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A GEOMETRY INEQUALITY WITH ONE PARAMETER IN ACUTE TRIANGLES

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ABSTRACT. With the help of Maple software, we prove a new geometry inequality with one parameter in acute triangles. We also propose several related conjectures as open problems after have been verified by computer.

1. INTRODUCTION AND MAIN RESULT

Let ABC be a triangle with side lengths a, b, c. Denote by R, r, s and S its circumradius, inradius, semiperimeter and area respectively; m_a, m_b, m_c the medians; h_a, h_b, h_c the altitudes; w_a, w_b, w_c the angle-bisectors, r_a, r_b, r_c the radii of excircles. In addition, we denote $\sum$ by cyclic sums.

We have known that some triangle inequalities can be generalized to the case with one parameter. For example, Klamkin generalized the following inequality

$$\sum \frac{a}{b+c-a} \geq 3 \tag{1.1}$$

to

$$\sum \frac{a}{k(b+c)-a} \geq 3, \tag{1.2}$$

where k is a real number such that $k \geq 1$ (see [1, p.148]). For another example, the author recently found that the following known inequality (see [2, inequality 6.21]):

$$\sum \frac{1}{h_a - 2r} \geq \frac{3}{r} \tag{1.3}$$

has the following generalization

$$\sum \frac{1}{h_a + kr} \geq \frac{3}{(k+3)r}, \tag{1.4}$$

where $-2 \leq k < 0$. And the inequality reversely holds when $k > 0$ (we omit the proof here).

Many years ago, with the help of the computer the author found that the following inequality

$$\sum \frac{m_a + h_a}{r_a + w_a} \leq 3 \tag{1.5}$$

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probably holds for the acute triangle ABC. Recently, the author studied this inequality again and found that it can be generalized to the case with one parameter. Specifically, we have the following conclusion:

Theorem 1.1. *Let ABC be an acute triangle and let $k \geq 1$ be a real number. Then*

$$\sum \frac{m_a + kh_a}{r_a + kw_a} \leq 3, \tag{1.6}$$

with equality if and only if the acute triangle ABC is equilateral.

The main purpose of this paper is to prove the above theorem. We also propose several related conjectures as open problems in the last section.

2. Lemmas

In order prove Theorem 1.1, we shall use the following lemmas.

Lemma 2.1. *In any triangle ABC the following inequality holds:*

$$m_a \leq \frac{8S^2 + bc(b-c)^2}{4aS}, \tag{2.1}$$

with equality if and only if $b = c$ or $A = \pi/2$.

Inequality is one of the equivalent form of Theorem 1.1 from [3].

Lemma 2.2. *In any triangle ABC, let*

$$\begin{aligned}
N_1 =&(b^2 + 6bc + c^2)a - (b+c)(b-c)^2,\\
N_2 =&(c^2 + 6ca + a^2)b - (c+a)(c-a)^2,\\
N_3 =&(a^2 + 6ab + b^2)c - (a+b)(a-b)^2,\\
M_1 =&(b^2 + 6bc + c^2)a^2 - (b+c)^2(b-c)^2,\\
M_2 =&(c^2 + 6ca + a^2)b^2 - (c+a)^2(c-a)^2,\\
M_3 =&(a^2 + 6ab + b^2)c^2 - (a+b)^2(a-b)^2.
\end{aligned}$$

Then

$$w_a \geq \frac{2SN_1}{M_1},\ w_b \geq \frac{2SN_2}{M_2},\ w_c \geq \frac{2SN_3}{M_3}, \tag{2.2}$$

where $M_i, N_i > 0 (i = 1, 2, 3)$.

Proof. First, it is easy to obtain the following identity:

$$w_a = \frac{r}{\sin\dfrac{A}{2}} + \frac{r}{\cos\dfrac{B-C}{2}}. \tag{2.3}$$

So, we have

$$w_a \geq r + \frac{r}{\sin\dfrac{A}{2}}. \tag{2.4}$$

Again, by the well-known formula

$$\sin\frac{A}{2} = \sqrt{\frac{(s-b)(s-c)}{bc}} \tag{2.5}$$

and the simplest arithmetic and geometric mean inequality, we get

$$2\sin\frac{A}{2} \le \frac{a}{b+c} + \frac{(b+c)(s-b)(s-c)}{abc}. \tag{2.6}$$

Hence

$$\begin{aligned}1+\frac{1}{\sin\frac{A}{2}} &\ge 1+\frac{2abc(b+c)}{bca^2+(s-b)(s-c)(b+c)^2}\\ &= \frac{4bca^2+(c+a-b)(a+b-c)(b+c)^2+8abc(b+c)}{4bca^2+(c+a-b)(a+b-c)(b+c)^2}.\end{aligned}$$

And, it is easy to verify the following two identities:

$$4bca^2+(c+a-b)(a+b-c)(b+c)^2 = M_1 \tag{2.7}$$

and

$$4bca^2+(c+a-b)(a+b-c)(b+c)^2+8abc(b+c) = 2sN_1. \tag{2.8}$$

So, we have

$$1+\frac{1}{\sin\frac{A}{2}} \ge \frac{2sN_1}{M_1}. \tag{2.9}$$

Four similar relations are also valid. Note that (2.4) and $S = rs$, we immediately deduce that

$$w_a \ge \frac{2SN_1}{M_1}$$

Similarly, one can obtain two inequalities for w_b and w_c. From (2.7) and (2.8), one sees that $M_1 > 0$ and $N_1 > 0$. The other four similar inequalities are also valid. The proof of Lemma 2.2 is complete. □

Lemma 2.3. *In any triangle ABC the following identities hold:*

$$\sum a^2 = 2s^2 - 8Rr - 2r^2, \tag{2.10}$$

$$\sum a^3 = 2s^3 - (12Rr + 6r^2)s, \tag{2.11}$$

$$\sum a^4 = 2s^4 - 4(4R+3r)rs^2 + 2(4R+r)^2r^2, \tag{2.12}$$

$$\sum a^5 = 2s^5 - 20(R+r)rs^3 + 10(2R+r)(4R+r)r^2s, \tag{2.13}$$

$$\begin{aligned}\sum a^6 = &2s^6 - 6(4R+5r)rs^4 + 6(24R^2+24Rr+5r^2)r^2s^2\\ &- 2(4R+r)^3r^3,\end{aligned} \tag{2.14}$$

$$\begin{aligned}\sum a^7 = &2s^7 - 14(2R+3r)rs^5 + 14(16R^2+20Rr+5r^2)r^2s^3\\ &- 14(2R+r)(4R+r)^2r^3s,\end{aligned} \tag{2.15}$$

$$\begin{aligned}\sum a^8 = &2s^8 - 8(4R+7r)rs^6 + 20(16R^2+24Rr+7r^2)r^2s^4\\ &- 8(4R+r)(32R^2+32Rr+7r^2)r^3s^2 + 2(4R+r)^4r^4,\end{aligned} \tag{2.16}$$

$$\begin{aligned}\sum a^9 =&2s^9 - 36(R+2r)rs^7 + 36(12R^2+21Rr+7r^2)r^2s^5\\ &- 12(160R^3+240R^2r+105Rr^2+14r^3)r^3s^3\\ &+ 18(2R+r)(4R+r)^3r^4s, \qquad (2.17)\end{aligned}$$

$$\begin{aligned}\sum a^{10} =&2s^{10} - 10(4R+9r)rs^8 + 140(2R+3r)(2R+r)r^2s^6\\ &- 20(160R^3+280R^2r+140Rr^2+21r^3)r^3s^4\\ &+ 10(40R^2+40Rr+9r^2)(4R+r)^2r^4s^2 - 2(4R+r)^5r^5, \qquad (2.18)\end{aligned}$$

$$\begin{aligned}\sum a^{11} =&2s^{11} - 22(2R+5r)rs^9 + 44(16R^2+36Rr+15r^2)r^2s^7\\ &- 308(2R+r)(8R^2+12Rr+3r^2)r^3s^5\\ &+ 22(4R+r)(160R^3+240R^2r+108Rr^2\\ &+ 15r^3)r^4s^3 - 22(2R+r)(4R+r)^4r^5s, \qquad (2.19)\end{aligned}$$

$$\begin{aligned}\sum a^{12} =&2s^{12} - 12(4R+11r)rs^{10} + 18(48R^2+120Rr+55r^2)r^2s^8\\ &- 56(128R^3+288R^2r+180Rr^2+33r^3)r^3s^6 + 6(4480R^4\\ &+ 8960R^3r+6048R^2r^2+1680Rr^3+165r^4)r^4s^4\\ &- 12(48R^2+48Rr+11r^2)(4R+r)^3r^5s^2 + 2(4R+r)^6r^6, \qquad (2.20)\end{aligned}$$

$$\begin{aligned}\sum a^{13} =&2s^{13} - 52(R+3r)rs^{11} + 130(8R^2+22Rr+11r^2)r^2s^9\\ &- 312(32R^3+80R^2r+55Rr^2+11r^3)r^3s^7 + 26(1792R^4\\ &+ 4032R^3r+3024R^2r^2+924Rr^3+99r^4)r^4s^5\\ &- 52(112R^3+168R^2r+77Rr^2+11r^3)(4R+r)^2r^5s^3\\ &+ 26(2R+r)(4R+r)^5r^6s, \qquad (2.21)\end{aligned}$$

$$\begin{aligned}\sum a^{14} =&2s^{14} - 14(4R+13r)rs^{12} + 154(8R^2+24Rr+13r^2)r^2s^{10}\\ &- 42(320R^3+880R^2r+660Rr^2+143r^3)r^3s^8\\ &+ 42(1792R^4+4480R^3r+3696R^2r^2+1232Rr^3\\ &+ 143r^4)r^4s^6 - 14(4R+r)(3584R^4+7168R^3r\\ &+ 4928R^2r^2+1408Rr^3+143r^4)r^5s^4 + 14(56R^2\\ &+ 56Rr+13r^2)(4R+r)^4r^6s^2 - 2(4R+r)^7r^7, \qquad (2.22)\end{aligned}$$

$$\begin{aligned}\sum a^{15} =&2s^{15} - 30(2R+7r)rs^{13} + 30(48R^2+156Rr+91r^2)r^2s^{11}\\ &- 110(160R^3+480R^2r+390Rr^2+91r^3)r^3s^9\\ &+ 90(1280R^4+3520R^3r+3168R^2r^2+1144Rr^3\\ &+ 143r^4)r^4s^7 - 6(64512R^5+161280R^4r+147840R^3r^2\\ &+ 63360R^2r^3+12870Rr^4+1001r^5)r^5s^5 + 10(896R^3\\ &+ 1344R^2r+624Rr^2+91r^3)(4R+r)^3r^6s^3\\ &- 30(2R+r)(4R+r)^6r^7s, \qquad (2.23)\end{aligned}$$

$$\begin{aligned}\sum a^{16} =&2s^{16} - 16(4R+15r)rs^{14} + 104(16R^2+56Rr+35r^2)r^2s^{12}\\ &- 176(128R^3+416R^2r+364Rr^2+91r^3)r^3s^{10}\\ &+ 132(1280R^4+3840R^3r+3744R^2r^2+1456Rr^3\\ &+ 195r^4)r^4s^8 - 16(43008R^5+118272R^4r+118272R^3r^2\\ &+ 54912R^2r^3+12012Rr^4+1001r^5)r^5s^6+8(10752R^4\\ &+ 21504R^3r+14976R^2r^2+4368Rr^3+455r^4)(4R+r)^2r^6s^4\\ &- 16(8R+5r)(8R+3r)(4R+r)^5r^7s^2+2(4R+r)^8r^8. \qquad (2.24)\end{aligned}$$

Proof. Identities (2.10)-(2.12) are familiar (see [1]). The proofs of identities (2.13)-(2.22) can be found in [4] and [5]. So, we only need to prove (2.23) and (2.24) here.
It is easy to get that

$$\sum a^{15} = \sum a^7 \sum a^8 + abc \sum b^6c^6 - \sum a \sum b^7c^7. \qquad (2.25)$$

With the help of Maple software, using $\sum a = 2s$, (2.15), (2.16), (2.33) and (2.34) below and the following known identity

$$abc = 4Rrs, \qquad (2.26)$$

we immediately obtain identity (2.23). In addition, we easily get

$$\sum a^{16} = \left(\sum a^8\right)^2 - 2\left(\sum b^4c^4\right)^2 + 4(abc)^4 \sum a^4. \qquad (2.27)$$

Then using identities (2.12), (2.16), (2.26) and (2.31) below, we immediately obtain (2.24). Lemma 2.3 is proved. □

Lemma 2.4. *In any triangle ABC the following identities hold:*

$$\sum bc = s^2+4Rr+r^2, \qquad (2.28)$$

$$\sum b^2c^2 = s^4 - 2(4R-r)rs^2 + (4R+r)^2r^2, \qquad (2.29)$$

$$\sum b^3c^3 = s^6 - 3(4R-r)rs^4 + 3r^4s^2 + (4R+r)^3r^3, \qquad (2.30)$$

$$\begin{aligned}\sum b^4c^4 =& s^8 - 4(4R-r)rs^6 + 2(16R^2-8Rr+3r^2)r^2s^4\\ &+ 4(4R+r)r^5s^2 + (4R+r)^4r^4, \qquad (2.31)\end{aligned}$$

$$\begin{aligned}\sum b^5c^5 =& s^{10} - 5(4R-r)rs^8 + 10(8R^2-4Rr+r^2)r^2s^6\\ &+ 10r^6s^4 + 5(4R+r)^2r^6s^2 + (4R+r)^5r^5, \qquad (2.32)\end{aligned}$$

$$\begin{aligned}\sum b^6c^6 =& s^{12} - 6(4R-r)rs^{10} + 3(48R^2-24Rr+5r^2)r^2s^8\\ &- 4(32R^3-24R^2r+12Rr^2-5r^3)r^3s^6\\ &+ 3(16R+5r)r^7s^4 + 6(4R+r)^3r^7s^2 + (4R+r)^6r^6, \qquad (2.33)\end{aligned}$$

$$\begin{aligned}\sum b^7c^7 =& s^{14} - 7(4R-r)rs^{12} + 7(32R^2 - 16Rr + 3r^2)r^2s^{10}\\ &- 7(64R^3 - 48R^2r + 20Rr^2 - 5r^3)r^3s^8 + 35r^8s^6\\ &+ 7(8R+3r)(4R+r)r^8s^4 + 7(4R+r)^4r^8s^2 + (4R+r)^7r^7, \end{aligned} \tag{2.34}$$

$$\begin{aligned}\sum b^8c^8 =& s^{16} - 8(4R-r)rs^{14} + 4(80R^2 - 40Rr + 7r^2)r^2s^{12}\\ &- 8(128R^3 - 96R^2r + 36Rr^2 - 7r^3)r^3s^{10}\\ &+ 2(256R^4 - 256R^3r + 160R^2r^2 - 80Rr^3 + 35r^4)r^4s^8\\ &+ 8(20R+7r)r^9s^6 + 4(16R+7r)(4R+r)^2r^9s^4\\ &+ 8(4R+r)^5r^9s^2 + (4R+r)^8r^8 \end{aligned} \tag{2.35}$$

Proof. Both identities (2.28) and (2.29) are known (see [1]). Identities (2.30)-(2.34) have been proved in [4] and [5]. It remains to show (2.35). Since

$$\sum b^8c^8 = \left(\sum b^4c^4\right)^2 - 2(abc)^4\sum a^4,$$

identity (2.35) follows easily by using (2.12), (2.26) and (2.31). Lemma 2.4 is proved. □

Lemma 2.5. *In the acute triangle ABC the following inequality holds:*

$$s^2 \geq 4R^2 - Rr + 13r^2 + \frac{(R-2r)r^3}{R^2}, \tag{2.36}$$

with equality if and only if $\triangle ABC$ is equilateral or isosceles.

Inequality (2.36) was obtained by the author in [6].

3. Proof of Theorem 1.1

In this section, we prove Theorem 1.1.

Proof. Since $k \geq 1$, we may assume that $k = 1 + t$ $(t \geq 0)$. Then inequality (1.6) becomes

$$\sum \frac{m_a + (1+t)h_a}{r_a + (1+t)w_a} \leq 3. \tag{3.1}$$

By Lemma 2.1 and 2.2, we have

$$\frac{m_a + (1+t)h_a}{r_a + (1+t)w_a} \leq \frac{\dfrac{8S^2 + bc(b-c)^2}{4aS} + (1+t)h_a}{r_a + (1+t)\dfrac{2SN_1}{M_1}}.$$

Then using the well-known formulas:

$$h_a = \frac{2S}{a}, \tag{3.2}$$

$$r_a = \frac{S}{s-a}, \tag{3.3}$$

we obtain

$$\frac{m_a + (1+t)h_a}{r_a + (1+t)w_a} \leq \frac{F_1}{4E_1S^2}, \tag{3.4}$$

where

$$\begin{aligned} E_1 =& a\left[M_1 + 2(s-a)(1+t)N_1\right], \\ F_1 =& (s-a)\left[8(2+t)S^2 + bc(b-c)^2\right]M_1. \end{aligned}$$

Similarly, we have

$$\frac{m_b + (1+t)h_b}{r_b + (1+t)w_b} \le \frac{F_2}{4E_2S^2}, \tag{3.5}$$

$$\frac{m_c + (1+t)h_c}{r_c + (1+t)w_c} \le \frac{F_3}{4E_3S^2}, \tag{3.6}$$

where

$$\begin{aligned} E_2 =& b\left[M_2 + 2(s-b)(1+t)N_2\right], \\ F_2 =& (s-b)\left[8(2+t)S^2 + ca(c-a)^2\right]M_2, \\ E_3 =& c\left[M_3 + 2(s-c)(1+t)N_3\right], \\ F_3 =& (s-c)\left[8(2+t)S^2 + ab(a-b)^2\right]M_3. \end{aligned}$$

Adding (3.4), (3.5) and (3.6) gives

$$\sum \frac{m_a + (1+t)h_a}{r_a + (1+t)w_a} \le \frac{1}{4S^2}\left(\frac{F_1}{E_1} + \frac{F_2}{E_2} + \frac{F_3}{E_3}\right). \tag{3.7}$$

Thus, to prove inequality (3.1) we only need to prove that

$$Q_0 \equiv 12S^2E_1E_2E_3 - (F_1E_2E_3 + F_2E_3E_1 + F_3E_1E_2) \ge 0. \tag{3.8}$$

We set $d = abc$. With the help of software Maple, using $s = (a+b+c)/2$ and Heron's formula:

$$S = \sqrt{s(s-a)(s-b)(s-c)} \tag{3.9}$$

we obtain the following complex identity (which can be verified by expanding):

$$4Q_0 = m_0x_3t^3 + n_0x_2t^2 + x_1t + x_0, \tag{3.10}$$

where

$$\begin{aligned} m_0 =& (a+b+c)(b+c-a)^2(c+a-b)^2(a+b-c)^2, \\ n_0 =& 2(b+c-a)(c+a-b)(a+b-c), \\ x_3 =& 120d^4 + \left(14\sum a\sum a^2 - 72\sum a^3\right)d^3 \\ &+ \left(21\sum a^6 + 35\sum a\sum a^5 - 40\sum b^3c^3 - 32\sum a^2\sum a^4\right)d^2 \\ &+ \left(10\sum a^2\sum a^7 + 12\sum a^3\sum a^6 - 15\sum a^4\sum a^5 - 7\sum a\sum a^8\right)d \\ &+ \sum a^5\sum a^7 - \sum a^3\sum a^9 - 4\sum b^6c^6 + 2\sum a^4\sum a^8 - 2\sum a^{12}, \end{aligned}$$

$$\begin{aligned}
x_2 =&790\left(\sum a\right) d^5 + \left(293\sum a^4 - 435\sum a\sum a^3 + 126\sum b^2c^2\right) d^4 \\
&+ \left(208\sum a^7 - 245\sum a^4\sum a^3 - 157\sum a^2\sum a^5 + 358\sum a\sum a^6\right) d^3 \\
&+ \left(192\sum a^2\sum a^8 + 178\sum a^3\sum a^7 - 154\sum b^5c^5 - 99\sum a\sum a^9\right. \\
&\left.-159\sum a^{10} - 160\sum a^4\sum a^6\right) d^2 + \left(58\sum a^4\sum a^9 - 50\sum a^6\sum a^7\right. \\
&\left.-31\sum a^3\sum a^{10} - 23\sum a^2\sum a^{11} + 14\sum a\sum a^{12} + 32\sum a^5\sum a^8\right) d \\
&- 7\sum a^{16} - 14\sum b^8c^8 + 2\sum a^3\sum a^{13} - 5\sum a^4\sum a^{12} - 2\sum a^5\sum a^{11} \\
&+ 12\sum a^6\sum a^{10}, \\
x_1 =&- 144\left(\sum a\right) d^6 + \left(1248\sum b^2c^2 - 2728\sum a^4 + 424\sum a\sum a^3\right) d^5 \\
&+ \left(92\sum a^3\sum a^4 - 1428\sum a^2\sum a^5 + 592\sum a\sum a^6 + 2408\sum a^7\right) d^4 \\
&- \left(1892\sum a^{10} + 304\sum a\sum a^9 + 2072\sum b^5c^5 + 240\sum a^4\sum a^6\right. \\
&\left.-880\sum a^2\sum a^8 - 1076\sum a^3\sum a^7\right) d^3 + \left(276\sum a^{13} + 12\sum a\sum a^{12}\right. \\
&-608\sum a^2\sum a^{11} + 24\sum a^3\sum a^{10} + 1084\sum a^4\sum a^9 + 96\sum a^5\sum a^8 \\
&\left.-732\sum a^6\sum a^7\right) d^2 + \left(96\sum a^2\sum a^{14} + 160\sum a^6\sum a^{10}\right. \\
&+188\sum a^5\sum a^{11} - 64\sum b^8c^8 - 224\sum a^4\sum a^{12} - 28\sum a^3\sum a^{13} \\
&\left.-132\sum a^7\sum a^9 - 32\sum a^{16} - 28\sum a\sum a^{15}\right) d + 60\sum a^7\sum a^{12} \\
&- 4\sum a^3\sum a^{16} - 16\sum a^5\sum a^{14} + 52\sum a^8\sum a^{11} - 48\sum a^6\sum a^{13} \\
&- 64\sum a^9\sum a^{10} + 20\sum a^4\sum a^{15}, \\
x_0 =&- 400\left(\sum a\right) d^6 + \left(304\sum a\sum a^3 - 912\sum a^4 + 304\sum b^2c^2\right) d^5 \\
&+ \left(48\sum a^7 - 8\sum a\sum a^6 - 72\sum a^2\sum a^5 + 64\sum a^3\sum a^4\right) d^4 \\
&+ \left(192\sum a^2\sum a^8 - 72\sum a^3\sum a^7 - 264\sum a^{10} - 144\sum b^5c^5\right. \\
&\left.+8\sum a^4\sum a^6 + 40\sum a\sum a^9\right) d^3 + \left(120\sum a^4\sum a^9 - 8\sum a^2\sum a^{11}\right. \\
&\left.-128\sum a^6\sum a^7 + 8\sum a^3\sum a^{10} - 24\sum a\sum a^{12} + 32\sum a^5\sum a^8\right) d^2 \\
&+ \left(56\sum a^{16} + 24\sum a^2\sum a^{14} - 24\sum a^4\sum a^{12} + 112\sum b^8c^8\right. \\
&\left.-80\sum a^7\sum a^9 - 56\sum a^6\sum a^{10} - 56\sum a^3\sum a^{13} + 136\sum a^5\sum a^{11}\right) d \\
&+ 40\sum a^7\sum a^{12} - 16\sum a^5\sum a^{14} + 16\sum a^8\sum a^{11} - 16\sum a^6\sum a^{13} \\
&- 32\sum a^9\sum a^{10} + 8\sum a^4\sum a^{15}
\end{aligned}$$

Note that $t \geq 0, m_0 > 0$ and $n_0 > 0$. For proving $Q_0 \geq 0$, we need to prove inequalities $x_3 \geq 0, x_2 \geq 0, x_1 \geq 0$ and $x_0 \geq 0$. Next, we shall prove these four inequalities in proper order. In fact, both inequalities $x_3 \geq 0$ and $x_2 \geq 0$ are valid for any triangle.
We now prove that $x_3 \geq 0$ holds for any triangle ABC. With the help of software Maple, using $\sum a = 2s, d = abc = 4Rrs$, and related identities given in Lemma 2.3 and 2.4, we obtain

$$x_3 = 64r^4s^2K_3, \tag{3.11}$$

where

$$\begin{aligned}K_3 =& -s^6 - (204R^2 + 42Rr + 3r^2)s^4 + (1280R^4 + 440R^3r - 148R^2r^2\\ &- 52Rr^3 - 3r^4)s^2 + (16R^2 + 2Rr - r^2)(4R + r)^3r.\end{aligned}$$

To prove $x_3 \geq 0$ we have to prove $K_3 \geq 0$.
We now recall that for any triangle ABC we have Euler's inequality

$$R \geq 2r \tag{3.12}$$

and the following fundamental triangle inequality (cf. [1] and [7]):

$$t_0 \equiv -s^4 + (4R^2 + 20Rr - 2r^2)s^2 - r(4R + r)^3 \geq 0, \tag{3.13}$$

with equality if and only if $triangle ABC$ is isosceles. We also have the following two Gerretsen inequalities:

$$g_1 \equiv s^2 - 16Rr + 5r^2 \geq 0, \tag{3.14}$$

$$g_2 \equiv 4R^2 + 4Rr + 3r^2 - s^2 \geq 0, \tag{3.15}$$

Based on the above four inequalities, after analysis we rewrite inequality $K_3 \geq 0$ as follows:

$$\begin{aligned}K_3 \equiv&(s^2 + 208R^2 + 62Rr + r^2)t_0 + 32R(7R + 2r)\left[(2R^2 + r^2)g_1 + 18Rrg_2\right.\\ &\left.+2r(4R + r)(3R + r)(R - 2r)\right] \geq 0.\end{aligned} \tag{3.16}$$

Then by (3.12)-(3.15) we deduce that $K_3 \geq 0$ holds for any triangle ABC.
Now, We show that inequality $x_2 \geq 0$ holds for any triangle ABC. With the help of Maple, using $\sum a = 2s, abc = 4Rrs$, Lemma 2.4 and 2.5, simplifying gives

$$x_2 = 256s^2r^5K_2, \tag{3.17}$$

where

$$\begin{aligned}K_2 =&(24R - 7r)s^8 - (1408R^3 + 1736R^2r + 248Rr^2 + 20r^3)s^6\\ &+ (8704R^5 + 3408R^4r - 920R^2r^3 - 288Rr^4 - 18r^5)s^4\\ &+ 4(504R^4 + 416R^3r + 138R^2r^2 + 10Rr^3 - r^4)(4R + r)^2rs^2\\ &+ (8R^2 + 4Rr + r^2)(4R + r)^5r^2.\end{aligned}$$

So we have to show $K_2 \geq 0$. After analysis, we obtain the following identity:

$$K_2 = t_0A_1 + g_1^2A_2 + 4r(A_3 + A_4), \tag{3.18}$$

where

$$\begin{aligned}A_1 =&7rs^4+(1408R^3+1764R^2r+388Rr^2+6r^3)s^2\\&+8(3976R^2+4196Rr+605r^2)R^2r,\\A_2 =&24Rg_1^2+96(16R-5r)Rrg_1+(3072R^5+36864R^3r^2\\&-23040R^2r^3+4052Rr^4+r^5),\\A_3 =&5R^2(4672R^4+7343r^4)g_1+(144480R^5+20832R^4r\\&+87500R^3r^2+4005Rr^4+2r^5)rg_2,\\A_4 =&(R-2r)(110208R^6+487616R^5r-511888R^4r^2\\&+206304R^3r^3+24672R^2r^4+13029Rr^5+6r^6)r.\end{aligned}$$

By Euler's inequality and Gerretsen's inequalities (3.14) and (3.15), we have $A_2 > 0$, $A_3 \geq 0$ and $A_4 \geq 0$. Thus inequality $K_2 \geq 0$ follows from identity (3.18) and inequality (3.13). So, we proved that inequality $x_2 \geq 0$ is valid for any triangle.
Next, we shall prove that $x_1 \geq 0$ holds for the acute triangle ABC. With the help of Maple software, using $\sum a = 2s$, Lemma 2.3 and 2.4, we easily obtain the following identity:

$$x_1 = 8192s^3r^6K_1, \tag{3.19}$$

where

$$\begin{aligned}K_1 =&-s^{10}-(12R^2-16Rr+7r^2)s^8+(608R^4-528R^3r\\&-812R^2r^2-112Rr^3-14r^4)s^6-(1024R^6-256R^5r\\&+3584R^4r^2+2368R^3r^3+620R^2r^4+120Rr^5+10r^6)s^4\\&+(128R^5+1472R^4r+1136R^3r^2+380R^2r^3+40Rr^4\\&-r^5)(4R+r)^2rs^2+(8R^2+4Rr+r^2)(4R+r)^5r^3.\end{aligned}$$

Thus we only need to show that $K_1 \geq 0$. According to Lemma 2.5 and Euler's inequality, for acute triangle ABC we have

$$v_0 \equiv s^2-4R^2+Rr-13r^2 \geq 0. \tag{3.20}$$

Based on (3.12), (3.13) and (3.20), after analysis we obtain the following identity:

$$K_1 = t_0B_1+g_2B_2+B_3v_0^2+B_4v_0+B_5, \tag{3.21}$$

where

$$\begin{aligned}B_1 =&s^6+(16R^2+4Rr+5r^2)s^4+(800R^3+832R^2r+192Rr^2+3r^3)rs^2\\&+8(128R^4+240R^3r+2736R^2r^2+2184Rr^3+313r^4)R^2,\\B_2 =&4096R^8+26112R^7r+48128R^6r^2+382080R^5r^3+237216R^4r^4\\&-9872R^3r^5-8972R^2r^6-260Rr^7-2r^8,\\B_3 =&544R^4v_0+6528R^6-1632R^5r+21216R^4r^2+268Rr^5+r^6,\end{aligned}$$

$$\begin{aligned}B_4 =&2(4R^2-Rr+13r^2)(3264R^6-816R^5r+10608R^4r^2\\&+268Rr^5+r^6),\\B_5 =&(R-2r)(18432R^9-44544R^8r+119680R^7r^2-216608R^6r^3\\&+276896R^5r^4-498592R^4r^5-83552R^3r^6-23468R^2r^7\\&-23083Rr^8-88r^9).\end{aligned}$$

It follows from Euler's inequality and inequality (3.20) that $B_2>0$ and $B_3>0$. Since $t_0\geq 0$ and $g_2\geq 0$, to prove $K_1\geq 0$ it remains to show that

$$B_4v_0+B_5\geq 0. \tag{3.22}$$

It is clear that $B_4>0$. Thus, by Lemma 2.5, to prove the above inequality we require the following inequality to be proved:

$$B_4\frac{(R-2r)r^3}{R^2}+B_5\geq 0. \tag{3.23}$$

Simplifying gives

$$\frac{R-2r}{R^2}B_6\geq 0, \tag{3.24}$$

where

$$\begin{aligned}B_6 =&18432R^{11}-44544R^{10}r+119680R^9r^2-190496R^8r^3\\&+263840R^7r^4-327232R^6r^5-125984R^5r^6+252340R^4r^7\\&-20939R^3r^8-616R^2r^9+6966Rr^{10}+26r^{11}.\end{aligned}$$

Putting $e=R-2r$, then $e\geq 0$ since we have Euler's inequality. Substituting $R=2r+e$ into B_6 and expanding gives

$$\begin{aligned}B_6 =&18432e^{11}+360960e^{10}r+3283840e^9r^2+18276064e^8r^3\\&+69008544e^7r^4+185286784e^6r^5+359837024e^5r^6\\&+502324340e^4r^7+488754389e^3r^8+310630838e^2r^9\\&+113348754er^{10}+17323470r^{11},\end{aligned}$$

so that $B_6>0$ and hence inequality (3.24) is proved. We therefore proved that $x_1\geq 0$ holds for the acute triangle ABC.

Finally, we prove that $x_0\geq 0$ holds for acute triangles. Firstly, with the help of soft ware Maple, it is not difficult to obtain the following identity:

$$x_0=4096s^3r^6K_0, \tag{3.25}$$

where

$$\begin{aligned}K_0 =&-3s^{10}-(64R^2+11r^2)s^8+(944R^4+960R^3r+280R^2r^2\\&-8Rr^3-14r^4)s^6-(2560R^6+5632R^5r+7408R^4r^2\\&+3264R^3r^3+408R^2r^4-8Rr^5+6r^6)s^4+(256R^5+1056R^4r\\&+736R^3r^2+248R^2r^3+32Rr^4+r^5)(4R+r)^2rs^2\\&+(8R^2+4Rr+r^2)(4R+r)^5r^3.\end{aligned}$$

Then we need to prove $K_0 \geq 0$. We easily obtain the following identity:

$$K_0 = t_0 C_1 s^2 + C_2, \tag{3.26}$$

where

$$\begin{aligned} C_1 =& 3s^4 + (76R^2 + 60Rr + 5r^2)s^2 + (608R^3 + 644R^2r + r^3)r, \\ C_2 =& 16R(40R^3 + 3r^3)s^6 - (2560R^6 + 10816R^3r^3 - 1912R^2r^4 \\ &+ 14656R^4r^2 + 3200R^5r - 108Rr^5 - r^6)s^4 + 2r(128R^5 + 1744R^4r \\ &+ 1960R^3r^2 + 446R^2r^3 + 18Rr^4 + r^5)(4R + r)^2s^2 \\ &+ (8R^2 + 4Rr + r^2)(4R + r)^5r^3. \end{aligned}$$

Since we have inequality (3.13), it remains to show that $C_2 \geq 0$. Noting that the following known inequality (see [2, inequality 5.5]):

$$(4R + r)^2 \geq 3s^2, \tag{3.27}$$

we only need to prove

$$\begin{aligned} &16R(40R^3 + 3r^3)s^4 - (2560R^6 + 10816R^3r^3 - 1912R^2r^4 \\ &+ 14656R^4r^2 + 3200R^5r - 108Rr^5 - r^6)s^2 + 2r(128R^5 + 1744R^4r \\ &+ 1960R^3r^2 + 446R^2r^3 + 18Rr^4 + r^5)(4R + r)^2 \\ &+ 3(8R^2 + 4Rr + r^2)(4R + r)^3r^3 \geq 0, \end{aligned}$$

i.e.,

$$\begin{aligned} P_0 \equiv& 16R(40R^3 + 3r^3)s^4 - (2560R^6 + 10816R^3r^3 - 1912R^2r^4 \\ &+ 14656R^4r^2 + 3200R^5r - 108Rr^5 - r^6)s^2 + r(256R^5 + 3488R^4r \\ &+ 4016R^3r^2 + 964R^2r^3 + 60Rr^4 + 5r^5)(4R + r)^2 \geq 0. \end{aligned} \tag{3.28}$$

We can write the above inequality as follows:

$$P_0 \equiv P_1 + P_2 + p_0 s^2 + q_0 \geq 0, \tag{3.29}$$

where

$$\begin{aligned} P_1 =& 16R(40R^3 + 3r^3)(s^2 - 4R^2 + Rr - 13r^2)^2, \\ P_2 =& 10432R^3r^3(4R^2 + 4Rr + 3r^2 - s^2), \\ p_0 =& 2560R^6 - 4480R^5r + 1984R^4r^2 + 1816R^2r^4 + 1356Rr^5 + r^6, \\ q_0 =& -10240R^8 + 9216R^7r - 9344R^6r^2 + 66560R^5r^3 - 98464R^4r^4 \\ &- 23648R^3r^5 + 2772R^2r^6 - 8012Rr^7 + 5r^8. \end{aligned}$$

Since $P_1 \geq 0$ and $P_2 \geq 0$ which follows from (3.15). It remains to prove that the following inequality

$$p_0 s^2 + q_0 \geq 0 \tag{3.30}$$

holds for the acute triangle ABC.

We shall consider the following two cases to finish the proof of inequality (3.30).

Case 1 R and r satisfy $5R - 12r \geq 0$.

We set $u_0 = s^2 - (2R + r)^2$. By the Ciamberlini's acute triangle inequality (see [8]):

$$s \geq 2R + r \tag{3.31}$$

we have $u_0 \geq 0$. It is easy to verify the following identity:

$$\begin{aligned} &78125(p_0 s^2 + q_0) \\ &= 78125 p_0 u_0 + 424038078 r^8 + 2(5R - 12r)\left[5(R - 2r)m_1 + 2535736082 r^6\right] r, \end{aligned} \tag{3.32}$$

where

$$\begin{aligned} m_1 =& 2400000R^5 - 15640000R^4 r + 29064000R^3 r^2 + 63553600R^2 r^3 \\ &+ 123003640 R r^4 + 251808736 r^5. \end{aligned}$$

Euler's inequality shows that $p_0 > 0$. Again, note that

$$\begin{aligned} &2400000R^5 - 15640000R^4 r + 29064000R^3 r^2 \\ &= 8000R^3(300R^2 - 1955Rr + 3633r^2) > 0. \end{aligned}$$

Consequently, from (3.32) we deduce that the strict inequality $p_0 s^2 + q_0 > 0$ holds for the acute triangle ABC under Case 1.

Case 2 R and r satisfy $5R - 12r < 0$.

In this case, we shall apply the acute triangle inequality (3.20). We set $e = R - 2r$ and $v_0 = s^2 - 4R^2 + Rr - 13r^2$, then it is easily verified that

$$15625(p_0 s^2 + q_0) = 15625 p_0 v_0 + er(12r - 5R)m_2 + 212019039 e r^7, \tag{3.33}$$

where

$$\begin{aligned} m_2 =& 35200000e^5 + 393280000e^4 r + 1731112000e^3 r^2 \\ &+ 3738844800e^2 r^3 + 3937362920 e r^4 + 1584482668 r^5. \end{aligned}$$

Note that $p_0 > 0, e \geq 0$ and the previous inequality (3.20), we conclude from (3.33) that $p_0 s^2 + q_0 \geq 0$ holds under Case 2.

Combining the discussions of the above two cases, we deduce that inequality (3.30) holds for all acute triangles. Therefore, we finished the proofs of inequalities $x_0 \geq 0$ and inequality (1.6). In addition, it is easy to determine that equality in (1.6) holds if and only triangle ABC is equilateral. This completes the proof of Theorem 1.1. □

4. Open problems

In this section, we give several conjectures related to inequality (1.6) as open problems.

Conjecture 4.1. *If $k \geq 2$, then for any triangle ABC the following inequality holds:*

$$\sum \frac{m_a + k h_a}{r_a + k w_a} \leq \sum \frac{m_a + h_a}{r_a + w_a}. \tag{4.1}$$

Conjecture 4.2. *If $k \geq \frac{3}{2}$, then for the acute triangle ABC the following inequality holds:*

$$\sum \frac{m_a + k h_a}{r_a + k w_a} \leq \frac{4}{3}\left(\sum \sin\frac{A}{2}\right)^2. \tag{4.2}$$

Conjecture 4.3. *If $0 < k \leq \frac{3}{4}$, then for any triangle ABC the following inequality holds:*

$$\sum \frac{m_a + kh_a}{r_a + kw_a} \geq 3. \tag{4.3}$$

Conjecture 4.4. *Let ABC be an acute triangle. If $k \geq 1.72$, then following inequality holds:*

$$\sum \left(\frac{m_a + kh_a}{r_a + kw_a} \right)^2 \leq 3. \tag{4.4}$$

If $0 < k \leq 1.29$, then the inequality reversely holds.

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