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ALMOST QUASI-YAMABE SOLITON ON 3-DIMENSIONAL LORENTZIAN PARA-KENMOTSU MANIFOLDS

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ABSTRACT. In this present paper, we have studied 3-dimensional Lorentzian para-Kenmotsu manifolds admitting an almost quasi-Yamabe soliton and gradient almost quasi-Yamabe solitons. It is shown that if a 3-dimensional Lorentzian para-Kenmotsu manifold M_3 admits an almost quasi-Yamabe soliton whose soliton vector field V such that $\varphi V \neq 0$, then the manifold M_3 is of constant sectional curvature 1, but converse is not true in general which has been proved by a concrete example. Next, it is proved that if the metric g of a 3-dimensional Lorentzian para-Kenmotsu manifold M_3 is a gradient quasi-Yamabe soliton, then either the manifold M_3 is of constant sectional curvature 1 or the almost quasi-Yamabe gradient soliton on M_3 is trivial. Finally, for an almost quasi-Yamabe soliton on a 3-dimensional Lorentzian para-Kenmotsu manifold M_3, we have shown that if the soliton vector field V is pointwise collinear with timelike smooth vector field ξ, then V is becoming a constant multiple of ξ.

1. INTRODUCTION

The concept of Yamabe flow was introduced by R. Hamilton [9] as a tool for constructing metrics of constant scalar curvature in a given conformal class of Riemannian metrics on a Riemannian manifold of dimension $n(\geq 3)$. The Yamabe flow on a n-dimensional Riemannian or pseudo-Riemannian manifold (M, g) is defined as the evolution equation of the metric $g = g(t)$ as follows

$$\frac{\partial}{\partial t}(g(t)) = -r(g(t)), \quad g(0) = g_0$$

where r is the well-known scalar curvature of the manifold. The Yamabe flow corresponds to the fast diffusion case of the plasma equation in mathematical physics.

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It is well-known that a Riemannian metric g of a complete Riemannian manifold (M, g) of dimension n is said to be a **Yamabe soliton** if, for some real constant ρ, there exists a smooth vector field V on M satisfying the equation

$$\frac{1}{2}\pounds_V g = (r - \rho)g \tag{1.1}$$

where $\pounds_V$ indicates the Lie-derivative in the direction of V. The Yamabe soliton is said to be shrinking, steady, or expanding according to ρ is positive, zero, or negative respectively. Moreover, the Yamabe soliton is said to be trivial if the soliton vector field V of the Yamabe soliton (g, V, ρ) is Killing. The Yamabe solitons have been investigated under some conditions by many authors such as ([4], [5] [6], [7], [11], [13]).

In 2013, E. Barbosa and E. Ribeiro [1] introduced the concept of **almost Yamabe soliton**, which is a generalization of the Yamabe soliton by setting ρ to be a smooth function on M, i.e., $\rho : M \to \mathbb{R}$ is a smooth function. Furthermore, T. Seko and S. Maeta in [17] completely classified almost Yamabe solitons in the context of hypersurfaces in Euclidean spaces. The so-called Yamabe soliton becomes the gradient soliton if $V = grad(h) = Dh$, for some smooth function $h : M \to \mathbb{R}$. In this case the Eq. (1.1) reduces to

$$\nabla^2 h = (r - \rho)g \tag{1.2}$$

where $\nabla^2 h$ is the Hessian of smooth function h on M.

In 2014, G. Huang and H. Li [12] introduced the notion of **quasi-Yamabe gradient soliton**, which is a generalization of gradient Yamabe soliton (see [12], [14]). The quasi-Yamabe gradient soliton equation is given by

$$\nabla^2 h = (r - \rho)g + \frac{1}{\beta} dh \otimes dh \tag{1.3}$$

where β is a positive constant and ρ is a real number. It is clear that if $\beta \to \infty$, the Eq. (1.3) recovers gradient Yamabe soliton. Huang-Li [12] proved that n-dimensional $(n \geq 3)$ complete quasi Yamabe gradient solitons with vanishing Weyl curvature tensor and positive sectional curvature must be rotationally symmetric.

In 2017, the notion of **gradient almost quasi-Yamabe soliton** was introduced by V. Pirhadi and A. Razavi [15]. They have proved that a necessary and sufficient condition under which an arbitrary compact almost Yamabe soliton is necessarily gradient [15]. In 2019, A. M. Blaga [2] studied almost quasi-Yamabe solitons on the warped product manifolds and derived a Bochner-type formula for a gradient almost quasi-Yamabe soliton. Moreover, in 2020, X. Chen [3] studied almost quasi-Yamabe solitons on almost cosymplectic manifolds. Currently, S. Ghosh et al. [8] have considered almost quasi-Yamabe solitons and gradient almost quasi-Yamabe solitons in the context of Kenmotsu manifolds.

According to Chen, a Riemannian metric g defined on a Riemannian manifold (M, g) is said to be an **almost quasi-Yamabe soliton** if, for some smooth function $\rho : M \to \mathbb{R}$, there exist a smooth vector field V and a positive constant β on M satisfying the equation

$$\frac{1}{2}\pounds_V g = (r - \rho)g + \frac{1}{\beta}V^b \otimes V^b \tag{1.4}$$

where V^b is the 1-form associated to V. The smooth vector field V is also called a soliton vector field for the almost quasi-Yamabe soliton (g, V, ρ, β). The almost quasi-Yamabe metric is closed if the 1-form V^b is closed and it is trivial if V is identically zero. Furthermore, if $\beta \to \infty$, the foregoing equation reduces to almost Yamabe soliton. Moreover, the preceding equation gives an almost quasi-Yamabe gradient soliton for $V = grad(h) = Dh$.

Motivated by the above studies, in the present manuscript we make the contribution to investigate the almost quasi-Yamabe soliton metric on 3-dimensional Lorentzian para-Kenmotsu manifolds.
The organization of the manuscript is as follows: Section 2 contains some preliminaries on a Lorentzian para-Kenmotsu manifold. Section 3 deals with 3-dimensional Lorentzian para-Kenmotsu manifolds admitting almost quasi-Yamabe solitons. Precisely we prove

if a 3-dimensional Lorentzian para-Kenmotsu manifold admits an almost quasi-Yamabe soliton and $\varphi V \neq 0$, then the manifold is of constant sectional curvature 1. Also, we construct an example to prove the converse is not true in general. Next, we study an almost quasi-Yamabe whose soliton vector field V is the gradient of some smooth function h on 3-dimensional Lorentzian para-Kenmotsu manifold M_3. Finally, we have shown that if a 3-dimensional Lorentzian para-Kenmotsu manifold M_3 admits an almost quasi-Yamabe soliton with the soliton vector field V is pointwise collinear with the timelike vector field ξ, then the soliton vector field V becomes a constant multiple of ξ.

2. Preliminaries

A Lorentzian almost para-contact metric structure [10] on a $(2n+1)$-dimensional smooth manifold M_{2n+1} is a quadruplet (φ, ξ, η, g), where φ is a $(1,1)$ fundamental tensor field, ξ a unit timelike smooth vector field, η a 1-form and a Lorentzian metric g, satisfying

$$\varphi^2(E) = E + \eta(E)\xi, \quad \eta(\xi) = -1, \tag{2.1}$$

$$\varphi(\xi) = 0, \quad rank(\varphi) = 2n, \quad \eta(\varphi E) = 0, \tag{2.2}$$

$$g(\varphi E, \varphi F) = g(E, F) + \eta(E)\eta(F), \tag{2.3}$$

$$g(\varphi E, F) = g(E, \varphi F), \tag{2.4}$$

$$g(E, \xi) = \eta(E), \tag{2.5}$$

for all smooth vector fields $E, F \in \chi(M_{2n+1})$. A Lorentzian almost para-contact metric manifold M_{2n+1} is called a Lorentzian para-Kenmotsu manifold [10], if it satisfies

$$(\nabla_E \varphi)F = -g(\varphi E, F)\xi - \eta(F)\varphi E, \tag{2.6}$$

where ∇ denotes the Levi-Civita connection of the metric g. From the antecedent equation, it is clear that

$$\nabla_E \xi = -E - \eta(E)\xi, \tag{2.7}$$

which gives

$$(\nabla_E \eta)F = -g(E,F) - \eta(E)\eta(F), \tag{2.8}$$

In a $(2n+1)$-dimensional Lorentzian para-Kenmotsu manifolds ([10], [16]), we have

$$R(E,F)\xi = \eta(F)E - \eta(E)F, \tag{2.9}$$

$$R(E,\xi)\xi = -E - \eta(E)\xi, \tag{2.10}$$

$$R(\xi,E)F = g(E,F)\xi - \eta(F)E, \tag{2.11}$$

$$S(E,\xi) = 2n\eta(E), \tag{2.12}$$

$$S(\xi,\xi) = -2n, \tag{2.13}$$

$$Q\xi = 2n\xi, \tag{2.14}$$

where R, S and Q are the curvature tensor, the Ricci tensor and the Ricci operator defined by $g(QE,F) = S(E,F)$, for all smooth vector fields $E, F \in \chi(M_{2n+1})$, respectively.

It is known that the second order Ricci tensor field S of a 3-dimensional Lorentzian para-Kenmotsu manifold M_3 is given by [16]

$$S(E,F) = (\frac{r}{2} - 1)g(E,F) + (\frac{r}{2} - 3)\eta(E)\eta(F). \tag{2.15}$$

This shows that a 3-dimensional Lorentzian para-Kenmotsu manifold is an η-Einstein manifold.

Lemma 2.1. [16] *For a 3-dimensional Lorentzian para-Kenmotsu manifold M_3, the curvature tensor R is given by*

$$\begin{aligned} R(E,F)Z = (\tfrac{r}{2} - 2)\{g(F,Z)E - g(E,Z)F\} + (\tfrac{r}{2} - 3)\{g(F,Z)\eta(E)\xi \\ -g(E,Z)\eta(F)\xi + \eta(F)\eta(Z)E - \eta(E)\eta(Z)F\}, \end{aligned} \tag{2.16}$$

for any smooth vector fields $E, F, Z \in \chi(M_3)$.

Lemma 2.2. [3] *For an almost quasi-Yamabe gradient soliton (M, g, Dh, β, ρ), the curvature tensor R is given by*

$$R(E,F)Dh = E(r-\rho)F - F(r-\rho)E - \frac{r-\rho}{\beta}\{E(h)F - F(h)E\}, \tag{2.17}$$

for all smooth vector fields $E, F \in \chi(M)$. where D stands for the gradient operator of the metric g.

3. Almost quasi-Yamabe solitons on 3-dimensional Lorentzian para-Kenmotsu manifolds

This section is devoted to the study of the 3-dimensional Lorentzian para-Kenmotsu manifold admitting an almost quasi-Yamabe soliton. Let M_3 be a 3-dimensional Lorentzian para-Kenmotsu manifold admitting a closed almost quasi-Yamabe soliton (g, V, ρ, β). In this regards our first theorem is

Theorem 3.1. *Let a 3-dimensional Lorentzian para-Kenmotsu manifold M_3 admit a closed almost quasi-Yamabe soliton (g, V, ρ, β) and $\varphi V \neq 0$. Then the manifold M_3 is of constant sectional curvature 1, but the converse is not true in general.*

Proof. Since V^b is closed, the Eq. (1.4) transforms to

$$\nabla_F V = (r-\rho)F + \frac{1}{\beta}g(V,F)V. \tag{3.1}$$

Executing covariant derivative of Eq. (3.1) along an arbitrary vector field E, we get

$$\begin{aligned}\nabla_E\nabla_F V = (E(r-\rho))F + (r-\rho)\nabla_E F + \tfrac{1}{\beta}\{g(\nabla_E V, F) + g(V, \nabla_E F)\}V \\ +\tfrac{1}{\beta}g(V,F)\nabla_E V.\end{aligned} \tag{3.2}$$

Exchanging E and F in the above Eq. (3.2) gives

$$\nabla_F \nabla_E V = (F(r-\rho))E + (r-\rho)\nabla_F E + \tfrac{1}{\beta}\{g(\nabla_F V, E) + g(V, \nabla_F E)\}V$$
$$+\tfrac{1}{\beta} g(V,E)\nabla_F V. \tag{3.3}$$

Again, replacing $F = [E, F]$ in Eq. (3.1) yields

$$\nabla_{[E,F]} V = (r-\rho)[E,F] + \frac{1}{\beta} g(V,[E,F])V. \tag{3.4}$$

Using the previous three Eqs. (3.2) , (3.3) and (3.4) in the well known formula $R(E,F)V = \nabla_E \nabla_F V - \nabla_F \nabla_E V - \nabla_{[E,F]} V$, we infer that

$$R(E,F)V = (E(r-\rho))F - (F(r-\rho))E + \frac{r-\rho}{\beta}\{g(V,F)E - g(V,E)F\}. \tag{3.5}$$

Executing inner product of Eq. (3.5) with timelike smooth vector field ξ yields

$$g(R(E,F)V,\xi) = \{E(r-\rho) - \tfrac{r-\rho}{\beta} g(E,V)\}\eta(F) - \{F(r-\rho)$$
$$-\tfrac{r-\rho}{\beta} g(F,V)\}\eta(E). \tag{3.6}$$

Again, in the view of Eq. (2.9) we obtain

$$g(R(E,F)V,\xi) = g(F,V)\eta(E) - g(E,V)\eta(F). \tag{3.7}$$

Equating the Eq. (3.6) and Eq. (3.7), we get

$$\{E(r-\rho) - \tfrac{r-\rho}{\beta} g(E,V)\}\eta(F) - \{F(r-\rho) - \tfrac{r-\rho}{\beta} g(F,V)\}\eta(E)$$
$$= g(F,V)\eta(E) - g(E,V)\eta(F). \tag{3.8}$$

Replacing E by φE and F by ξ in the above Eq. (3.8) and using Eq. (2.1), Eq. (2.2) and Eq. (2.4), we have

$$\varphi D(r-\rho) = (\frac{r-\rho}{\beta} - 1)\varphi V. \tag{3.9}$$

Let us consider an orthonormal basis $\{e_1, e_2, e_3\}$ of the tangent space at each point of 3-dimensional Lorentzian para-Kenmotsu manifold M_3. Contracting Eq. (3.5) over F and making use of Eq. (2.15) entails that

$$\{\frac{r}{2} - 1 - \frac{2(r-\rho)}{\beta}\}V + (\frac{r}{2} - 3)\eta(V)\xi = -2D(r-\rho). \tag{3.10}$$

Now applying φ on the both sides of the above Eq. (3.10) and using Eq. (2.2) leads

$$\{\frac{r}{2} - 1 - \frac{2(r-\rho)}{\beta}\}\varphi V = -2\varphi D(r-\rho). \tag{3.11}$$

In view of the Eq. (3.9) and Eq. (3.11), we get

$$(\frac{r}{2} - 3)\varphi V = 0. \tag{3.12}$$

If we assume $\varphi V \neq 0$, then Eq. (3.12) gives us $r = 6$. Now using $r = 6$ in Eq. (2.16) we get

$$R(E,F)Z = \{g(F,Z)E - g(E,Z)F\}.$$

Thus, the manifold M_3 is of constant sectional curvature 1. But the converse is not true in general, for example, Let us consider the 3-dimensional smooth manifold $M_3 = \{(u,v,w) \in \mathbb{R}^3 : w \neq 0\}$ with the standard coordinate system (u,v,w) of $\mathbb{R}^3$.
Let us consider the smooth vector fields E_1, E_2, E_3 of M_3 be such that

$$[E_1, E_2] = 0, \quad [E_2, E_3] = -E_2, \quad [E_1, E_3] = -E_1.$$

Let us define a Lorentzian metric g on M_3 by

$$g(E_i, E_j) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$$

and a $(1,1)$ tensor field φ on M_3 by

$$\varphi(E_1) = -E_2, \quad \varphi(E_2) = -E_1, \quad \varphi(E_3) = 0.$$

Now considering $E_3 = \xi$, let η be the 1-form on M_3, defined by

$$g(X, E_3) = \eta(X), \quad \forall X \in \chi(M_3)$$

Then it can be observed that $\eta(\xi) = -1$.
Using the linearity property of φ and g we obtain

$$\varphi^2 X = X + \eta(X)\xi, \quad g(\varphi X, \varphi Y) = g(X, Y) + \eta(X)\eta(Y), \forall X, Y \in \chi(M_3)$$

Hence the structure (g, φ, ξ, η) defines a Lorentzian almost para-contact metric structure on M_3.
Let ∇ be a Riemannian connection with respect to g. Utilizing the well-known Koszul's formula given by

$$2g(\nabla_X Y, Z) = Xg(Y, Z) + Yg(Z, X) - Zg(X, Y) - g(X, [Y, Z]) \\ -g(Y, [X, Z]) + g(Z, [X, Y]),$$

we calculate the following:

$$\nabla_{E_1} E_1 = -E_3, \quad \nabla_{E_1} E_2 = 0, \quad \nabla_{E_1} E_3 = -E_1,$$

$$\nabla_{E_2} E_1 = 0, \quad \nabla_{E_2} E_2 = E_3, \quad \nabla_{E_2} E_3 = -E_2,$$

$$\nabla_{E_3} E_1 = 0, \quad \nabla_{E_3} E_2 = 0, \quad \nabla_{E_3} E_3 = 0.$$

In view of the above results it is clear that the manifold M_3 satisfies

$$\nabla_X \xi = -X - \eta(X)\xi, \quad \forall X \in \chi(M_3)$$

Hence $M_3(g, \varphi, \xi, \eta)$ is a 3-dimensional Lorentzian para-Kenmotsu manifold.
Thus from the above computations and using the well-known formula

$$R(X, Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z$$

the non-vanishing components of the curvature tensor R as follows:

$$R(E_1, E_2)E_1 = -E_2, \quad R(E_2, E_1)E_1 = E_2, \quad R(E_1, E_3)E_1 = -E_3,$$

$$R(E_3, E_1)E_1 = E_3, \quad R(E_1, E_2)E_2 = E_1, \quad R(E_2, E_1)E_2 = -E_1,$$

$$R(E_2, E_3)E_2 = -E_3, \quad R(E_3, E_2)E_2 = E_3, \quad R(E_1, E_3)E_3 = -E_1,$$

$$R(E_3, E_1)E_3 = E_1, \quad R(E_2, E_3)E_3 = -E_2, \quad R(E_3, E_2)E_3 = E_2.$$

Using the well-known formula $S(X, Y) = \sum_{i=1}^{3} g(E_i, E_i)g(R(E_i, X)Y, E_i)$ the non-vanishing components of the Ricci tensor S can be easily be calculated as

$$S(E_1, E_1) = 2, \quad S(E_2, E_2) = 2, \quad S(E_3, E_3) = -2.$$

Again, the scalar curvature r of the given Lorentzian para-Kenmotsu manifold can be calculated as under:

$$r = \sum_{i=1}^{3} g(E_i, E_i)S(E_i, E_i) = S(E_1, E_1) + S(E_2, E_2) - S(E_3, E_3) = 6.$$

Let $X = X^1E_1 + X^2E_2 + X^3E_3$, $Y = Y^1E_1 + Y^2E_2 + Y^3E_3$ and $Z = Z^1E_1 + Z^2E_2 + Z^3E_3$. Then we can easily verify that

$$R(X, Y)Z = [g(Y, Z)X - g(X, Z)Y]$$

Thus, the manifold M_3 is of constant sectional curvature 1.
Now if we take a soliton vector field $V = V^1E_1 + V^2E_2 + V^3E_3$ such that $(V^1)^2 + (V^2)^2 \neq 0$, then we have $\nabla_{E_1}V = -V^1E_3 - V^3E_1$ and $(r - \rho)E_1 + \frac{1}{\beta}g(V, E_1)V = (6 - \rho)E_1 + \frac{1}{\beta}V^1(V^1E_1 + V^2E_2 + V^3E_3)$. Hence

$$\nabla_{E_1}V \neq (r - \rho)E_1 + \frac{1}{\beta}g(V, E_1)V$$

for any values of ρ, β and any choice of soliton vector field V.

Thus a 3-dimensional Lorentzian para-Kenmotsu manifold M_3 can not admit an almost quasi-Yamabe soliton although M_3 is a 3-dimensional Lorentzian para-Kenmotsu manifold with constant sectional curvature 1. This completes the proof. □

Furthermore, if $\beta \to \infty$, the Eq. (3.9) and Eq. (3.11) reduces to

$$\varphi D(r-\rho) = -\varphi V, \tag{3.13}$$

and

$$(\frac{r}{2} - 1)\varphi V = -2\varphi D(r-\rho), \tag{3.14}$$

respectively. In view of Eq. (3.13), the Eq. (3.14) reduces to

$$(\frac{r}{2} - 3)\varphi V = 0. \tag{3.15}$$

The foregoing equation eventually implies that either $r = 6$ or $\varphi V = 0$. Now utilizing the value of $r = 6$ in the Eq. (2.16) we get $R(E,F)Z = \{g(F,Z)E - g(E,Z)F\}$. This shows that manifold M is of constant sectional curvature 1. Again if we consider $\varphi V = 0$ and using the Eq. (2.1) one can easily obtain $V = -\eta(V)\xi \implies V$ is pointwise collinear with the timelike smooth vector field ξ. Thus we conclude the following:

Corollary 3.1. *If a* 3*-dimensional Lorentzian para-Kenmotsu manifold M_3 admits a closed almost Yamabe soliton (g, V, ρ), then either the manifold M_3 is of constant sectional curvature* 1 *or the soliton vector field V is pointwise collinear with the timelike smooth vector field ξ.*

Again, if ρ is a constant function on manifold, then closed almost quasi-Yamabe soliton becomes closed quasi-Yamabe soliton. Thus, maintaining the same process as in the proof of Corollary 3.2, we can state the following:

Corollary 3.2. *If a* 3*-dimensional Lorentzian para-Kenmotsu manifold M_3 admits a closed quasi-Yamabe soliton (g, V, ρ, β), then either the manifold M_3 is of constant sectional curvature* 1 *or the soliton vector field V is pointwise collinear with the timelike smooth vector field ξ.*

Next, we consider a 3-dimensional Lorentzian para-Kenmotsu manifold M_3 admitting an almost quasi-Yamabe whose soliton vector field V is the gradient of some smooth function $h : M_3 \to \mathbb{R}$, i.e., $V = grad(h) = Dh$. In this regard our next theorem is

Theorem 3.2. *If a* 3*-dimensional Lorentzian para-Kenmotsu manifold* M_3 *admits an almost quasi-Yamabe gradient soliton* (g, Dh, ρ, β)*, then either the manifold* M_3 *is of constant sectional curvature* 1 *or the almost quasi-Yamabe gradient soliton on* M_3 *is trivial.*

Proof. Let us assume that M_3 be a 3-dimensional Lorentzian para-Kenmotsu manifold admitting an almost quasi-Yamabe gradient soliton (g, Dh, ρ, β) and the soliton vector field $V = grad(h) = Dh$ for some smooth function $h : M_3 \to \mathbb{R}$.
Taking inner product both sides of the Eq. (2.17) with timelike smooth vector field ξ, we get

$$\begin{aligned} g(R(E,F)Dh,\xi) = E(r-\rho)\eta(F) - F(r-\rho)\eta(E) \\ -\tfrac{r-\rho}{\beta}\{E(h)\eta(F) - F(h)\eta(E)\}. \end{aligned} \tag{3.16}$$

Again recalling the Eq. (2.9) and taking inner product both sides with Dh yields

$$g(R(E,F)Dh,\xi) = F(h)\eta(E) - E(h)\eta(F). \tag{3.17}$$

Comparing the Eq. (3.16) and Eq. (3.17) we get

$$\begin{aligned} E(r-\rho)\eta(F) - F(r-\rho)\eta(E) - \tfrac{r-\rho}{\beta}\{E(h)\eta(F) - F(h)\eta(E)\} \\ = F(h)\eta(E) - E(h)\eta(F) \end{aligned}. \tag{3.18}$$

Replacing E and F by φE and ξ respectively in the above Eq. (3.18) and taking reference of the Eq.(2.1), Eq. (2.2) and Eq. (2.4) entails that

$$(\frac{r-\rho}{\beta} - 1)(\varphi E)(h) = (\varphi E)(r-\rho). \tag{3.19}$$

Let $\{e_1, e_2, e_3\}$ be an orthonormal basis of the tangent space at each point of 3-dimensional para-Kenmotsu manifold M_3. Then contracting Eq. (2.17) over the vector field F and making use of Eq. (2.15), we obtain

$$\{\frac{r}{2} - 1 - \frac{2(r-\rho)}{\beta}\}E(h) + (\frac{r}{2} - 3)\eta(E)\xi(h) = -2E(r-\rho). \tag{3.20}$$

Now replacing E by φE in the above Eq. (3.20) and making use of Eq. (2.2) we get

$$\{\frac{r}{2} - 1 - \frac{2(r-\rho)}{\beta}\}(\varphi E)(h) = -2(\varphi E)(r-\rho). \tag{3.21}$$

Now comparing the Eq. (3.19) and Eq. (3.21), we have

$$(\frac{r}{2} - 3)(\varphi E)(h) = 0. \tag{3.22}$$

It follows that, either $r = 6$ or $(\varphi E)(h) = 0$

Case (I): If $r = 6$, then the Eq. (2.16) reduces to

$$R(E,F)Z = \{g(F,Z)E - g(E,Z)F\},$$

which implies that the manifold M_3 is of constant sectional curvature 1.

Case (II): If $(\varphi E)(h) = 0$, then operating φ on both sides and making use of Eq. (2.1) we get

$$Dh = -\xi(h)\xi. \tag{3.23}$$

Taking covariant derivative of Eq. (3.23) along an arbitrary vector field E and taking reference of Eq. (3.1) and Eq. (2.7), we have

$$(r-\rho)E + \frac{1}{\beta}g(E,Dh)Dh = -E(\xi(h))\xi + \xi(h)(E + \eta(E)\xi). \tag{3.24}$$

Using Eq. (3.23) in Eq. (3.24), we get

$$(r-\rho)E + \frac{1}{\beta}(\xi(h))^2\eta(E)\xi = -E(\xi(h))\xi + \xi(h)(E + \eta(E)\xi). \tag{3.25}$$

Setting $E = \xi$ in Eq. (3.25) yields

$$(r-\rho) - \frac{1}{\beta}(\xi(h))^2 = -\xi(\xi(h)). \tag{3.26}$$

Now choosing the local orthonormal frame $\{e_1, e_2, e_3\}$ on a 3-dimensional Lorentzian para-Kenmotsu manifold M_3. Contracting Eq. (3.25), we obtain

$$3(r-\rho) - \frac{1}{\beta}(\xi(h))^2 = -\xi(\xi(h)) + 2\xi(h). \tag{3.27}$$

By virtue of Eq. (3.26) and Eq. (3.27), we have

$$\xi(h) = (r-\rho), \tag{3.28}$$

which is equivalent to

$$dh = (\rho - r)\eta, \tag{3.29}$$

where d stands for the exterior derivative.
Executing exterior derivative of Eq. (3.29) and using Poincare lemma: $d^2 \equiv 0$, we get

$$(\rho - r)d\eta + (d\rho)\eta - (dr)\eta = 0. \tag{3.30}$$

Taking wedge product operator both sides of Eq. (3.30) with η, we get

$$(\rho - r)\eta \wedge d\eta = 0, \tag{3.31}$$

which becomes

$$(\rho - r) = 0, \tag{3.32}$$

since $\eta \wedge d\eta \neq 0$, in a Lorentzian para-Kenmotsu manifold.
Using Eq. (3.32) in Eq. (3.29) gives us $dh = 0 \implies h = \text{constant}$. This means almost quasi-Yamabe gradient soliton is trivial. This completes the proof. □

Now recalling the Eqs. (3.26), (3.27) and (3.32), one can easily obtain that

$$\xi(\xi(h)) = \frac{1}{\beta}(\xi(h))^2. \tag{3.33}$$

If $\xi = \frac{\partial}{\partial x}$, then Eq. (3.33) transforms to

$$\frac{\partial^2 f}{\partial x^2} = \frac{1}{\beta}(\frac{\partial f}{\partial x})^2. \tag{3.34}$$

Then it is easy to observe that $h = -\beta ln x, x > 0$ is satisfies the above partial differential equation (3.34).

Thus we state the following:

Corollary 3.3. *If a 3-dimensional Lorentzian para-Kenmotsu manifold M_3 admits an almost quasi-Yamabe gradient soliton (g, Dh, ρ, β), then the manifold M_3 is of scalar curvature $r = \rho$.*

Corollary 3.4. *A 3-dimensional Lorentzian para-Kenmotsu manifold M_3 admitting an almost quasi-Yamabe gradient soliton (g, Dh, ρ, β) satisfies the differential equation $\frac{\partial^2 h}{\partial x^2} = \frac{1}{\beta}(\frac{\partial h}{\partial x})^2$ and the soliton function h is given by $h = -\beta ln x, x > 0$.*

In the last part of this section, we consider a 3-dimensional Lorentzian para-Kenmotsu manifold M_3 admitting an almost quasi-Yamabe whose soliton vector field V is pointwise collinear with the timelike smooth vector field ξ. In this regard our next theorem is

Theorem 3.3. *If a 3-dimensional Lorentzian para-Kenmotsu manifold M_3 admits an almost quasi-Yamabe soliton (g, V, ρ, β) with the soliton vector field V is pointwise collinear with the timelike vector field ξ, then the soliton vector field V becomes a constant multiple of ξ.*

Proof. Let M_3 be a 3-dimensional Lorentzian para-Kenmotsu manifold admitting an almost quasi-Yamabe soliton (g, V, ρ, β) such that the vector field V is pointwise collinear with ξ, then there exists a non-vanishing smooth function $\omega : M_3 \to \mathbb{R}$ such that $V = \omega\xi$. Then from Eq. (1.4) we derive

$$g(\nabla_E \omega\xi, F) + g(E, \nabla_F \omega\xi) = 2(r - \rho)g(E, F) + \frac{2\omega^2}{\beta}\eta(E)\eta(F), \tag{3.35}$$

for all smooth vector fields $E, n\chi(M_3)$.

which becomes

$$\omega\{g(\nabla_E\xi, F) + g(E, \nabla_F\xi)\} + E(\omega)\eta(F) + F(\omega)\eta(E)$$

$$= 2(r-\rho)g(E,F) + \tfrac{2\omega^2}{\beta}\eta(E)\eta(F). \tag{3.36}$$

Using Eq. (2.7) in Eq. (3.36) we get

$$-2\omega\{g(E,F) + \eta(E)\eta(F)\} + E(\omega)\eta(F) + F(\omega)\eta(E)$$

$$= 2(r-\rho)g(E,F) + \tfrac{2\omega^2}{\beta}\eta(E)\eta(F). \tag{3.37}$$

Now taking $F = \xi$ in Eq. (3.37) and using Eq. (2.1) we lead

$$E(\omega) = \{\xi(\omega) - 2(r - \rho - \frac{\omega^2}{\beta})\}\eta(E). \tag{3.38}$$

Again putting $E = \xi$ in the above Eq. (3.38) and using (2.1) yields

$$\xi(\omega) = (r - \rho - \frac{\omega^2}{\beta}). \tag{3.39}$$

Next, we shall consider a local orthonormal basis $\{e_i : i = 1, 2, 3\}$ of the tangent space at each point of M_3. The contraction of the above Eq. (3.37) over F gives

$$\xi(\omega) = 2(r - \rho + \omega) + (r - \rho - \frac{\omega^2}{\beta}). \tag{3.40}$$

Comparing Eq. (3.39) with Eq. (3.40) we get

$$\omega = -(r - \rho). \tag{3.41}$$

Now recalling the Eqs. (3.38), (3.39) and (3.41), one can easily find that

$$E(\omega) = (\omega + \frac{\omega^2}{\beta})\eta(E) \iff d\omega = (\omega + \frac{\omega^2}{\beta})\eta \tag{3.42}$$

Taking exterior derivative of both sides of Eq. (3.42) and using Poincare lemma: $d^2 \equiv 0$ yields

$$(\omega + \frac{\omega^2}{\beta})d\eta + d(\omega + \frac{\omega^2}{\beta})\eta = 0. \tag{3.43}$$

Taking wedge product operator both sides of Eq. (3.43) with η, we get

$$(\omega + \frac{\omega^2}{\beta})\eta \wedge d\eta = 0. \tag{3.44}$$

Since $\eta \wedge d\eta \neq 0$, we immediately have $\omega + \frac{\omega^2}{\beta} = 0 \implies \omega = -\beta$, since ω is non-vanishing function. Thus we conclude that ω is a constant function on M_3. Therefore, the soliton vector field V is constant multiple of the timelike vector field ξ. This completes the proof. □

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