps on $ABC$ (Fig. 6). With similar arguments, if $C > \frac{\pi}{3}$ it follows that the vertex $C$ lies both beetwen $E$ and $B_1$

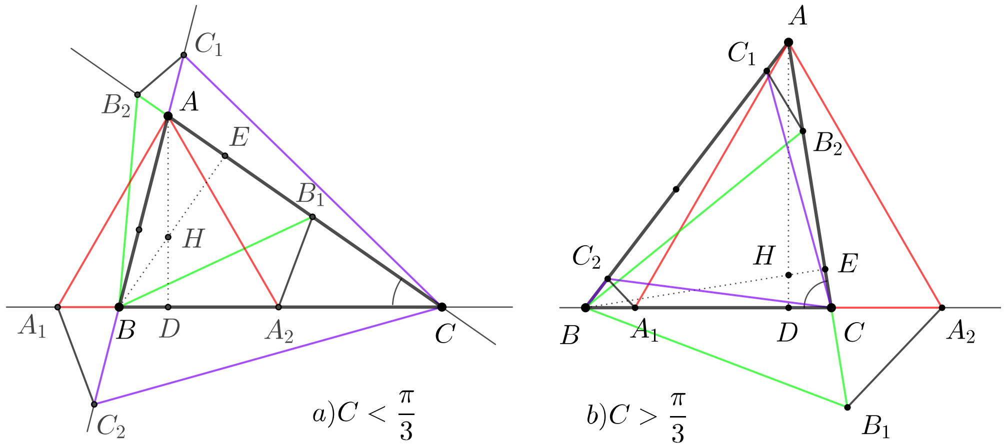


Fig. 6

and beetwen $D$ and $A_2$. In this case, the triangle $CB_1A_2$ is outside of $ABC$ (Fig. 6). The case $C = \frac{\pi}{3}$ is trivial: the points $B_1$ and $A_2$ coincide with $C$, i.e. the triangle $CB_1A_2$ degenerates at point $C$.

The triangle $ABC$ can have no more than two angles smaller than $\frac{\pi}{3}$. Hence, one or two of the triangles $AC_1B_2$, $BA_1C_2$, $CB_1A_2$ overlaps the triangle $ABC$, as one or two angles of the given triangle is smaller than $\frac{\pi}{3}$. The proofs of the preceding sentences have been given where the triangle $ABC$ has only one angle smaller than $\frac{\pi}{3}$, but can be easily adapted to the remaining case. This strategy will also be adopted below. So, *we will assume in the sequel that* $A > \frac{\pi}{3}$, $B > \frac{\pi}{3}$, *and* $C < \frac{\pi}{3}$.

## 3. Orthocentroidal circle and points $H_+$ and $H_-$

The *orthocentroidal circle* $\mathcal{C}_{HG}$ (i.e. the circle having the segment $HG$ as diameter) and the *symmedian point* $K$ play an important role in the study of the configuration $\mathcal{F}$. Obviously, $\mathcal{C}_{HG}$ contains the orthogonal projections of the centroid $G$ on the altitudes of the triangle $ABC$; the triangle determined by these projections is called the *orthocentroidal triangle of* $ABC$ (Fig. 7). We recall four notable results in this regard: 1) $F_+$ and $F_-$ are inverse points in the orthocentroidal circle ([9]), 2) $F_+$ and $F_-$ are the isodynamic points of the orthocentroidal triangle of $ABC$ ([5, p. 3],[2, p. 11]), 3) the Napoleon triangles $N_+^a N_+^b N_+^c$ and $N_-^a N_-^b N_-^c$ are perspective from the circumcenter $O$, 4) the lines $F_+F_-$ and $N_+N_-$ intersect at the point $K$ ([8, p. 129]).
Recall that by $O_a, O_b, O_c$ we have denoted the centers of the equilateral triangles $AA_1A_2$, $BB_1B_2$, $CC_1C_2$, respectively.

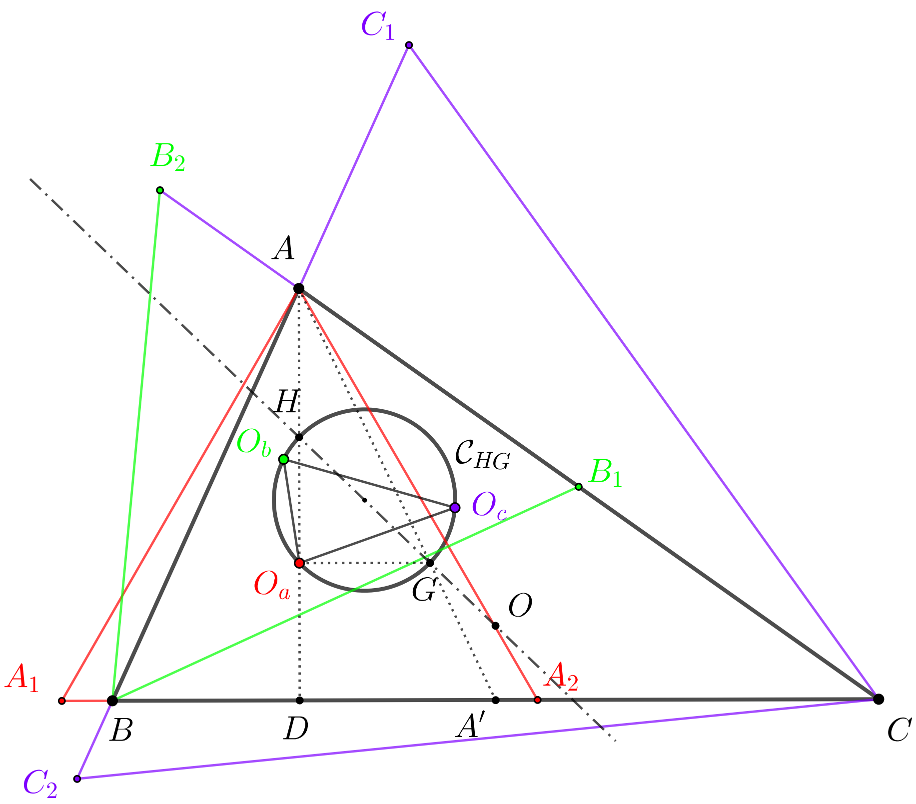


Fig. 7

**Proposition 3.1.** *The triangle $O_aO_bO_c$ is the orthocentroidal triangle of $ABC$. The circle determined by the centers $O_a, O_b, O_c$ is the orthocentroidal circle of $ABC$.*

*Proof.* It suffice to show that $O_a$ is the projection of $G$ on the altitude from $A$ of the triangle $ABC$. For this, we consider the triangle determined by $BC$ and the support lines of altitude and median from $A$ and take into account that $G$ divides the median in the ratio 2:1. Since $G$ divides the median in the ratio 2:1, its projection divides the altitude by $A$ in the same ratio and, and thus coincides with $O_a$. □

Next, we will introduce two points that will play an important role in the study of the configuration $\mathcal{F}$.

**Proposition 3.2.** *1) The lines $A_1O_a$, $B_1O_b$, $C_1O_c$ are concurrent on the orthocentroidal circle. 2) The lines $A_2O_a$, $B_2O_b$, $C_2O_c$ are concurrent on the orthocentroidal circle (Fig. 8).*

*Proof.* 1) Let $X$ denote the intersection point of the lines $A_1O_a$ and $B_1O_b$. Then, $\widehat{O_aXO_b} = \widehat{A_1XO_b} = \pi - \widehat{A_1XB_1}$. In the quadrilateral $A_1XB_1C$ we have: $\widehat{A_1XB_1} = 2\pi - \widehat{XA_1C} - C - \widehat{CB_1X} = 2\pi - \dfrac{\pi}{6} - C - \left(\widehat{CB_1B} + \widehat{BB_1O_b}\right) = 2\pi - \dfrac{\pi}{6} - C - \left(\dfrac{2\pi}{3} + \dfrac{\pi}{6}\right) = \pi - C$. So, we get: $\widehat{O_aXO_b} = \pi - (\pi - C) = C$. On the other hand, $\widehat{O_aHO_b} = \widehat{DHB} = C$. Hence, it

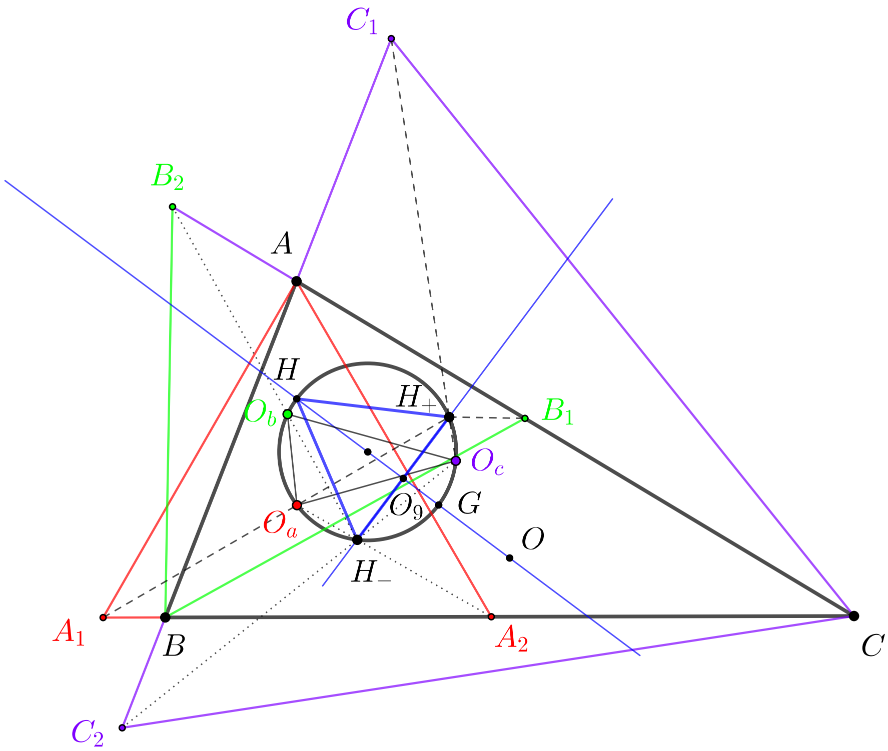


Fig. 8

follows that $\widehat{O_aXO_b} = \widehat{O_aHO_b}$, that is $X \in \mathcal{C}_{HG}$. Therefore, $A_1O_a$,and $B_1O_b$ intersect at a point on the circle $\mathcal{C}_{HG}$.
Similarly, it is shown that $B_1O_b$ and $C_1O_c$ intersect on the circle $\mathcal{C}_{HG}$. Consequently, the three lines $A_1O_a$, $B_1O_b$ and $C_1O_c$ are concurrent in a point located on $\mathcal{C}_{HG}$.
2) The statement is shown with the same arguments. □

According to this proposition, we define the points $H_+$ and $H_-$ by

$$H_+ := A_1O_a \cap B_1O_b \cap C_1O_c \quad \text{and} \quad H_- := A_2O_a \cap B_2O_b \cap C_2O_c,$$

and call them the *orthocentroidal points* of the triangle $ABC$.

**Proposition 3.3.** *The following statements are true:*
*1) the triangle $HH_+H_-$ is equilateral;*
*2) the orthocentroidal points $H_+, H_-$ are symmetric with respect to the Euler line and the midpoint of $H_1H_2$ is the nine-point center of the given triangle (Fig. 8).*

*Proof.* 1) We have: $\widehat{HH_+H_-} = \widehat{HO_cH_-} = \pi - \widehat{H_-O_cC} = \pi - \widehat{C_2O_cC} = \pi - \dfrac{2\pi}{3} = \dfrac{\pi}{3}$.
Similarly, we get: $\widehat{HH_-H_+} = \dfrac{\pi}{3}$. Thus, $\Delta HH_+H_-$ is equilateral.
2) The assertions are consequences of the fact that $H$ is on the Euler line and the triangle $HH_+H_-$ is equilateral (Fig. 8). □

**Remark 3.1.** *The sidelength of the equilateral triangle $HH_+H_-$ is obtained immediately from the formula $OH^2 = 9R^2 - (a^2+b^2+c^2)$* [4, p. 20]. *Indeed, since HG is the diameter of its circumcircle, it follows that $H_+H_-^2 = \frac{3}{4}HG^2 = \frac{3}{4}\left(\frac{2}{3}OH\right)^2$, hence*

$$H_+H_-^2 = 3R^2 - \frac{1}{3}\left(a^2+b^2+c^2\right).$$

On the other hand, the preceding proposition makes it possible to define the points $H_+$ and $H_-$ independently of the configuration $\mathcal{F}$. Thus, the points $H_+$ and $H_-$ can be constructed as the vertices of the equilateral triangle inscribed in the orthocentroidal circle $\mathcal{C}_{HG}$ and having the orthocentre $H$ as one of its vertices or as the points of intersection of the circle $\mathcal{C}_{HG}$ with the perpendicular to Euler line at nine-point center (in both these definitions the resulting triangle $HH_+H_-$ must be of the opposite orientation to that of the triangle $ABC$).
Another way to define the points $H_+$ and $H_-$ is given by the following sentence:

**Proposition 3.4.** *The following statements are true:*
*1) $H_+ = (AB_1C_1) \cap (BC_1A_1) \cap (CA_1B_1)$,*
*2) $H_- = (AB_2C_2) \cap (BC_2A_2) \cap (CA_2B_2)$ (Fig. 9).*

*Proof.* 1) Denote by $H_1$ the intersection point of the circles $(BC_1A_1)$ and $(CA_1B_1)$ other than $A_1$. Obviously,

$$\widehat{B_1H_1C_1} = 2\pi - \widehat{A_1H_1B_1} - \widehat{A_1H_1C_1}.$$

Since the quadrilateral $A_1H_1B_1C$ is cyclic, we have: $\widehat{A_1H_1B_1} = \pi - C$. Also, because $A_1BH_1C_1$ is cyclic quadrilateral, we have that $\widehat{A_1H_1C_1} = \widehat{A_1BC_1} = \pi - B$. Therefore, $\widehat{B_1H_1C_1} = 2\pi - (\pi - C) - (\pi - B) = B + C = \pi - A$. Hence, $\widehat{B_1H_1C_1} = \widehat{B_1AC_1}$, i.e. the quadrilateral $AH_1B_1C_1$ is cyclic, and $H_1$ lies on the circle $(AB_1C_1)$. We conclude that the three circles have in common the point $H_1$.
It remains to show that $H_+$ coincides with $H_1$. First, from the fact that the orthocentroidal triangle is similar to the given one, we have: $\widehat{O_aO_cO_b} = C$. Then, $\widehat{O_aH_+O_b} = C$, and therefore $\widehat{A_1H_+O_b} = C$. It follows that $\widehat{A_1H_+B_1} = \pi - \widehat{A_1H_+O_b} = \pi - C$, whence $\widehat{A_1H_+B_1} = \pi - C$. Combining the last relation with the relation $\widehat{A_1H_1B_1} = \pi - C$ established above, we get: $\widehat{A_1H_+B_1} = \widehat{A_1H_1B_1}$. Hence $H_+$ lies on the circle $(CA_1B_1)$. In the same way we show that $H_+$ is also on the circles $(AB_1C_1)$, and $(BC_1A_1)$. So, $H_+$ and $H_1$ coincide.
2) A similar argument works to prove this statement. □

**Proposition 3.5.** *With the preceding notation and conventions, we have the following sets of collinear points:*
*(i) $A_1, O_a, H_+, N_+^b, N_-^c$ on the line $A_1O_a$,*
*(ii) $A_2, O_a, H_-, N_+^c, N_-^b$ on the line $A_2O_a$,*
*and analogous sets of points relative to the lines $B_1O_b, B_2O_b$ and $C_1O_c, C_2O_c$.*

*Proof.* (i) Indeed, by Proposition 2.3, $N_+^b, N_-^c \in A_1O_a$. Also, by Proposition 3.2, $H_+ \in A_1O_a$.
(ii) It is done in the same way. □

Now, we will complete the property 3) stated at the beginning of this section. First, we will need the following elementary and well-known result.

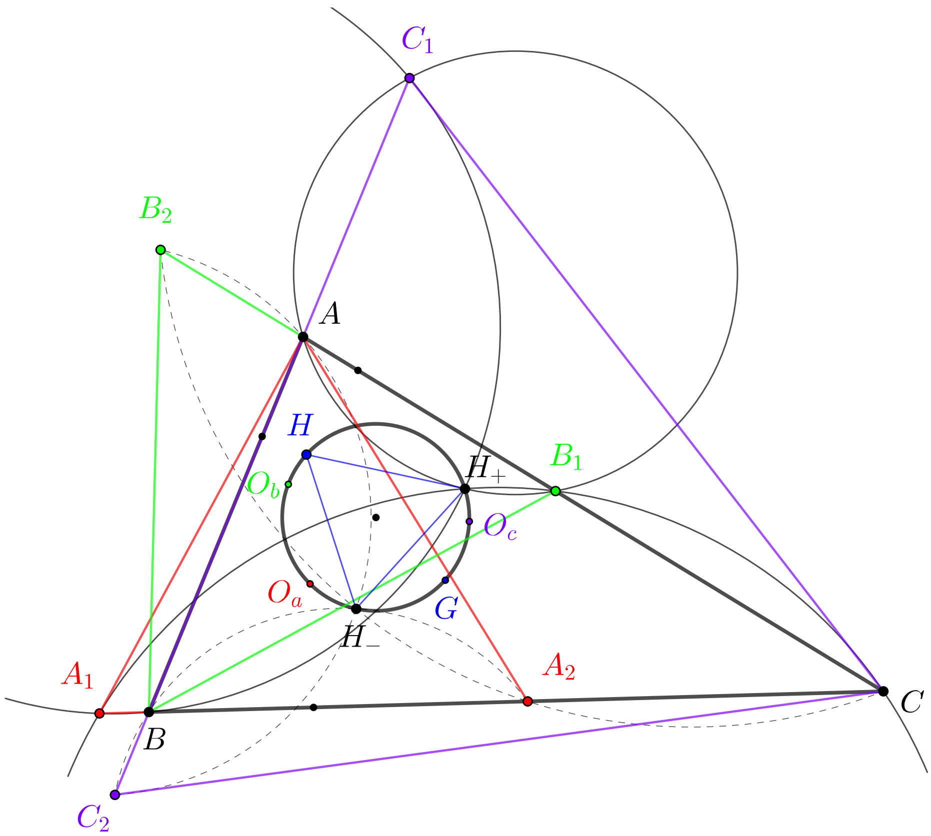


Fig. 9

**Lemma 3.1.** *Let $ABC$ be an equilateral triangle and let $X_1, X_2 \in BC$, $Y_1, Y_2 \in CA$, $Z_1, Z_2 \in AB$ such that $BX_1 = CY_1 = AZ_1$ and $BZ_2 = CX_2 = AY_2$. Then, the triangle determined by the lines $X_2Y_1, Y_2Z_1, Z_2X_1$ is equilateral and has the same center as the given one.*

**Proposition 3.6.** *(i) the Napoleon triangles are perspective in three manners:*

$$\begin{pmatrix} N^a_+ N^b_+ N^c_+ \\ N^a_- N^b_- N^c_- \end{pmatrix}, \begin{pmatrix} N^a_+ N^b_+ N^c_+ \\ N^b_- N^c_- N^a_- \end{pmatrix}, \text{ and } \begin{pmatrix} N^a_+ N^b_+ N^c_+ \\ N^c_- N^a_- N^b_- \end{pmatrix};$$

*(ii) the three centres of perspective are the points $O$, $H_+$ and $H_-$ respectively;*
*(iii) the three axes of perspective determine an equilateral triangle, $T_aT_bT_c$, with the center $G$ (Fig. 10).*

*Proof.* (i)-(ii) It is known that $O \in N^a_+N^a_- \cap N^b_+N^b_- \cap N^c_+N^c_-$. By Proposition 3.5, we have: $H_+ \in N^a_+N^b_- \cap N^b_+N^c_- \cap N^c_+N^a_-$ and $H_- \in N^a_+N^c_- \cap N^b_+N^a_- \cap N^c_+N^b_-$.
(iii) Denote the points of intersection of the sidelines of triangle $N^a_+N^b_+N^c_+$ and $N^a_-N^b_-N^c_-$ as in Fig. 10. Then, the line $UV$ (or $T_aT_b$) is the axis of perspective of triangle $N^a_+N^b_+N^c_+$ and $N^a_-N^b_-N^c_-$, $WX$ (or $T_bT_c$) is the axis of $N^a_+N^b_+N^c_+$ and $N^b_-N^c_-N^a_-$, and $YZ$ (or $T_cT_a$) is the axis of $N^a_+N^b_+N^c_+$ and $N^c_-N^a_-N^b_-$. By a counterclockwise rotation about $G$ through $\frac{2\pi}{3}$, we obtain that $N^a_+X = N^b_+Z = N^c_+V$ and $N^a_+W = N^b_+Y = N^c_+U$. Then, by applying Lemma 3.1 to the triangle $N^a_+N^b_+N^c_+$, we get the desired result. $\square$

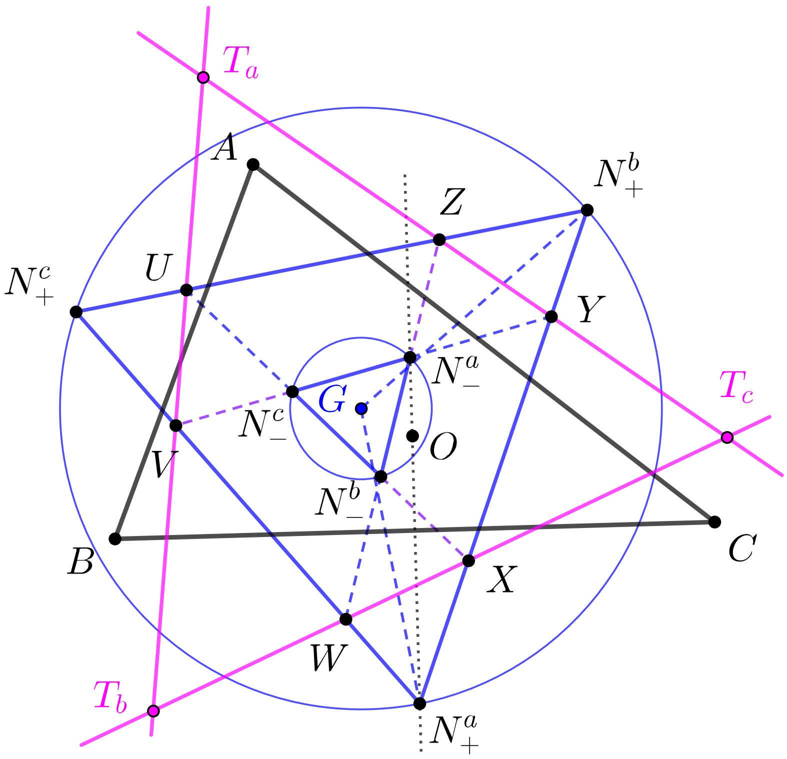


Fig. 10

The triangles $HH_+H_-$ and $OH_+H_-$ are symmetric with respect to the line $H_+H_-$. We note that $OH_+H_-$ is equilateral and has the center $G$, the same as the Napoleon triangles (Fig. 11).

**Proposition 3.7.** *(i) $N_+^a, N_+^b, N_+^c$ are the perspective centers of the pairs of triangles $(OH_+H_-$, $N_-^aN_-^bN_-^c)$, $(OH_+H_-$, $N_-^bN_-^cN_-^a)$ and $(OH_+H_-$, $N_-^cN_-^aN_-^b)$, respectively.*
*(ii) $N_-^a, N_-^b, N_-^c$ are the perspective centers of the pairs of triangles $(OH_+H_-$, $N_+^aN_+^cN_+^b)$, $(OH_+H_-$, $N_+^bN_+^aN_+^c)$, and $(OH_+H_-$, $N_+^cN_+^bN_+^a)$, respectively (Fig. 11).*

*Proof.* It follows directly from the fact that $O \in N_+^aN_-^a, O \in N_+^bN_-^b, O \in N_+^cN_-^c$ and Proposition 3.5. □

## 4. Similarity properties of triangles in triads $(AB_2C_1, BC_2A_1, CA_2B_1)$ and $(AB_1C_2, BC_1A_2, CA_1B_2)$

We have seen that the configuration $\mathcal{F}$ can be built starting with the equilateral triangles $CBA_+, BCA_-$ and those obtained cyclically from them or starting with the equilateral triangles $AA_1A_2, BB_1B_2, CC_1C_2$. The second way allows us to imagine within the configuration $\mathcal{F}$ new and varied figures with interesting properties.

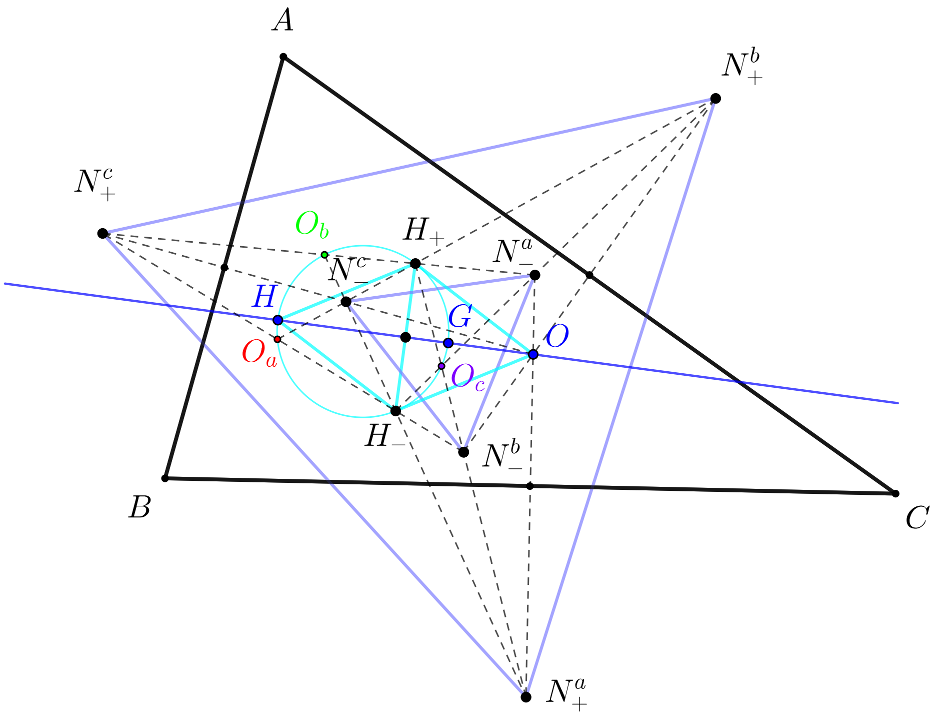


Fig. 11

In this sense, we will focus our attention on the properties of dual triads $(AB_2C_1, BC_2A_1, CA_2B_1)$ and $(AB_1C_2, BC_1A_2, CA_1B_2)$. Two triangles that have a common vertex which is also the vertex of the triangle $ABC$ are called *duals* (for example, $AB_2C_1$ and $AB_1C_2$ are dual to each other).
We start with an elementary but important result for the study that follows.

**Proposition 4.1.** *The six triangles $AB_2C_1$, $BC_2A_1$, $CA_2B_1$; $AB_1C_2$, $BC_1A_2$, $CA_1B_2$ have the properties:*
*(i) each of them is inversely similar to the given triangle $ABC$;*
*(ii) any two of them are directly similar.*

*Proof.* (i) We limit ourselves to seeing that one of them is inversely similar to the triangle $ABC$. For example, let's show that the triangles $ABC$ and $AB_2C_1$ are inversely similar. But, it is clear that we have: $\widehat{A} = \widehat{A}, \widehat{B_2} = \widehat{B}, \widehat{C_1} = \widehat{C}$ (the last two result from the properties of the cyclic quadrilateral $BCC_1B_2$). Then, $AB_2C_1 \sim ABC$ and, obviously, they are inversely similar.
(ii) follows from (i). □

**Remark 4.1.** *Positions of points $A_1$, $A_2$; $B_1$, $B_2$; $C_1$, $C_2$ on the sidelines of the triangle $ABC$ depend on the shape of this triangle. If $A > B > C$, we have three cases to consider. It is easy to specify the position of $A_i$, $B_i$, $C_i$ $(i = 1, 2)$ in each of these cases (Fig. 12):*
*I.* $A > \dfrac{2\pi}{3}$, $\dfrac{\pi}{3} > B > C$ $(B - A_1 - A_2 - C,\ C - A - B_1 - B_2,\ C_1 - C_2 - A - B)$;

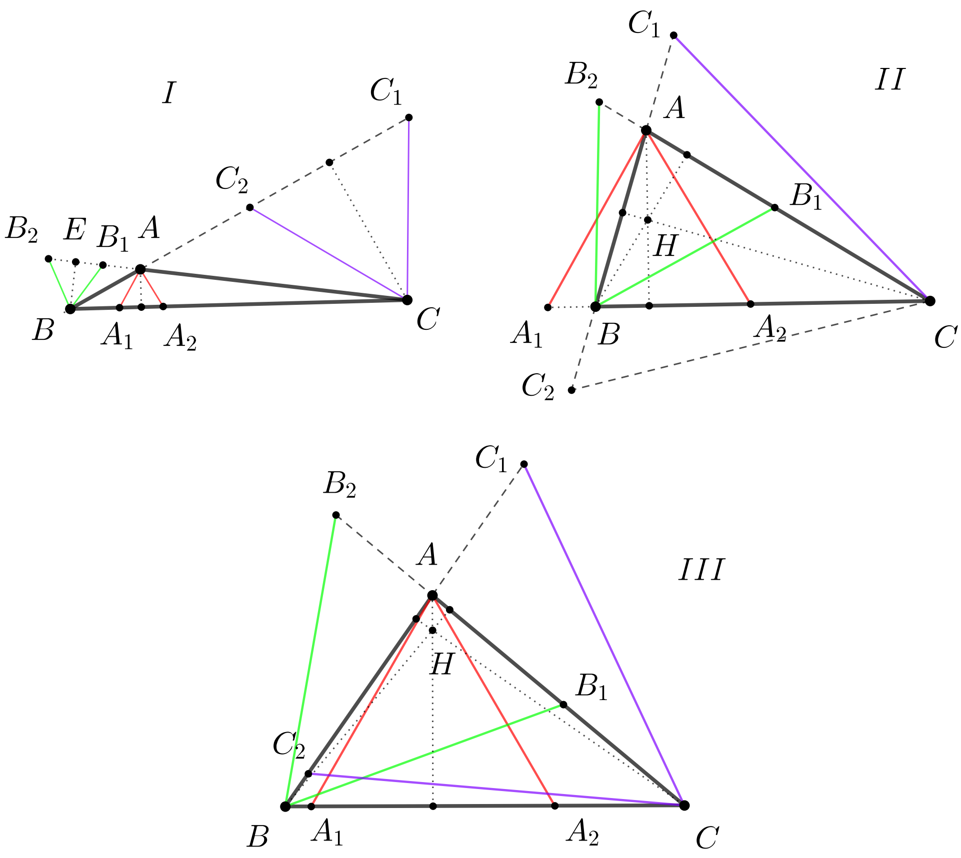


Fig. 12

*II.* $A > B > \frac{\pi}{3} > C$ $(A_1 - B - A_2 - C,\ C - B_1 - A - B_2,\ C_1 - A - B - C_2)$;
*III.* $A > \frac{\pi}{3} > B > C$ $(B - A_1 - A_2 - C,\ C - B_1 - A - B_2,\ C_1 - A - C_2 - B)$.

**Remark 4.2.** *With regard to the triad* $(AB_2C_1,\ BC_2A_1,\ CA_2B_1)$ *we find that: 1) the triangle* $AB_2C_1$ *is external to the triangle* $ABC$ *in all cases, 2) the triangle* $BC_2A_1$ *falls on it in cases I and III, and 3) the triangle* $CA_2B_1$ *falls on it in all cases. With regard to the dual triad* $(AB_1C_2, BC_1A_2, CA_1B_2)$ *the situation is simpler: only the triangle* $AB_1C_2$ *can be external to the triangle* $ABC$ *and this only happens in the case I. So, if* $A > B > C$*, we have the following table:*

| | $AB_2C_1$ | $BC_2A_1$ | $CA_2B_1$ | $AB_1C_2$ | $BC_1A_2$ | $CA_1B_2$ |
|---|---|---|---|---|---|---|
| *I* | *e* | *f* | *f* | *e* | *f* | *f* |
| *II* | *e* | *e* | *f* | *f* | *f* | *f* |
| *III* | *e* | *f* | *f* | *f* | *f* | *f* |

*(e indicates that the triangle is external to* $ABC$*, and* $f$ *that it falls on* $ABC$*).*

We adopt some notations. Relative to the dual triangles $AB_2C_1$ and $AB_1C_2$, let $O^a_+$ and $O^a_-$ be their circumcenters, respectively. The circumcircles $(AB_2C_1)$ and $(AB_1C_2)$ are also denoted $(O^a_+), (O^a_-)$ and their radii are denoted $r^a_+$ and $r^a_-$, respectively. Define $O^b_+, O^b_-$; $O^c_+, O^c_-$ and $r^b_+, r^b_-$; $r^c_+, r^c_-$ cyclically.
We denote $\mathcal{S}(d)$ the reflection in the line $d$ and $\mathcal{H}(V, \rho)$ the homothety with center $V$ and rotio $\rho$. Also, $\mathcal{R}(P, \alpha)$ denotes the rotation about $P$ and angle $\alpha$; the angle of rotation $\alpha$ will be oriented only positively (counterclockwise).

According to the statement (i) in Proposition 4.1, $ABC$ is inversely similar to all triangles in the two triads; we have six pairs of such triangles: $(AB_2C_1, ABC)$, $(BC_2A_1, BCA)$, $(CA_2B_1, CAB)$; $(AB_1C_2, ABC)$, $(BC_1A_2, BCA)$, $(CA_1B_2, CAB)$. The theory of similitude ([1], [7], [10]) guarantees that a triangle in any of these pairs is the image of the other by a reflexion in a line followed by a homothety.

**Proposition 4.2.** *We have:*
*(i)* $ABC = \mathcal{H}\left(A, \dfrac{R}{r^a_+}\right) \cdot \mathcal{S}(w'_A)(AB_2C_1)$;
*(ii)* $BCA = \mathcal{H}\left(B, \dfrac{R}{r^b_+}\right) \cdot \mathcal{S}(w_B)(BC_2A_1)$ *in cases I and III, and*
$BCA = \mathcal{H}\left(B, \dfrac{R}{r^b_+}\right) \cdot \mathcal{S}(w'_B)(BC_2A_1)$ *in case II;*
*(iii)* $CAB = \mathcal{H}\left(C, \dfrac{R}{r^c_+}\right) \cdot \mathcal{S}(w_C)(CA_2B_1)$;
*(iv)* $ABC = \mathcal{H}\left(A, \dfrac{R}{r^a_-}\right) \cdot \mathcal{S}(w'_A)(AB_1C_2)$ *in case I, and* $ABC = \mathcal{H}\left(A, \dfrac{R}{r^a_-}\right) \cdot \mathcal{S}(w_A)(AB_1C_2)$ *in cases II and III;*
*(v)* $BCA = \mathcal{H}\left(B, \dfrac{R}{r^b_-}\right) \cdot \mathcal{S}(w_B)(BC_1A_2)$;
*(vi)* $CAB = \mathcal{H}\left(C, \dfrac{R}{r^c_-}\right) \cdot \mathcal{S}(w_C)(CA_1B_2)$,
*where $w_A$ and $w'_A$ denote the internal and external bisectors of angle $A$ etc.*

*Proof.* These statements are proven in the same way. The external or internal bisector will be used depending on whether the source triangle is external or not to the triangle $ABC$.
We detail only for (i). The triangles $AB_2C_1$ and $ABC$ have in common the vertex $A$ and $AB_2C_1$ is external to the triangle $ABC$. We will use the external bisector of the angle $A$. Define $B'_2 = \mathcal{S}(w'_A)(B_2)$ and $C'_1 = \mathcal{S}(w'_A)(C_1)$. Obviously, $B'_2 \in AB$, $C'_1 \in AC$ and $B'_2C'_1 \parallel BC$. Hence, $\mathcal{S}(w'_A)(AB_2C_1) = AB'_2C'_1$. On the other hand, because $AB'_2C'_1 \sim ABC$ and $B'_2C'_1 \parallel BC$, it follows that $\mathcal{H}\left(A, \dfrac{R}{r^a_+}\right)(AB'_2C'_1) = ABC$.
Combining these two partial results, we get $ABC = \mathcal{H}\left(A, \dfrac{R}{r^a_+}\right) \cdot \mathcal{S}(w'_A)(AB_2C_1)$. □

Now, let's examine two triangles chosen from the directly similar triangles $AB_2C_1$, $BC_2A_1$, $CA_2B_1$ and $AB_1C_2$, $BC_1A_2$, $CA_1B_2$. The dual triangles have in common a vertex of the

triangle $ABC$. It is easy to verify that two triangles that are not dual, one from the first triad and the other from the second, have in common a vertex which is among the points $A_1, A_2, B_1, B_2, C_1, C_2$. It is obvious that two triangles in the same triad have no common vertices.
The simplest is the case of dual triangles. There are three pairs of dual triangles.

**Proposition 4.3.** *The dual triangles $AB_2C_1$ and $AB_1C_2$, $BC_2A_1$ and $BC_1A_2$, $CA_2B_1$ and $CA_1B_2$ are homothetic. More specifically, we have:*
*(i)* $AB_1C_2 = \mathcal{H}\left(A, \varepsilon_a \frac{r_-^a}{r_+^a}\right)(AB_2C_1)$,
*(ii)* $BC_1A_2 = \mathcal{H}\left(B, \varepsilon_b \frac{r_-^b}{r_+^b}\right)(BC_2A_1)$,
*(iii)* $CA_1B_2 = \mathcal{H}\left(C, \varepsilon_c \frac{r_-^c}{r_+^c}\right)(CA_2B_1)$,
*where $\varepsilon_a, \varepsilon_b, \varepsilon_c$ will be taken $-1$ or $+1$ depending on whether the dual triangles are opposite in their common vertex or not.*

*Proof.* By Proposition 4.1, two dual triangle are directly similar. It is used that $B_2C_1 \| B_1C_2$, $C_2A_1 \| C_1A_2$, and $A_2B_1 \| A_1B_2$. □

More complicated is the study of pairs of triangles chosen from different triads and which are not dual. We have six such pairs: $(AB_2C_1,\ BC_1A_2)$, $(AB_2C_1, CA_1B_2)$, $(BC_2A_1,\ CA_1B_2)$, $(BC_2A_1,\ AB_1C_2)$, $(CA_2B_1,\ AB_1C_2)$, $(CA_2B_1,\ BC_1A_2)$. The triangles in any pair are directly similar. Thus, each of them is the image of the other through a rotation followed by a homothety of the same center. Let's write these pairs again highlighting the homologous elements of triangles: $(AB_2C_1, A_2BC_1)$, $(AB_2C_1, A_1B_2C)$, $(BC_2A_1, B_2CA_1)$, $(BC_2A_1, B_1C_2A)$, $(CA_2B_1, C_2AB_1)$, $(CA_2B_1, C_1A_2B)$.
We will see that the two geometric transformations have as their center one of the vertices $A_1, A_2, B_1, B_2, C_1, C_2$ and that the angle of rotation is simply expressed by the angles $A, B, C$ of the given triangle, but their expressions depend on the position of the two triangles in relation to it (see Fig. 12 and the above table).

**Proposition 4.4.** *Relative to the pairs of triangles above we have:*
*(i)* $A_2BC_1 = \mathcal{H}\left(C_1, \frac{r_-^b}{r_+^a}\right) \cdot \mathcal{R}(C_1, C)\,(AB_2C_1)$ *in all cases;*
*(ii)* $A_1B_2C = \mathcal{H}\left(B_2, \frac{r_-^c}{r_+^a}\right) \cdot \mathcal{R}(B_2, 2\pi - B)\,(AB_2C_1)$ *in all cases;*
*(iii)* $B_2CA_1 = \mathcal{H}\left(A_1, \frac{r_-^c}{r_+^b}\right) \cdot \mathcal{R}(A_1, \pi + A)\,(BC_2A_1)$ *in cases I and III,*
*and* $B_2CA_1 = \mathcal{H}\left(A_1, \frac{r_-^c}{r_+^b}\right) \cdot \mathcal{R}(A_1, A)\,(BC_2A_1)$ *in case II;*
*(iv)* $B_1C_2A = \mathcal{H}\left(C_2, \frac{r_-^a}{r_+^b}\right) \cdot \mathcal{R}(C_2, 2\pi - C)\,(BC_2A_1)$ *in cases I and II,*
*and* $B_1C_2A = \mathcal{H}\left(C_2, \frac{r_-^a}{r_+^b}\right) \cdot \mathcal{R}(C_2, \pi - C)\,(BC_2A_1)$ *in case III;*

*(v)* $C_2AB_1 = \mathcal{H}\left(B_1, \frac{r_-^a}{r_+^c}\right) \cdot \mathcal{R}(B_1, B)\,(CA_2B_1)$ *in case I, and*

$C_2AB_1 = \mathcal{H}\left(B_1, \frac{r_-^a}{r_+^c}\right) \cdot \mathcal{R}(B_1, \pi + B)\,(CA_2B_1)$ *in cases II and III;*

*(vi)* $C_1A_2B = \mathcal{H}\left(A_2, \frac{r_-^b}{r_+^c}\right) \cdot \mathcal{R}(A_2, \pi - A)\,(CA_2B_1)$ *in all cases.*

*Proof.* Normally, these statements are proven in the same way. Therefore, we limit ourselves to proving only one of them. For example, let's show the second part of statement (iii).
The triangles $BC_2A_1$ and $B_2CA_1$ have a common vertex, namely, the point $A_1$. It is easy to see that the (positive) angle between $A_1B$ and $A_1B_2$ is equal to $A$ and that the value of the (positive) angle between $A_1C_2$ and $A_1C$ is also $A$ (Fig. 12). We denote $B'$ and $C_2'$ the images of the points $B$ and $C_2$ by rotation $\mathcal{R}(A_1, A)$. It is clair that $B' \in A_1B_2, C_2' \in A_1C$, and $B'C_2' \parallel BC$. As a result, we get: $B'C_2'A_1 = \mathcal{R}(A_1, A)\,(BC_2A_1)$ and $\mathcal{H}\left(A_1, \frac{r_-^c}{r_+^b}\right)(B'C_2'A_1) = B_2CA_1$. By combining these results, we achieve the required equality. □

More interesting results are obtained when the pairs are formed by triangles taken from the same triad. $F_+$ and $F_-$ will be the centers of rotation and homothety as the pairs are formed with triangles of the second or first triad. The next two sentences are demonstrated with the same arguments as the previous ones. In fact, the Torricelli circles of the triangle ABC are used appropriately. Therefore, we will give some details only for the first of them.

**Proposition 4.5.** *The Fermat point $F_-$ is the center of rotation and homothety of the pair of triangles $(AB_2C_1, BC_2A_1)$, $(BC_2A_1, CA_2B_1)$, $(CA_2B_1, AB_2C_1)$ (Fig. 13). We have:*
*(i)* $A_1BC_2 = \mathcal{H}\left(F_-, \frac{r_+^b}{r_+^a}\right) \cdot \mathcal{R}(F_-, C)\,(AB_2C_1)$ *in cases I and III, and* $A_1BC_2 = \mathcal{H}\left(F_-, \frac{r_+^b}{r_+^a}\right) \cdot \mathcal{R}(F_-, \pi + C)\,(AB_2C_1)$ *in cases II;*
*(ii)* $B_1CA_2 = \mathcal{H}\left(F_-, \frac{r_+^c}{r_+^b}\right) \cdot \mathcal{R}(F_-, \pi + A)\,(BC_2A_1)$ *in cases I and III, and*

$B_1CA_2 = \mathcal{H}\left(F_-, \frac{r_+^c}{r_+^b}\right) \cdot \mathcal{R}(F_-, A)\,(BC_2A_1)$ *in case II;*

*(iii)* $C_1AB_2 = \mathcal{H}\left(F_-, \frac{r_+^a}{r_+^c}\right) \cdot \mathcal{R}(F_-, B)\,(CA_2B_1)$ *in all cases.*

*Proof.* We will give some details for the second part of the statement (i). Let $(A, A_1)$ and $(B_2, B)$ be two pairs of homologous points of the directly similar triangles $AB_2C_1$ and $A_1BC_2$ (Fig. 13). Obviously, $C$ is the point of intersection of the lines $AB_2$ and $A_1B$. According to the general theory of similitude, circles $(CAA_1)$ and $(CB_2B)$ intersect at the center of rotation of triangles. But these circles are $\mathcal{T}_-^b$ and $\mathcal{T}_-^a$, respectively, and their intersection (other than C) is $F_-$. So, $F_-$ is the center of rotation.

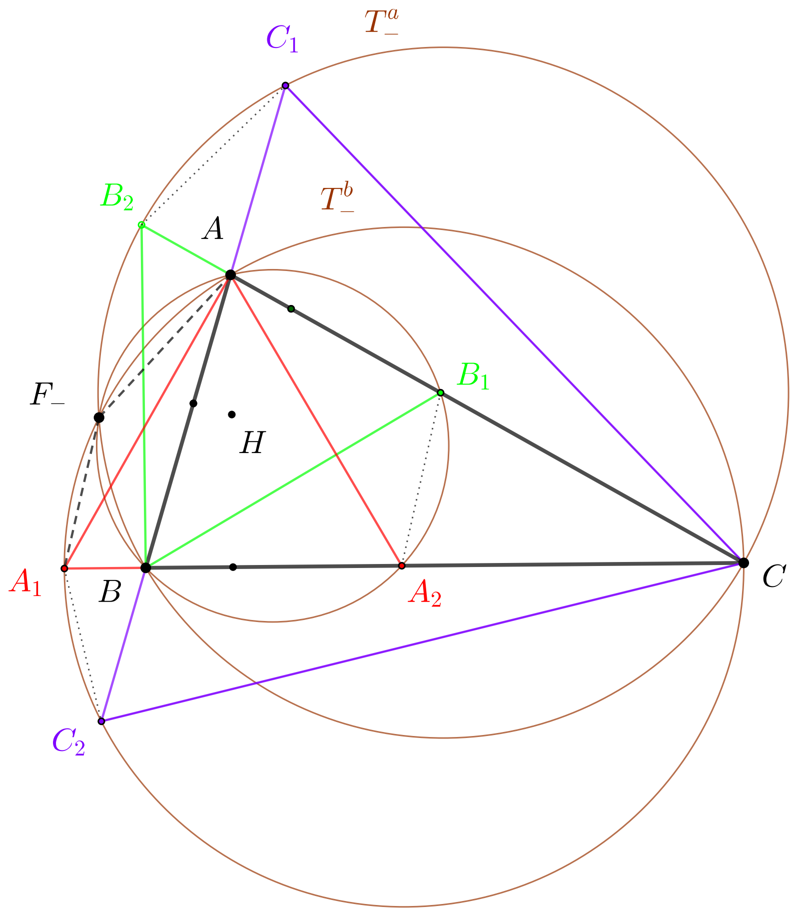


Fig. 13

On the other hand, the angle of rotation is, for example, the positive oriented angle $\angle(F_-A, F_-A_1) = 2\pi - \angle(F_-A_1, F_-A) = 2\pi - \widehat{AF_-A_1} = 2\pi - \left(\pi - \widehat{ACA_1}\right) = 2\pi - (\pi - C) = \pi + C$ (I used the fact that the qudrilateral $AF_-A_1C$ is inscribed in $\mathcal{T}_-^b$).
Now, it is easy to see that the triangle $A_1BC_2$ is the image of the triangle $AB_2C_1$ by the rotation $\mathcal{R}(F_-, \pi + C)$ followed by the homorhety $\mathcal{H}\left(F_-, \dfrac{r_+^b}{r_+^a}\right)$.
The remaining statements are shown in the same way. $\square$

**Proposition 4.6.** *The Fermat point $F_+$ is the center of rotation and homothety of the pair of triangles $(AB_1C_2, BC_1A_2)$, $(BC_1A_2, CA_1B_2)$, $(CA_1B_2, AB_1C_2)$. We have:*
*(i) $A_2BC_1 = \mathcal{H}(F_+, \dfrac{r_-^b}{r_-^a}) \cdot \mathcal{R}(F_+, C)(AB_1C_2)$ in case I and $A_2BC_1 = \mathcal{H}\left(F_+, \dfrac{r_-^b}{r_-^a}\right) \cdot \mathcal{R}(F_+, \pi + C)\ (AB_1C_2)$ in cases II and III;*

*(ii)* $B_2CA_1 = \mathcal{H}\left(F_+, \frac{r^c_-}{r^b_-}\right) \cdot \mathcal{R}(F_+, \pi + A)\,(BC_1A_2)$ *in all cases;*

*(iii)* $C_2AB_1 = \mathcal{H}\left(F_+, \frac{r^a_-}{r^c_-}\right) \cdot \mathcal{R}(F_+, B)\,(CA_1B_2)$ *in case I, and* $C_2AB_1 = \mathcal{H}\left(F_+, \frac{r^a_-}{r^c_-}\right) \cdot \mathcal{R}(F_+, \pi + B)\,(CA_1B_2)$ *in cases II and III.*

Now, let's find the formulas for the circumradii of the six triangles in these triads.

**Proposition 4.7.** *We have the following formulas:*

$$r^a_+ = \frac{2\sqrt{3}}{3} R\sin\left(A - \frac{\pi}{3}\right),\ r^b_+ = \varepsilon\frac{2\sqrt{3}}{3} R\sin\left(B - \frac{\pi}{3}\right),\ r^c_+ = -\frac{2\sqrt{3}}{3} R\sin\left(C - \frac{\pi}{3}\right)$$

*(where* $\varepsilon$ *is* $-1$ *in cases I, III and* $+1$ *in case II), and*

$$r^a_- = \eta\frac{2\sqrt{3}}{3} R\sin\left(A + \frac{\pi}{3}\right),\ r^b_- = \frac{2\sqrt{3}}{3} R\sin\left(B + \frac{\pi}{3}\right),\ r^c_- = \frac{2\sqrt{3}}{3} R\sin\left(C + \frac{\pi}{3}\right),$$

*(where* $\eta$ *is* $-1$ *in case I and* $+1$ *in cases II, III).*

*Proof.* Let us prove only the first formula. According to Proposition 4.1, the triangles $AB_2C_1$ and $ABC$ are similar. Hence, we have (Fig. 12):

$$r^a_+ = R\frac{AB_2}{AB}.$$

Let $E$ be the foot of the altitude from $B$ of the triangle $ABC$. Note that $B_2 - E - A$ in the case I and $B_2 - A - E$ in the cases II and III. So, in the case I, we have:

$$AB_2 = AE + EB_2 = c\cos(\pi - A) + h_b \tan\frac{\pi}{6} = \frac{2\sqrt{3}}{3} c\sin\left(A - \frac{\pi}{3}\right).$$

Therefore, in this case, we get:

$$r^a_+ = \frac{2\sqrt{3}}{3} R\sin\left(A - \frac{\pi}{3}\right)$$

the desired formula. In cases II and III we make a similar calculation.
The others formulas are shown in the same way. □

**Remark 4.3.** *The parameters* $\varepsilon$ *and* $\eta$ *agree with the respective arguments of the function* sin *and ensure the positivity of the expression in the above formulas.*

**Corollary 4.1.** *(i) The points* $H, A, O^a_+, O^a_-$ *are collinear.*
*(ii)* $HO^a_+ = r^a_-$, $HO^a_- = r^a_+$.
*(iii)* $AH = |r^a_+ - r^a_-|$.
*(iv)* $O^a_+, O^a_-$ *are isotomic points with respect to the segment* $AH$.
*Analogous results are valid relative to vertices Band C.*

*Proof.* (i) Since the triangles $AB_2C_1$ and $ABC$ are inversely similar, we have $\widehat{O^a_+AC_1} = \widehat{OAC}$. But $\widehat{OAC} = \widehat{HAB}$. Hence, $\widehat{O^a_+AC_1} = \widehat{HAB}$ and the points $O^a_+, A, H$ are collinear. Etc.
(ii) We will use Proposition 4.7 and the fact that $AH = 2R\,|\cos A|$. Let us show the first formula in the case III (Fig. 12). We have:

$$HO^a_+ = AO^a_+ + AH = r^a_+ + 2R\cos A$$

$$= \frac{2\sqrt{3}}{3}R\sin\left(A-\frac{\pi}{3}\right)+2R\cos A = \frac{2\sqrt{3}}{3}R\left[\sin\left(A-\frac{\pi}{3}\right)+\sqrt{3}\cos A\right]$$
$$= \frac{2\sqrt{3}}{3}R\sin\left(A+\frac{\pi}{3}\right) = r_-^a.$$

In cases I and II we adapt this calculation by taking into account the order of the points $A, H, O_+^a$ on the line $AH$.
(iii) and (iv) immediately follows from (i) and (ii). □

**Corollary 4.2.** *The formulas*

$$r_+^a = ON_-^a,\ r_-^a = ON_+^a$$

*and their analogues hold.*

*Proof.* Let $A'$ be the midpoint of the segment $BC$. It is known that $OA' = R\,|\cos A|$. On the other hand, $A'N_+^a = A'N_-^a = \frac{1}{3}A_+A' = \frac{1}{3}\cdot\frac{\sqrt{3}}{2}a = \frac{\sqrt{3}}{3}R\sin A$. With these preparations, it is easy to calculate $ON_+^a$ and $ON_-^a$ in each of the cases I-III. The required result is obtained. □

## 5. Radical axes and centers

We will study from this point of view the circumcircles of the triangles in the two triads, i.e. the circles $(O_+^a), (O_+^b), (O_+^c); (O_-^a), (O_-^b), (O_-^c)$. We will see that the Fermat points $F_+$ and $F_-$ have un important role.
We need some preparations. First, let's introduce a few points: $B' := AA_1 \cap (O_+^a), C'' := AA_2 \cap (O_+^a)$, and, ciclycally, the points $C', A''$ on $(O_+^b)$, and $A', B''$ on $(O_+^c)$ (Fig. 14).

The following lemmas are easy to prove.

**Lemma 5.1.** *(i) $B'C''\|BC$, $C'A''\|CA$, $A'B''\|AB$.*
*(ii) The triangles $AB'C''$, $BC'A''$, $CA'B''$ are equilateral.*

**Lemma 5.2.** *The points in the systems: 1) $A_-$, $B_+$, $C_1$, $C''$, 2) $A_-$, $B_2$, $B'$, $C_+$, and their analogues relative to the vertices B and C, are collinear.*

**Proposition 5.1.** *The following statement are true:*
*(i) the radical axis of the circles $(O_+^b)$ and $(O_+^c)$ is the Fermat cevian $AF_+$ (or line $AA_+$);*
*(ii) the radical axis of the circles $(O_-^b)$ and $(O_-^c)$ is the Fermat cevian $AF_-$ (or line $AA_-$);*
*(iii) the radical axis of the circles $(O_+^a)$ and $(O_-^a)$ is the parallel through vertex A to the sideline BC;*
*(iv) the radical axis of the circles $(O_+^b)$ and $(O_-^c)$ is the line $A_1C_+$,*
*as well as similar ones (Fig. 14).*

*Proof.* (i) Since quadrilateral $BC_2CB_1$ is cyclic, we have $AB\cdot AC_2 = AB_1\cdot AC$, and so $A$ is on the radical axis of the circles $(O_+^b)$ and $(O_+^c)$. On the other hand, the quadrilaterals $BA_1C'C_2$ and $CB_1A_2B''$ are cyclic and therefore $\widehat{BC'C_2} = \widehat{BA_1C_2}$, $\widehat{CB''B_1} = \widehat{CA_2B_1}$. But, according to Proposition 4.1, $ABC \sim A_1BC_2 \sim A_2B_1C$, and therefore $\widehat{BA_1C_2} = \widehat{B_1A_2C} = A$. Combining the previous relations, we obtain that $\widehat{BC'C_2} = \widehat{CB''B_1} = A$. So, $B_1B''C_2C'$ is a cyclic quadrilateral. It follows that $A_+C_2\cdot A_+C' = A_+B''\cdot A_+B_1$, i.e. $A_+$ is on the

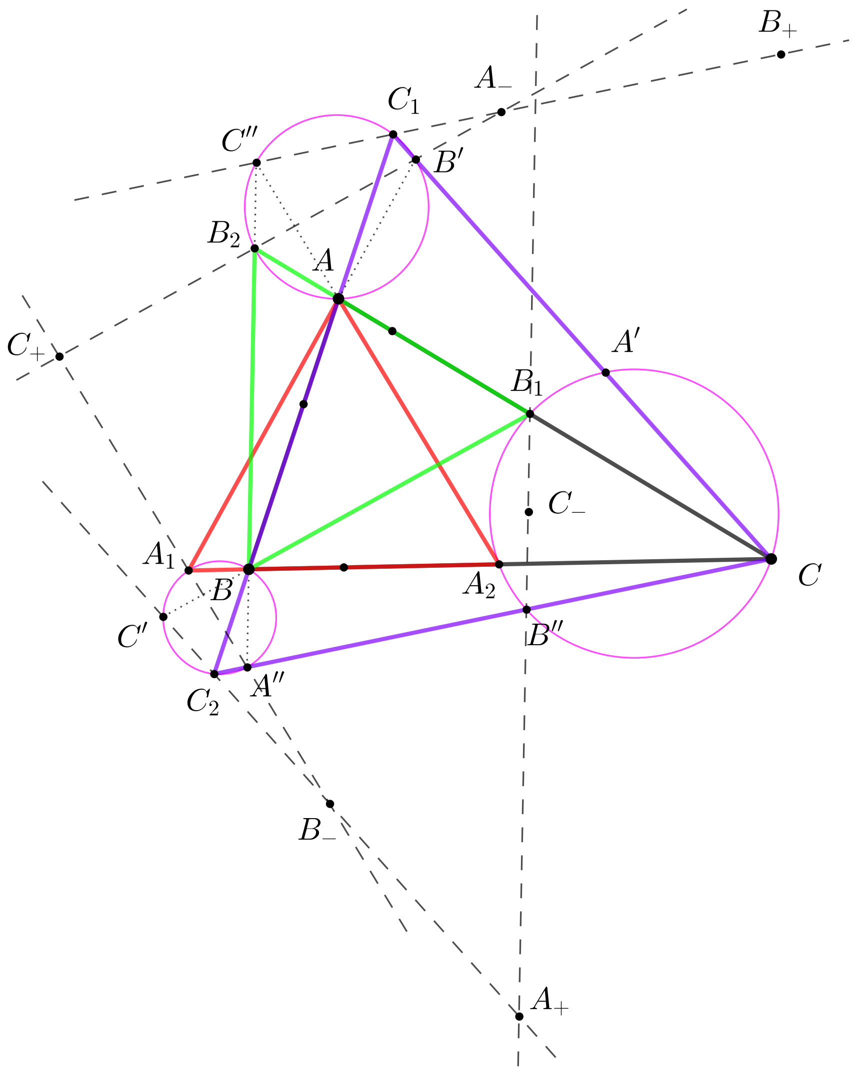


Fig. 14

radical axis of the circles $(O^b_+)$ and $(O^c_+)$. In the end, $AA_+$ is the radical axis of the circles $(O^b_+)$ and $(O^c_+)$.
(ii) It is proved in the same way as for (i).
(iii) It is easy to see that the centers of the circles $(O^a_+)$ and $(O^a_-)$ lie on the perpendicular through $A$ to $BC$. It follows that these circles are tangent at $A$. Their radical axis is the common tangent.
(iv) Obviously, the point $A_1$ lies on both circles $(O^b_+)$ and $(O^c_-)$, and $A'' \in (O^b_+)$. Let us show that we have and $A'' \in (O^c_-)$. For this purpose, it is sufficient to show that the quadrilateral $B_2A_1A''C$ is cyclic. But it is easy to see that we have $\widehat{A''B_2C} = \widehat{A''A_1C} = \dfrac{\pi}{3}$.

So, the common chord $A''A_1$ is the radical axis of the two circles. It remains to observe that, by Lemma 5.2, the lines $A''A_1$ and $A_1C_+$ coincide. □

The following proposition is a direct consequence of Proposition 5.1.

**Proposition 5.2.** *The following statement are true:*
*(i) the radical center of the circles $(O^a_+), (O^b_+), (O^c_+)$ is the Fermat point $F_+$;*
*(ii) the radical center of the circles $(O^a_-), (O^b_-), (O^c_-)$ is the Fermat point $F_-$;*
*(iii) the radical center of the circles $(O^a_+), (O^b_+), (O^c_-)$ is the point $C_+$;*
*(iv) the radical center of the circles $(O^a_+), (O^b_-), (O^c_-)$ is the point $A_-$,*
*as well as similar ones.*

## 6. Trilinear coordinates of $H_+$ and $H_-$

We choose triangle ABC as the reference triangle. The barycentric coordinates of the vertices of the orthocentroidal triangle are $O_a(a^2, a^2+b^2-c^2, a^2-b^2+c^2)$, $O_b(a^2+b^2-c^2, b^2, -a^2+b^2+c^2)$, $O_c(a^2-b^2+c^2, -a^2+b^2+c^2, c^2)$ ([9, #X(5476)]) and can be easily found by the reader. So, using the cosine formula, their trilinear coordinates are

$$O_a\left(1, 2\cos C, 2\cos B\right),\ O_b\left(2\cos C, 1, 2\cos A\right),\ O_c\left(2\cos B, 2\cos A, 1\right).$$

Let us now find the trilinear coordinates of the points $A_1, A_2$; $B_1, B_2$; $C_1, C_2$ (the vertices of the equilateral triangles $AA_1A_2$, $BB_1B_2$, $CC_1C_2$ other than $A, B, C$). Thus, for $A_1$ we have (Fig. 1b): $d\left(A_1, BC\right) = 0$, $d\left(A_1, CA\right) = A_1C\sin C = \dfrac{AA_1\sin\widehat{A_1AC}}{\sin C}\sin C = AA_1\sin\widehat{A_1AC} = AA_1\sin\left(C+\dfrac{\pi}{3}\right)$, and $d\left(A_1, AB\right) = AA_1\sin\widehat{A_1AB} = AA_1\sin\left(B-\dfrac{\pi}{3}\right)$. So, we get $A_1\left(0, \sin\left(C+\dfrac{\pi}{3}\right),\ -\sin\left(B-\dfrac{\pi}{3}\right)\right)$. In the previous calculations, the following order on the sideline $BC$ was assumed: $A_1 - B - A_2 - C$. We mention that this result is valid and in the other possible positions of $A_1, A_2$ on $BC$: $A_1 - A_2 - B - C$, $B - A_1 - A_2 - C$, $A_1 - B - A_2 - C$, $B - C - A_1 - A_2$.
Similarly, the trilinear coordinates of $A_2$ and the other remaining vertices are obtained. Finally, we have:

$$\begin{aligned}
&A_1\left(0, \sin\left(C+\frac{\pi}{3}\right), -\sin\left(B-\frac{\pi}{3}\right)\right), \quad A_2\left(0, -\sin\left(C-\frac{\pi}{3}\right), \sin\left(B+\frac{\pi}{3}\right)\right),\\
&B_1\left(-\sin\left(C-\frac{\pi}{3}\right), 0, \sin\left(A+\frac{\pi}{3}\right)\right), \quad B_2\left(\sin\left(C+\frac{\pi}{3}\right), 0, -\sin\left(A-\frac{\pi}{3}\right)\right),\\
&C_1\left(\sin\left(B+\frac{\pi}{3}\right), -\sin\left(A-\frac{\pi}{3}\right), 0\right), \quad C_2\left(-\sin\left(B-\frac{\pi}{3}\right), \sin\left(A+\frac{\pi}{3}\right), 0\right).
\end{aligned}$$

Now, the equations of the lines $A_1O_a, B_1O_b, C_1O_c$ are given by

$$\begin{aligned}
(A_1O_a)&\quad -\alpha\sin A + \beta\sin\left(B-\frac{\pi}{3}\right) + \gamma\sin\left(C+\frac{\pi}{3}\right) &= 0,\\
(B_1O_b)&\quad \alpha\sin\left(A+\frac{\pi}{3}\right) - \beta\sin B + \gamma\sin\left(C-\frac{\pi}{3}\right) &= 0,\\
(C_1O_c)&\quad \alpha\sin\left(A-\frac{\pi}{3}\right) + \beta\sin\left(B+\frac{\pi}{3}\right) - \gamma\sin C &= 0.
\end{aligned}$$

Similarly, the lines $A_2O_a, B_2O_b, C_2O_c$ have the equations:

$$\begin{array}{lrcl} (A_2O_a) & -\alpha\sin A+\beta\sin\left(B+\dfrac{\pi}{3}\right)+\gamma\sin\left(C-\dfrac{\pi}{3}\right) & = & 0,\\ (B_2O_b) & \alpha\sin\left(A-\dfrac{\pi}{3}\right)-\beta\sin B+\gamma\sin\left(C+\dfrac{\pi}{3}\right) & = & 0,\\ (C_2O_c) & \alpha\sin\left(A+\dfrac{\pi}{3}\right)+\beta\sin\left(B-\dfrac{\pi}{3}\right)-\gamma\sin C & = & 0. \end{array}$$

**Remark 6.1.** *Using trilinear coordinates, the proof of Propositions 2.2 and 3.2 returns to a routine calculation.*

**Proposition 6.1.** *The points $H_+$ and $H_-$ have the following trilinear coordinates:*

$$H_+\left(\cos\left(B-C-\frac{\pi}{6}\right),\cos\left(C-A-\frac{\pi}{6}\right),\cos\left(A-B-\frac{\pi}{6}\right)\right),$$

$$H_-\left(\cos\left(B-C+\frac{\pi}{6}\right),\cos\left(C-A+\frac{\pi}{6}\right),\cos\left(A-B+\frac{\pi}{6}\right)\right).$$

*Proof.* Taking into account that $H_+ = A_1O_a \cap B_1O_b$, the trilinear coordinates of $H_1$ are the second-order minors of the matrix

$$\begin{bmatrix} -\sin A & \sin\left(B-\frac{\pi}{3}\right) & \sin\left(C+\frac{\pi}{3}\right)\\ \sin\left(A+\frac{\pi}{3}\right) & -\sin B & \sin\left(C-\frac{\pi}{3}\right) \end{bmatrix},$$

that is

$$H_+\left(\begin{vmatrix} \sin\left(B-\frac{\pi}{3}\right) & \sin\left(C+\frac{\pi}{3}\right)\\ -\sin B & \sin\left(C-\frac{\pi}{3}\right) \end{vmatrix},\begin{vmatrix} \sin\left(C+\frac{\pi}{3}\right) & -\sin A\\ \sin\left(C-\frac{\pi}{3}\right) & \sin\left(A+\frac{\pi}{3}\right) \end{vmatrix},\right.$$
$$\left.\begin{vmatrix} -\sin A & \sin\left(B-\frac{\pi}{3}\right)\\ \sin\left(A+\frac{\pi}{3}\right) & -\sin B \end{vmatrix}\right).$$

Using the formula $2\sin x\sin y = \cos(x-y)-\cos(x+y)$, we finally get the required result.
The trilinear coordinates of $H_-$ are calculated similarly. □

**Remark 6.2.** *The trilinear coordinates of the points $H_+$ and $H_-$ can be written in the form*

$$H_+\left(\sin\left(B-C+\frac{\pi}{3}\right),\sin\left(C-A+\frac{\pi}{3}\right),\sin\left(A-B+\frac{\pi}{3}\right)\right),$$

$$H_-\left(\sin\left(B-C-\frac{\pi}{3}\right),\sin\left(C-A-\frac{\pi}{3}\right),\sin\left(A-B-\frac{\pi}{3}\right)\right).$$

**Remark 6.3.** *$H_+$ and $H_-$ are central points which apparently does not appear in Kimberling's list* [9]. *A decision in this regard is necessary.*

**Proposition 6.2.** *The line $H_+H_-$ has the equation*

$$\alpha\sin 3A+\beta\sin 3B+\gamma\sin 3C=0.$$

*Proof.* By direct calculation. □

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