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SPECTRA IN THE ANTI-DE SITTER GEOMETRY

SIMON DAVIS

ABSTRACT. When the dimensions of string theory in anti-de Sitter space-time are defined such there is a massless graviton in the physical spectrum, it coincides with the critical dimension of Liouville theory if the kinetic term for the time coordinate of the string in the embedding space is not present in the action. If it is restored, the critical dimension is consistent with the zero-curvature limit of strings propagating in flat space-time. The quantization of modes of various spin in a conformally related Einstein static universe with reflective boundary conditions at a timelike infinity is sufficient to introduce spin for fields in a locally anti-de Sitter region.

1. **Introduction**

The string worldsheet action in curved space

$$I = -\frac{T}{2}\int_{\Sigma} d^2\sigma \sqrt{h} h^{\alpha\beta} g^{\mu\nu} \partial_\alpha X_\mu \partial_\beta X_\nu \tag{1.1}$$

Is invariant under worldsheet reparameterizations, Weyl scaling and isometries of the target space-time metric. When the worldsheet admits a globally flat metric, the derivation of the equations of motion and the quantization of the string proceed according to the calculation in flat space. The worldsheet can be curved, however, and the expansion of the string coordinate field in a mode expansion consists of a set set of terms. Nevertheless, the leading terms are equal to those on a flat worldsheet and the remaining terms have a similar ofrm determined by the uniformizing group of the surface with Euclidean signature, such that the derivation of the critical dimension and the normal ordering constant in the Lorentz algebra is identical.
A similar conclusion is reached for conformally flat target space metrics. The equations of motion for the coordinate field X^μ are equivalent, and the operatorsin the quantization of the theory will have the same expansion. Consequently, the commutators occurring in the Lorentz algebra will yield equal values of the dimension and normal ordering constant. It follows that the critical dimension of a string in a conformally flat space-time would be equal to that derived for the propagation in flat space-time. Consequently, the critical dimension of the bosonic string would be equal in Minkowski and anti-de Sitter space-time.

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The expected values are supported by calculations of the expectation values of massless gravitons. It is demonstrated that the critical dimensions for these state remains 26. A Weyl anomaly cancellation produces a result which can be explained through other investigations of string theory in anti-de Sitter space. This dimension would be affected by the presence of the conformal factor in the metric. There exists also a conformal factor which interpolates flat and anti-de Sitter geometries confirming the same critical dimensions for both space-times.

2. **Expectation Values of Operators in Anti-de Sitter Space**

The reparameterizations of the bosonic string action are given by

$$\begin{aligned} \delta X^\mu &= \xi^\alpha \partial_\alpha X^\mu \\ \delta h^{\alpha\beta} &= \xi^\gamma \partial_\gamma h^{\gamma\beta} - \partial_\gamma \xi^\beta h^{\alpha\gamma} \\ \delta(\sqrt{h}) &= \partial_\alpha(\xi^\alpha \sqrt{h}) \end{aligned} \tag{2.1}$$

$$\begin{aligned} \delta S = &-\frac{T}{2}\int_\Sigma d^2\sigma \Big[\delta(\sqrt{h}) h^{\alpha\beta} g^{\mu\nu} \partial_\alpha X_\mu \partial_\beta X_\nu + \sqrt{h}\delta(h^{\alpha\beta}) g^{\mu\nu} \partial_\alpha X_\mu \partial_\beta x_\nu \\ &\quad + \sqrt{h} h^{\alpha\beta} g^{\mu\nu} \partial_\alpha(\delta X_\mu) \partial_\beta X_\nu + \sqrt{h} h^{\alpha\beta} g^{\mu\nu} \partial_\alpha X_\mu \partial_\beta(\delta X_\nu) \Big] \\ = &-\frac{T}{2}\int_\Sigma d^2\sigma \Big[-\xi^\gamma \sqrt{h} \partial_\gamma h^{\alpha\beta}\, \partial_\alpha X_\mu \partial_\beta X_\nu - \xi^\gamma \sqrt{h} h^{\alpha\beta} g^{\mu\nu} \partial_\gamma \partial_\alpha X_\mu \partial_\beta X_\nu \\ &\quad - \xi^\gamma \sqrt{h} h^{\alpha\beta} g^{\mu\nu} \partial_\alpha X_\mu \partial_\gamma \partial_\beta X_\nu + \partial_\gamma(\xi^\gamma \sqrt{h} h^{\alpha\beta} g^{\mu\nu} \partial_\alpha X_\mu \partial_\beta X_\nu) \\ &\quad + \sqrt{h} \xi^\gamma \partial_\gamma h^{\alpha\beta} g^{\mu\nu} \partial_\alpha X_\mu \partial_\beta X_\nu - \sqrt{h} \xi^\gamma \partial_\gamma h^{\alpha\beta} g^{\mu\nu} \partial_\alpha X_\mu \partial_\beta X_\nu \\ &\quad - \sqrt{h} h^{\alpha\gamma} g^{\mu\nu} \partial_\gamma \xi^\beta \partial_\alpha X_\mu \partial_\beta X_\nu + \sqrt{h} h^{\alpha\beta} g^{\mu\nu} (\partial_\alpha \xi^\gamma)(\partial_\gamma X_\mu)(\partial_\beta X_\nu) \\ &\quad + \sqrt{h} h^{\alpha\beta} g^{\mu\nu} (\partial_\beta \xi^\gamma)(\partial_\alpha X_\mu)(\partial_\gamma X_\nu) + \sqrt{h} h^{\alpha\beta} g^{\mu\nu} \xi^\gamma \partial_\alpha \partial_\gamma X_\mu \partial_\beta X_\nu \\ &\quad + \sqrt{h} h^{\alpha\beta} g^{\mu\nu} \partial_\alpha X_\mu \xi^\gamma \partial_\beta \partial_\gamma X_\nu \Big] \\ = &-\frac{T}{2}\int_{\partial\Sigma} d\ell_\gamma \xi^\gamma \sqrt{h} h^{\alpha\beta} g^{\mu\nu} \partial_\alpha X_\mu \partial_\beta \partial_\gamma X_\nu. \end{aligned} \tag{2.2}$$

This integral vanishes if the surface has no boundary or $X_\alpha \to 0$ at the boundary. The transformation of the metric under Weyl scaling is

$$\begin{aligned} \delta h^{\alpha\beta} &= \Lambda h^{\alpha\beta} \\ \delta h_{\alpha\beta} &= -\Lambda h_{\alpha\beta} \\ \delta(\sqrt{h}) &= -\Lambda \sqrt{h} \end{aligned} \tag{2.3}$$

and

$$\delta S = -\frac{T}{2}\int_{\Sigma} d^2\sigma\ \sqrt{h}h^{\alpha\beta}g^{\mu\nu}\partial_\alpha X_\mu \partial_\beta X_\nu - \frac{T}{2}\int_{\Sigma} d^2\sigma\ \sqrt{h}\ \delta h^{\alpha\beta}\partial_\alpha X_\mu \partial_\beta X_\nu \tag{2.4}$$
$$= -\frac{T}{2}\int_{\Sigma} d^2\sigma\ (-\Lambda\sqrt{h})j^{\alpha\beta}g^{\mu\nu}\partial_\alpha X_\mu \partial_\beta X_\nu - \frac{T}{2}\int_{\Sigma} d^2\sigma\ \sqrt{h}\ \Lambda h^{\alpha\beta}g^{\mu\nu}\partial_\alpha X_\mu \partial_\beta X_\nu$$
$$= 0.$$

On anti-de Sitter space in d dimensions,

$$\delta X^\mu = M^\mu{}_\nu X^\nu \tag{2.5}$$
$$\delta g^{\mu\nu} = M_\rho{}^\mu g^{\rho\nu} + M_\rho{}^\nu g^{\mu\rho}$$
$$\delta X_\mu = -M_\mu{}^\nu X_\nu.$$

The operators $M_{\mu\nu}$ are generators of the $so(d-1,2)$ algebra and

$$\delta S = -\frac{T}{2}\int_{\Sigma} d^2\sigma\ \sqrt{h}h^{\alpha\beta}\delta g^{\mu\nu}\partial_\alpha X_\nu \partial_\beta X_\nu \tag{2.6}$$
$$-\frac{T}{2}\int_{\Sigma} d^2\sigma\ \sqrt{h}h^{\alpha\beta}g^{\mu\nu}\partial_\alpha (\delta X_\mu)\partial_\beta X_\nu$$
$$-\frac{T}{2}\int_{\Sigma} d^2\sigma\ \sqrt{h}h^{\alpha\beta}g^{muv}\partial_\alpha X_\mu \partial_\beta (\delta X_\nu)$$
$$= -\frac{T}{2}\int_{\Sigma} d^2\sigma\ \sqrt{h}h^{\alpha\beta}(M_\rho{}^\mu g^{\rho\nu} + M_{\rho\nu}g^{\mu\rho})\partial_\alpha X_\mu \partial_\beta X_\nu$$
$$+\frac{T}{2}\int_{\Sigma} d^2\sigma\ \sqrt{h}h^{\alpha\beta}g^{\rho\nu}M_\rho{}^\mu \partial_\alpha X_\mu \partial_\beta X_\nu$$
$$+\frac{T}{2}\int_{\Sigma} d^2\sigma\ \sqrt{h}h^{\alpha\beta}g^{\mu\rho}\partial_\alpha X_{\mu\beta} X_\nu$$
$$= 0.$$

In flat space, the functional derivatives of the action on a surface Σ with a metric $h_{\alpha\beta}$ are

$$\frac{\delta S}{\delta X^\mu} = 0 \tag{2.7}$$
$$\frac{\delta S}{\delta \partial_\alpha X^\mu} = -T\sqrt{h}h^{\alpha\beta}\partial_\beta X_| u$$

and the equation of motion is

$$\Delta X_\mu = \frac{1}{\sqrt{h}}\partial_(\sqrt{h}h^{\alpha\beta}\partial_\beta X_\mu) = 0. \tag{2.8}$$

Locally, the metric on the surface can be set equal to $\eta_{\alpha\beta}$ and the Euclidean form of the equation reduces to

$$\Delta X_\mu = \left(\frac{\partial^2}{\partial\sigma^2} + \frac{\partial^2}{\partial\tau^2}\right) X_\mu = 0. \tag{2.9}$$

which does not represent the global dynamics over a closed Riemann surface.
The Lorentzian equation is

$$\Box X_\mu = \left(\frac{\partial^2}{\partial\sigma^2} - \frac{\partial^2}{\partial\tau^2}\right) X_\mu = 0. \tag{2.10}$$

For a light-cone diagram, the worldsheet is flat except for curvature at the interaction points. Given that the worldsheet can be represented as the quotient of two-dimensional Minkowski space-time by Γ_{Lor}. The solution to the wave equation then would be given by
The solution to the wave equation on a Riemann surface would be given by the automorphic functions on the covering space. The sphere can be stereographically projected to the extended complex plane and the universal covering of the torus is $\mathbb{C}$ with coordinates $\sigma^+ = \sigma + i\tau$ and $\sigma^- = \sigma - i\tau$. The mode expansion is

$$X^\mu(\sigma, i\tau) = x^\mu + p^\mu(i\tau) + i\sum_{n\neq 0} \frac{1}{n}\alpha_n^\mu e^{-in(i\tau)} cos\ n\sigma. \tag{2.11}$$

Then

$$X^\mu(\sigma, i\tau) = \frac{x^\mu}{2} + \frac{p^\mu}{2}(\sigma + i\tau) + \frac{x^\mu}{2} + \frac{p^\mu}{2}(i\tau - \sigma) + i\sum_{n\neq 0}\frac{1}{n}\alpha_n^\mu(cosh\ n\tau + sinh\ n\tau)\ cos\ n\sigma \tag{2.12}$$

$$= \frac{x^\mu}{2} + \frac{p^\mu}{2}\sigma^+ + \frac{x^\mu}{2} + \frac{p^\mu}{2}\sigma^- + i\sum_{n\neq 0}\frac{1}{n}\alpha_n^\mu[cos(in\tau) + i\ sin(in\tau)]\ cos(n\sigma)$$

$$= \frac{x^\mu}{2} + \frac{p^\mu}{2}\sigma^+ + \frac{x^\mu}{2} + \frac{p^\mu}{2}\sigma^- + \frac{i}{2}\sum_{n\neq 0}\alpha_n^\mu[e^{in\sigma^+} + e^{in\sigma^-}].$$

On the torus, this mode expansion has to be modified to be

$$X^\mu(\sigma, i\tau) = \sum_{\gamma\in\Gamma}\left[x^\mu + \frac{p^\mu}{2}(\gamma\cdot\sigma^+) + \frac{p^\mu}{2}(\gamma\cdot\sigma^-)\right] + \frac{i}{2}\sum_{n\neq 0}\alpha_n^\mu[e^{in\gamma\cdot\sigma^+} + e^{in\gamma\cdot\sigma^-}]. \tag{2.13}$$

In the light-cone gauge, in flat space, $X^+(\sigma, i\tau) = x^+ + p^+(i\tau)$. On the torus,

$$X^+(\sigma, i\tau) = x^+ + p^+(i\tau) + \sum_{\gamma\neq I}(x^+ + p^+(\gamma\cdot(i\tau))). \tag{2.14}$$

Similarly,

$$X^-(\sigma, i\tau) = x^- + p^-(i\tau) + \sum_{n\neq 0}\frac{1}{n}\alpha_n^- e^{n\tau} cos\ n\sigma \tag{2.15}$$

$$+ \sum_{\gamma\neq I}\left[x^- + p^-(\gamma\cdot i\tau) + i\sum_{n\neq 0}\frac{1}{n}\alpha_n^- e^{n\gamma\cdot\tau} cos\ n(\gamma\cdot\sigma^-)\right].$$

The Lorentzian mode expansions would be

$$X^+(\sigma, \tau) = x^+ + p^+\tau + \sum_{\gamma_{Lor}\neq I}(x^+ + p^+(\gamma\cdot\tau)) \tag{2.16}$$

and

$$X^-(\sigma, \tau) = x^- + p^-\tau + i\sum_{n\neq 0}\frac{1}{n}\alpha_n^\mu e^{-in\tau} cos\ n\sigma \tag{2.17}$$

$$+ \sum_{\gamma_{Lor}\neq I}\left[x^- + p^-(\gamma\cdot\tau) + i\sum_{n\neq 0}\frac{1}{n}\alpha_n^- e^{-in(\gamma\cdot\tau)} cos(n\gamma\cdot\sigma)\right]$$

where

$$\alpha_n^- = \frac{1}{p^+}\left(\frac{1}{2}\sum_{i=1}^{d-2}\sum_{n=-\infty}^{\infty} : \alpha_{n-m}^I \alpha_m^i : -a\delta\right) \tag{2.18}$$

With a being a normal ordering constant. Since the expansion of the light-cone coordinates has the Same leading terms and the remaining terms have a form determined by the uniformizing group Γ_{Lor}, with the same coefficients α_n^-, the derivation of d and a is identical. With $J^{\mu\nu} = \ell^{\mu\nu} + E^{\mu\nu}$, $\ell^{\mu\nu} = x^\mu p^\nu - x^\nu p^\mu$ and $E^{\mu\nu} = -i\sum_{n=1}^{\infty}\frac{1}{n}(\alpha_{-n}^\mu \alpha_n^\nu - \alpha_{-n}^\nu \alpha_n^\mu)$, and the normal ordering $\frac{1}{2}\sum_{i=1}^{d-2}\sum_{n=-\infty}^{\infty}\alpha_{-n}^I\alpha_n^I = \frac{1}{2}\sum_{n=1} d - 2\sum_{n=-\infty}^{\infty} : \alpha_{-n}^I \alpha_n^i : + \frac{d-2}{2}\sum_{n=1}^{\infty} n = \frac{1}{2}\sum_{n=1}^{d-2}\sum_{n=-\infty}^{\infty} : \alpha_{-n}^I\alpha_n^i : -\frac{d-2}{24}$, it follows that $[J^{i-}, J^{j-}] = \frac{1}{(p^+)^2}\sum_{n=1}^{\infty}\Delta_m)\alpha_{-m}^i\alpha_m^j - \alpha_{-m}^j\alpha_m^i)$, where $\Delta_m = m\left(\frac{26-d}{2}\right) + \frac{1}{m}\left[\frac{d-26}{12} + 2(1-a)\right]$ and $\Delta_m = 0$ if $d = 26$ and $a = 1$.

The effect of introducing a conformally flat metric is similar. Given that $g^{\mu\nu} = \Omega^{-2}\eta^{\mu\nu}$ and $g_{\mu\nu} = \Omega^2\eta_{\mu\nu}$, the action is

$$S = -\frac{T}{2}\int_\Sigma d^2\sigma\,\sqrt{h}h^{\gamma\delta}\Omega^{-2}\eta^{\mu\nu}\partial_\gamma X_\mu \partial_\delta X_\nu \tag{2.19}$$

and

$$\partial_\alpha \frac{\delta S}{\delta\partial_\alpha X^\mu} = -T\partial_\alpha(\sqrt{h}h^{\alpha\beta}\Omega^2\eta_{\mu\nu}\partial_\beta X^\nu) = -T\Omega^2\eta_{\mu\nu}\partial_\alpha(\sqrt{h}h^{\alpha\beta}\partial_\beta X^\nu) \tag{2.20}$$

Since $\eta_{\mu\nu} = diag(-1, 1, ..., 1)$, each component separately solves the Laplace equation $\Delta X^\mu = 0$. The mode expansion, the coefficients and the operators will be identical. By normal ordering, $d = 26$ and $a = 1$. It follows that the critical dimension will be 26 for the Minkowski space-time, anti-de Sitter space and the geometry with a smooth interpolating factor.

Quantization in anti-de Sitter space is derived from

$$S^{(2)} = -\frac{1}{2\pi\alpha'}\int d\tau d\sigma\left[\sum_{R=1}^{d-1}\eta^{ab}(\delta x^R)_{,a}(\delta x^R)_{,b} = \frac{m^2\alpha'^2}{\ell^2}\delta x^R \delta x_R\right] \tag{2.21}$$

When the time coordinate δx^0 of the string in the embedding space is not included. The mass formula would be given by

$$m^2\alpha'^2 = \sum_{n>0}\left(2n^2 + \frac{m^2\alpha'^2}{\ell^2}\right)\sum_{R=1}^{d-1}\left[(a_n^R)^\dagger(a_n^R) + (a_n^R)(a_n^R)^\dagger + (\tilde{a}_n^R)^\dagger(\tilde{a}_n^R) + (\tilde{a}_n^R)(\tilde{a}_n^R)^\dagger\right]$$
$$(L_0 - 2\pi\alpha' a)|\psi\rangle = (\tilde{L}_0 - 2\pi\alpha' a)|\psi\rangle = 0,$$

where the commutation relations of the annihilation and creation operators are

$$[a_m^R, (a_n^S)^\dagger] = \frac{\alpha'}{2\sqrt{n^2 + \frac{m^2\alpha'^2}{\ell^2}}}\delta_{mn}\delta^{RS} \qquad [\tilde{a}_m^R, (\tilde{a}_n^S)^\dagger] = \frac{\alpha'}{2\sqrt{n^2 + \frac{m^2\alpha'^2}{\ell^2}}}\delta_{mn}\delta^{RS} \tag{2.22}$$
$$[a_m^R, a_n^S] = 0 \qquad [\tilde{a}_m^R, \tilde{a}_n^S] = 0$$
$$[(a_m^R)^\dagger, (a_n^S)^\dagger] = 0 \qquad [\tilde{a}_m^R)^\dagger, (\tilde{a}_n^S)^\dagger] = 0,$$

and, with $\lambda = \frac{\alpha'}{\ell^2}$,

$$(L_0 + \bar{L}_0)|\psi\rangle = \Big\{\pi \sum_n 2(2n^2 + m^2\alpha'\lambda) \sum_{R=1}^{d-1} \frac{2\alpha'}{\sqrt{n^2 + m^2\alpha'\lambda}} [(\alpha_n^R)^\dagger \alpha_n^R + (\tilde{\alpha}) n^R)^\dagger \tilde{\alpha}_n^R] \tag{2.23}$$
$$+ \pi \sum_{n>0} 2(2n^2 + m^2\alpha'\lambda) \frac{2\alpha'}{\sqrt{n^2 + m^2\alpha'\lambda}} (d-1) \Big\} |\psi\rangle.$$
$$= 4\pi\alpha' a|\psi\rangle$$

Setting $a = m^2\alpha'$,

$$m^2\alpha' = (d-1)\sum_{n>0} \frac{2n^2 + m^2\alpha'\lambda}{\sqrt{n^2 + m^2\alpha'\lambda}} + \sum_{n>0} \frac{2n^2 + m^2\alpha'\lambda}{\sqrt{n^2 + m^2\alpha'\lambda}} \sum_{R=1}^{d-1} [(\alpha_n^R)^\dagger \alpha_n^R + (\tilde{\alpha}_n^R)^\dagger \tilde{\alpha}_n^R]. \tag{2.24}$$

Since $\alpha_n^R|0\rangle = \tilde{\alpha} - n^R|0\rangle = 0$ for the vacuum, the expectation value of $m^2\alpha'$ in the spin-2 state $(\tilde{\alpha}_1^R)^\dagger(\alpha_1^S)^\dagger|0\rangle = |\Omega_{11}^{RS}\rangle$ is

$$\langle 0|\alpha_1^S \tilde{\alpha}_1^R m^2\alpha' (\tilde{\alpha}_1^R)^\dagger (\alpha_1^S)^\dagger|0\rangle \tag{2.25}$$
$$= \langle 0|\alpha_1^2 \tilde{\alpha}_1^R (d-1)\Big[-\frac{1}{6} + \frac{(m^2\alpha'^2)^2}{4}\zeta(3)\lambda^2 - \frac{(m^2\alpha')^3}{4}\zeta(5)\lambda^3 + O(\lambda^4)\Big](\tilde{\alpha}_1^R)^\dagger(\alpha_1^S)^\dagger|0\rangle$$
$$\sum_{n>0} \frac{2n^2 + m^2\alpha'\lambda}{\sqrt{n^2 + m^2\alpha'\lambda}} \sum_{Q=1}^{d-1} \langle 0|\alpha_1^S \tilde{\alpha}_1^R [(\alpha_n^Q)^\dagger \alpha_n^Q + (\tilde{\alpha}_n^Q)^\dagger \tilde{\alpha}_n^Q](\tilde{\alpha}_1^R)^\dagger(\alpha_1^S)^\dagger|0\rangle$$
$$= (d-1)\Big[-\frac{1}{6} + \frac{(m^2\alpha')^2}{4}\zeta(3)\lambda^2 - \frac{(m^2\alpha')^3}{4}\zeta(5)\lambda^3 + O(\lambda^4)\Big] + \sum_{n>0} \frac{2n^2 + m^2\alpha'\lambda}{\sqrt{n^2 + m^2\alpha'\lambda}} \cdot 2\delta_{1n}$$
$$= (d-1)\Big[-\frac{1}{6} + \frac{(m^2\alpha')^2}{4}\zeta(3)\lambda^2 - \frac{(m^2\alpha')^3}{4}\zeta(5)\lambda^3 + O(\lambda^4)\Big] + 2\frac{2 + m^2\alpha'\lambda}{\sqrt{1 + m^2\alpha'\lambda}}$$

and, setting the expectation value equal to zero,

$$-\frac{d-1}{6} + 4 = 0 \tag{2.26}$$
$$d = 25.$$

However, if $m^2\alpha' = 0$, a would be zero and the normal ordering would be dependent on the state. By contrast, the value of a required for the masslessness of the graviton and the cancellation of anomalies in Minkowski space-time is constant.
If the term $-\eta^{ab}(\delta x^0)_{,a}(\delta x^0)_{,b}$ is added to the action, the contribution To the expectation value of $L_o + \bar{L}_0$ is $-\sum_{n+0} \frac{2n^2+m^2\alpha'\lambda}{\sqrt{n^2+m^2\alpha'\lambda}}$. yielding the equation $-\frac{d-2}{6} + 4 = 0$ and $d = 26$.
The expectation value of $m^2\alpha'$ in a vector state $(\alpha_1^R)^\dagger|0\rangle = |\Omega_1^R\rangle$ in the initial theory without the time coordinate would be

$$\begin{aligned}
&\langle 0|\alpha_1^R m^2\alpha'(\alpha_1^R)^\dagger|0\rangle \\
&= \langle 0|\alpha_1^R(d-1)\Big[-\frac{1}{6}+\frac{(m^2\alpha')^2}{4}\zeta(3)\lambda^2-\frac{(m^2\alpha')^3}{4}\zeta(5)\lambda^3+O(\lambda^4)\Big](\alpha_1^R)^\dagger|0\rangle \\
&\quad+\sum_{n>0}\frac{2n^2+m^2\alpha'\lambda}{\sqrt{n^2+m^2\alpha'\lambda}}|0|\alpha_1^R[(\alpha_n^Q)^\dagger\alpha_n^Q+(\tilde\alpha_n^Q)^\dagger\tilde\alpha_n^Q](\tilde\alpha_n^R)^\dagger\Big]|0\rangle \\
&= (d-1)\Big[-\frac{1}{6}+\frac{(m^2\alpha')^2}{4}\zeta(3)\lambda^2-\frac{(m^2\alpha')^2}{4}\zeta(5)\lambda^3+O(\lambda^4)\Big]+\frac{2+m^2\alpha'\lambda}{\sqrt{1+m^2\alpha'\lambda}}.
\end{aligned} \tag{2.27}$$

Then the condition for vanishing of the expectation value is

$$\begin{aligned}
-\frac{d-1}{6}+2=0 \\
d=13.
\end{aligned} \tag{2.28}$$

When the time coordinate is included, the expectation value of the square mass vanishes if $-\frac{d-2}{6}+2=0$ or $d=14$. A massless vector field is likely to arise in a super-Yang-Mills theory, and therefore, it could be included in a superstring theory. There is a difference, however, between this dimension and the critical dimension for the formulation of the path integral of the superstring in anti-de Sitter space. The added dimensions will be explained in §4,

3. **The Liouville Theory**

Consider the partition function of string theory

$$Z\sim\sum_{topologies}\int Dh;DXe^{-S} \tag{3.1}$$

Gauge fixing the worldsheet metric to be $h_{\alpha\beta}=e^{\phi}\delta_{\alpha\beta}$, ϕ is a Liouville field. Then

$$\begin{aligned}
Z&=\int[d\tau]D_h\phi D_h(gh.)D_hX\, e^{-S_\mu-S_{gh}-\frac{\mu g}{2\pi}\int d^2\xi\sqrt{h}} \\
&=\int[d\tau]J(,\hat h)D_{\hat h}D_{\hat h}(gh.)D_h(X)e^{-S_\mu-S_{gh.}-\frac{\mu g}{2\pi}\int d^2\xi\sqrt{h}}.
\end{aligned} \tag{3.2}$$

The Jacobian has the form

$$J(\phi,\hat h)=e^{-\int d^2\xi\sqrt{h}\left(\tilde a\tilde h^{ab}\partial_a\phi\partial_b\phi+\tilde b\hat R\phi+\mu e^{\tilde c\phi}\right)}. \tag{3.3}$$

Under the variations $\delta\hat h=\epsilon(\xi)\hat h$ and $\delta\phi=-\epsilon(\xi)$, the variation produces the term $\left(\frac{d-25}{48\pi}+\tilde b\right)\int d^2\xi\hat R\xi$ And $(2\tilde a-b)\int d^2\xi\sqrt{\hat h}\,\epsilon\Box\phi$ and invariance requires $\tilde a-\frac{1}{2}\tilde b$ and $\tilde b=\frac{25-d}{48\pi}$. When $d=25$, the Jacobian is reduced to $e^{-\mu\int d^2\xi\sqrt{\hat h}e^{\tilde c\phi}}$. However, it is not necessary for the dimension to be 25, and the above coefficients yield

$$J(\phi,\hat h)=e^{-\frac{25-D}{48\pi}\int d^2\xi\sqrt{h}\left(\frac{1}{2}\tilde h^{ab}\partial_a\phi\partial_b\phi+\tilde R\phi\right)-\mu\int d^2\xi\,\sqrt{h}\,e^{\tilde c\phi}} \tag{3.4}$$

When these considerations are extended to the supersymmetric Liouville model, the superspace integral arising from the transformation of the metric in the partition function introduces the factor

$$J(\Phi, H) = e^{-\frac{d-9}{32\pi}\int d^2x d^2\theta\ (sdet\ H)^{\frac{1}{2}}\ \left(\frac{1}{2}D^{\alpha}\Phi D_{\alpha}\Phi + \hat{R}\Phi\right) + 2i\mu \int d^2x d^2\theta\ (sdet\ H)^{\frac{1}{2}} e^{\frac{\Phi}{2}}} \tag{3.5}$$

Again, the Jacobian is a numerical factor in the $d = 9$, which is one less than the critical dimension of the superstring. Invariance of the Jaoobian under transformations of the metric on the super-Riemann surface, H, and the scalar superfield Φ, also may be achieved in dimensions other than 9.

The Liouville model is constructed with the two-dimensional metrics restricted to be conformally flat, and this restriction generates the Jacobian in the path integral. All surfaces have a locally conformally flat metric. However, the field must not be defined globally on the surface, and the dynamics are no longer directly connected to the path integral of string theory. Therefore, reduction of the critical dimension by one in the Liouville field theory does not immediately reflect the conditions for a consistent quantization of strings.

It may be demonstrated that the bosonic string action in three dimensional anti-de Sitter space is equivalent to a Liouville action [14]. The embedding of the string worldsheets in the conformally flat space-time allows the introduction of a global scalar field determining the dynamics of the string. However, a set of scalar fields representing coordinates of the string is not defined, and therefore, the invariances of the path integral will not determine a parameter representing the dimension of the embedding space-time. Instead, the model can be quantized consistently in three dimensions.

Furthermore, In the critical dimension of string theory, there is a decoupling of the Weyl field from the other fields in the Liouville action. The role of this field is similar to the conformal factor in Euclidean gravity. The decoupling allows the formulation of the path integral which would be free from divergences arising from the integration over this variable.

4. A Resolution to the Problem of the Dimension of String Propagation in Anti-de Sitter Space

The theoretical derivation of the dimension of a consistent quantum theory of strings in anti-de Sitters pace may be given in terms of the path integral. The conformal factor yields a coefficient multiplying the Liouville action in the path integral which vanishes in a critical dimension. However, this factor will be altered in anti-de Sitters space, and it is zero in another dimension.

The partition function for the bosonic string in anti-de Sitter space is

$$\begin{aligned} Z &= \int DhDXDy\ e^{-S} \\ S &= \frac{R^2}{4\pi}\int d^2\xi\sqrt{h}h^{ab}\frac{\partial_a X^I \partial_b X^i}{y^2} + \frac{R^2}{4\pi}\int d^2\xi\sqrt{h}h^{ab}\frac{\partial_a y \partial_b y}{y^2}. \end{aligned} \tag{4.1}$$

in conformally flat coordinates with the line element

$$ds^2 = \frac{R^2}{y^2}\left(dy^2 + \sum_{i=1}^{d}(dX^i)^2\right) \tag{4.2}$$

Then

$$\begin{aligned} Z &= \int D\phi \int Df\, e^{-\frac{d+2}{8\pi}\int d^2\xi\left[f\left(1-ln\left(\frac{f}{\Lambda^2}\right)+\phi\right)-\frac{1}{2}(\partial_a\phi)^2+\lambda e^{\phi}\right]} Det_{FP}^{\frac{1}{2}} \\ &= \int D\phi\, e^{d-2496\pi\int d^2\xi[(\partial_a\xi)^2+\lambda e^{\phi}]} \int Df\, e^{-\frac{d+2}{8\pi}\int d^2\xi\, f\left(1-ln\left(\frac{f}{\Lambda^2}\right)+\phi\right)}, \end{aligned} \tag{4.3}$$

where $y(\xi) = e^{\chi(\xi)}$ and $f(\xi) = h^{ab}\partial_a\chi\partial_b\chi + \Delta\chi$ and $h_{ab} = e^{\phi}\delta_{ab}$ [7]. A diagrammatic evaluation of the series expansion of the integral over f yields a formula for the coefficient multiplying the Liouville action

$$C = \frac{d_c - 24}{96\pi} + \frac{1}{24\pi} - 0.003 + ... \tag{4.4}$$

which tends to zero as $d_c \to 21$ or $d_c + 1 \to 22$. The propagation of the string a decoupling of the conformal ode would be valid in 22 dimensions.

The difference between theis valud and that found through the presence of a massless graviton In the physical spectrum can be explained through the physical effect of flattening the space-time Through which a string propagates in a maximally curved space-time. It may be concluded that the Hamiltonian of a shift in the four-dimensional mass and momentum scales in the bosonic string Hamiltonian in anti-de Sitter space is cancelled by a ground state contribution, and it follows that propagation of the string in the four dimensions can be regarded as being equivalent to that of flat Space [8]. Consider adding four coordinates representing anti-de Sitter space. The dynamics of the string in the space spanned by these coordinates would be equivalent to that of flat space. Removal of the time coordinate, with a restriction to the three-dimensional conformally flat subspace, and the union with the 22 coordinates of the initial anti-de Sitter space yields 25 dimensions. When the time coordinate is preserved, together with a Wick rotation to ensure that the entire space-time has a Lorentz metric, the sum of the dimensions equals 26. The mass of the graviton would be zero in these extra dimensions, and therefore, consistency of the physical spectrum with the vanishing of the conformal anomaly is restored. This method may be used only to remove four anti-de Sitter coordinates, and it cannot be iterated, since the flattening effect resulting from the cancellation in the effect of the curvature on the Hamiltonian energy and the zero-point of the mass scale cancel in four dimensions.

The algebra of the annihilation and creation of string and the existence yields a massless vector in the physical spectrum in 14 dimension, with the time coordinate of the string in the embedding space-time In the worldsheet action, and a reduction by four dimensions would give the 10 dimensions of superstring theory. It is the critical dimension in flat space, and quantum consistency of superstring in $AdS_5 \times S^5$ may be established through other method [2]. These Techniques include the formulatuion of a Green-Schwarz action with manifest space-time supersymmetry, The κ-symmetry of this action is equivalent to the classical BRST invariance of a pure spinor action that differs by an integral of a

BRST variation. It is claimed that classical BRST invariance of the pure spinor produces equations that are sufficient to prove cancellation of the Weyl anomaly and conformal invariance at one loop [4]. However, an extra term $\frac{1}{2\pi\alpha'}\int d^2z\, \frac{1}{2}\alpha'\Phi(Z)$ is necessary at the quantum level, and the relation with κ-symmetry of the Green-Schwarz action and classical BRST invariance of the pure spinor action cannot be proven directly. The $AdS_5 \times S^5$ pure spinor action without worldsheet curvature term

$$\begin{aligned} S_{AdS_5\times S^5} &= S_{GS} + S_{gh} + \int d^2z\ \left(d_\alpha \bar{L}^\alpha + d_{\dot{\alpha}} L^{\dot{\alpha}} - \frac{1}{2} d_\alpha \bar{d}_{\dot{\beta}} F^{\alpha\dot{\beta}}\right) \\ S_{GS} &= \int d^2z\ \left[\frac{1}{2}\eta_{mn} L^m \bar{L}^n + \int dy\ \epsilon^{IJK}(\gamma_{m\alpha\beta} L_I^m L_J^\alpha L_K^\beta + \gamma_{m\dot{\alpha}\dot{\beta}} L_I^m L_J^{\dot{\alpha}} L_K^{\dot{\beta}}\right] \\ S_{gh} &= \int d^2z \left[L_{gh}^{free} + \frac{1}{2} N_{mn}\bar{L}^{mn} + \frac{1}{2}\hat{N}_{mn} L^{mn} + \frac{1}{4} N_{mn}\hat{N}_{pq} R^{mnpq}\right] \\ F^{\alpha\dot{\beta}} &= \frac{1}{120} F^{m_1\ldots m_5}(\gamma_{m_1\ldots m_5})^{\alpha\dot{\beta}} \\ R^{mnpq}(\gamma_{pq})_\alpha{}^\beta &= \gamma^m{}_{\alpha\gamma} F^{\gamma\dot{\delta}} \gamma^n{}_{\dot{\delta}\kappa} F^{\beta\kappa} - \gamma^n{}_{\alpha\gamma} F^{\gamma\dot{\delta}}\gamma^m{}_{\dot{\delta}\kappa} F^{\beta\kappa} \end{aligned} \tag{4.5}$$

is classically invariant because $\bar{\partial}(\lambda^\alpha d_\alpha) = \bar{p}artial(\hat{\lambda}^{\dot{\alpha}} d_{\dot{\alpha}}) = 0$ and $\{Q_{BRST}, Q_{BRST}\} = \{\hat{Q}_{BRST}, \hat{Q}_{BRST}\} = \{Q_{BRST}, \hat{Q}_{BRST}\} = 0$. However, this invariance Is no longer proven similarly when the worldsheet curvature term is included.

5. **Finite-Volume Corrections to the Masses**

There exists a curvature expansion of the kinetic and Wess-Zumino terms in the Green-Schwarz action of a superstring on $AdS_5 \times S^5$. The energy levels can be evaluated by calculating the expectation values of each of the bosonic and fermionic contributions to the Hamiltonian [15]. The terms in the expansion of the energy in powers of $\frac{1}{R}$, where R is the radius of curvature, would generate fine structure for sufficiently large R. A truncated series reduces the symmetry from the conformal group SO(4, 2) to the anti-de Sitter group SO(3, 2), and the corrections do not represent the energy levels of an N = 4 super-Yang-Mills theory precisely. Finite-volume corrections to the mass of a stable state are proportional to $e^{-\frac{\sqrt{3}}{2}\bar{m}L}$, where $\bar{m}$ is the mass gap and L is the linear extent, with the proportionality constant being given by the forward scattering amplitude of the lightest particles in the infinite-volume limit, such that mass splitting provides information about the phase shifts in an infinite volume and the width of the resonance because unstable particles can mix with the scattering states of their decay products in a finite volume [13]. Finite-volume effects would be reduced exponentially as L $\to \infty$.
Analyticity of Schwinger functions in the coupling depends on the finiteness of $\frac{\epsilon}{m_0^2}$, which is necessary for the cluster expansion [5]. The cluster expansion is used in the proof of the existence of the infinite-volume limit [11]. With the mass gap, certain Schwarz distributions can be shown to decrease rapidly. In Wightman theory with a mass $M > 0$, the support of the Wightman function $W^T(p_1, ..., p_n)$ is contained in $\sum_{\ell=0}^n p_\ell]inV_+^n, \sum_{\ell=1}^n p_\ell =$

0, Inequalities such as $\| \frac{d}{dt} \prod_{i=1}^{n} A^*(f_i, t)|0]\rangle \| < C_N(1 + |t|)6-N$, $C_N < 0$, for a nonoverlapping set of functions $\{\tilde{f}_1, \ldots \tilde{f}_n\}$, $\tilde{f}_j \in S(G_m)$ require a positive mass, since $|\tilde{f}_1, ..., \tilde{f}_n^{ex}\rangle$ may be approximated by $\prod_j = 1^m A^*(f_j, t)|0\rangle$. The existence of a Poincare-invariant vacuum and generalized physical states,

$$|p_{1\alpha_1}, \ldots, p_{F\alpha_F}; q_1\rho_1, ..., q_n\rho_n\rangle = b^{*(\epsilon_1)}_{\lambda_F\zeta_F}(p_1)...b^{*(\epsilon_F}_{\lambda_F\zeta_F}(p_F)a^*_{\rho_1}(q_1)...a^*_{\rho_n}(q_n)|0\rangle \quad (5.1)$$
$$\alpha_i = \{\lambda_i, \zeta_i, \epsilon_i\},\ \epsilon_i = 1,$$

depends on positive masses and energies [?]. The cluster expansion, which is necessary for the infinite-volume limit, requires the analyticity of the Schwinger function with a finite mas gap. If there is no mass gap, the validity of the cluster expansion at finite volumes is unproven.

Finiteness conditions for the integral of the stochastic measure generate inequalities required for a non-zero probability of an interaction. The extension to Yang-Mills theories follows from the existence of a stochastic measure analogous to a scalar field theory. Given that a similar constraint follows in a Yang-Mills theory, the finiteness of the interaction region may be explained. The integrals in the Kallen representation would give infinite energies for black body radiation even if the energies are quantized. Therefore, is physically viable to use finite volumes.

Quantum field theory in a finite expanding space-time would introduce particles with a positive mass. The method of generating bound states through local anti-de Sitter regions would yield a spectrum beginning with an infimum bounded away from zero. If the radius of curvature R in the $AdS_5 \times S^5$ solution is allowed to decrease, the series for the energy does not converge rapidly and the entire formula for the Hamiltonian is necessary. An analysis of the energy spectrum may be used to establish the most accurate match with the data. The existence of a gap between the ground state energy and zero does not follow from supersymmetry, but it is a consequence of the finite size of the interaction region. The existence of a mass gap in Yang-Mills theories refers to an infinite-volume limit and not the vanishing of the Luscher effect when L $\rightarrow \infty$. In the present model, however, the finite-size region could be viewed as a conformal compactification of $AdS_5 \times S^5$. Timelike infinity is related conformally to Minkowski space-time, and this model may be sufficient to provide a further description of the phenomenology of Yang-Mills theories.

Reflecting boundary conditions at I in conformally compactified four-dimensional anti-de Sitter space are sufficient to define the hyperbolic problem of determining the time-development of Cauchy data on a spacelike surface in anti-de Sitter space [3]. The energies then would be quantized for a conformally coupled spin-s in the $D(s+1, s)$ representation: $\omega = n + \ell + 1, n = 0,\ 1,\ 2,\ \ldots, \ell = s,\ s+1,\ s+2,\ \ldots,$

Therefore, the compactness of the conformally related space-time in the embedding space would cause the masses to be non-zero, although a singularity in the transformation signals a phase transition. The generators of the anti-de Sitter group include the angular momentum, and the phase transition could describe the transitions from non-spinning to spinning states. The conformally coupled Scalar modes in anti-de Sitter space are given by the product of $\Omega^{-1} = cos\ \rho$ and the modes of the Einstein static universe

$$\phi^{AdS}_{\omega\ell m} = cos\ \rho\ \phi^{E}_{\omega\ell m} \tag{5.2}$$
$$\phi^{E}_{\omega\ell m} = N_{\omega\ell} e^{-i\omega t} (sin\ \rho)^{\ell} F\left(\frac{1}{2}(\ell+1-\omega), \frac{1}{2}(\ell+1+\omega); \ell+\frac{3}{2}; sin^2\rho\right) Y_{\ell m}(\theta,\phi)$$

in the coordinates (t, ρ, θ, ϕ) such that the line element is $ds^2_{AdS} = \frac{1}{cos^2\rho} ds^2_E = \frac{a^2}{cos^2\rho}(dt^2 - d\rho^2 - sin^2\rho(d\theta^2 + sin^2\theta\ d\phi^2))$ [6]. Then the radial derivative of this scalar mode is

$$\begin{aligned}
& -sin\ \rho\ \phi^E_{\omega\ell m} + cos\ \rho \frac{d}{d\rho}\phi^E_{\omega\ell m} = N_{\omega\ell} e^{-i\omega t} Y_{\ell m}(\theta,\phi) \qquad (5.3)\\
& \Big[-(sin\ \rho)^{\ell+1} F\left(\frac{1}{2}(\ell+1-\omega); \frac{1}{2}(\ell+1+\omega); \ell+\frac{3}{2}; sin^2\rho\right) \\
& \quad + \ell (sin\ \rho)^{\ell-1}(cos\ \rho)^2 F\left(\frac{1}{2}(\ell+1-\omega), \frac{1}{2}(\ell+1+\omega); \ell+\frac{3}{2}; sin^2\rho\right) \\
& \quad + (sin\ \rho)^{\ell} cos\ \rho\ \frac{d}{d\rho} F\left(\frac{1}{2}(\ell+1-\omega), \frac{1}{2}(\ell+1+\omega); \ell+\frac{3}{2}; sin^2\rho\right) \Big] \\
& \underset{\rho\to\frac{\pi}{2}}{\longrightarrow} N_{\omega\ell} e^{-i\omega t} Y_{\ell m}(\theta,\phi) \Big[-(sin\ \rho)^{\ell+1} F\left(\frac{1}{2}(\ell+1-\omega), \frac{1}{2}(\ell+1+\omega); \ell+\frac{3}{2}; sin^2\rho\right) \\
& \quad + 2(sin\ \rho)^{\ell+1}(cos\ \rho)^2 \frac{d}{dz} F\left(\frac{1}{2}(\ell+1-\omega), \frac{1}{2}(\ell+1+\omega); \ell+\frac{3}{2}; z\right)\Big|_{z=sin^2\rho} \Big] \\
& = N_{\omega\ell} e^{-i\omega t} Y_{\ell m}(\theta,\phi) \Big[-(sin\ \rho)^{\ell+1} F\left(\frac{1}{2}(\ell+1-\omega), \frac{1}{2}(\ell+1+\omega), \ell+\frac{3}{2}; sin^2\rho\right) \\
& \quad + \frac{1}{2}(sin\ \rho)^{\ell+1}(cos\ \rho)^2 \frac{(\ell+1)^2-\omega^2}{\ell+\frac{3}{2}} F\left(\frac{1}{2}(\ell+3-\omega), \frac{1}{2}(\ell+1+\omega); \ell+\frac{5}{2}; sin^2\rho\right) \Big].
\end{aligned}$$

By the identities

$$\begin{aligned}
&(a-b)c\ F(a,b;c;z) - a(c-b)\ F(a+1,b,c+1;z) + (c-a)b\ F(a,b+1;c+1;z) = 0 \\
&(a-c)F(a-1;b;c;z) - a(z-1)F(a+1,b;c;z) + (c-2a+(a-b)z)F(a,b;c;z) = 0 \\
&(b-c)F(a,b-1;c;z) - b(z-1)F(a,b+1;c;z) + (c-2b+(b-a)z)F(a,b;c;z) = 0,
\end{aligned}$$

it follows that the radial derivative tends to

$$\begin{aligned}
& -N_{\omega\ell} e^{-i\omega t} Y_{\ell m}(\theta,\phi)(sin\ \rho)^{\ell+1}\left(\ell+\frac{3}{2} - \frac{1}{2}(\ell+1-\omega) - \frac{1}{2}(\ell+1+\omega)\right) \qquad (5.4)\\
& \qquad F\left(\frac{1}{2}(\ell+3-\omega)), \frac{1}{2}(\ell+3+\omega) + \ell + \frac{5}{2}; sin^2\rho\right). \\
& = -\frac{1}{2} N_{\omega\ell} e^{-i\omega t} Y_{\ell m}(\theta,\phi)(sin\ \rho)^{\ell+1} F\left(\frac{1}{2}(\ell+3-\omega)), \frac{1}{2}(\ell+3+\omega) + \ell + \frac{5}{2}; sin^2\rho\right)
\end{aligned}$$

as $\rho \to \frac{\pi}{2}$, and there is an increase in the exponent of $(sin\ \rho)$ and the argument in the hypergeometric function corresponding to $\Delta\ell = 1$. The transition from anti-de Sitter space to the Einstein static universe and the negative scalar flux at timelike infinity I can introduce spin. This result is consistent with the existence of a mass gap in membrane

theories related to the minimum eigenvalue of an inertia matrix with respect to the centre of massrepresenting rotational energy [10].
The linear extent of the interaction region differs from the radius of curvature of $AdS_5 \times S^5$. The sum of the energies of the Hamiltonian derived from the Green-Schwarz Lagrangian, the Luscher Effect and the quantized frequencies in the conformally compactified space-time is

$$\begin{aligned} E &= E_1 + E_2 + E_3 \\ E_2 &= -b\, e^{-\frac{\sqrt{3}}{2}(E_{1\ ground\ state}+E_3)} \\ E_3 &= c\frac{s+1}{R}. \end{aligned} \tag{5.5}$$

The energies derived from the string Hamiltonian is expected to have a minimum equal to zero, and the higher-order terms in the series for E_1 must vanish in this state or the symmetry would be reduced. Then

$$E_{ground\ state} = E_2 + E_3 = -b\, e^{-\frac{\sqrt{3}}{2}\frac{c(s+1)}{R}L} + \frac{c(s+1)}{R} \tag{5.6}$$

and

$$\frac{dE_{ground\ state}}{dR} = b\left(\frac{\sqrt{3}}{2}\frac{c(s+1)}{R}L\right) e^{-\frac{\sqrt{3}}{2}\frac{c(s+1)}{R}L} - \frac{c(s+1)}{R^2} = 0. \tag{5.7}$$

This transcendental equations can be solved algorithmically through the recursion $R_{n+1} = R_n - \frac{f(R_n)}{f'(R_n)}$ where

$$f(R) = e^{\frac{\sqrt{3}}{2}L\frac{c(s+1)}{R}} - \frac{\sqrt{3}}{2}bLR \tag{5.8}$$

and

$$R_0 = \frac{2}{\sqrt{3}bL}\left[1 + \left(1 + 3bc(s+1)L^2\right)^{\frac{1}{2}}\right]. \tag{5.9}$$

Then $lim_{n\to\infty}E(R_n)$ will be the minimized energy of a configuration representing a particle state with spin s.
This model can be extended to include singleton representations of SO(3,2), designated $D\left(s+\frac{1}{2}, s\right)$, do not have a classical limit in Minkowski space-time. The energies

$$E_{singleton} = -b\, e^{-\frac{\sqrt{3}}{2}\frac{c\left(s+\frac{1}{2}\right)}{R}L} + \frac{c\left(s+\frac{1}{2}\right)}{R}, \tag{5.10}$$

which would be minimized at the solution to

$$f_s(R) = e^{\frac{\sqrt{3}}{2}L\frac{c\left(s+\frac{1}{2}\right)}{R}} - \frac{\sqrt{3}}{2}blR = 0 \tag{5.11}$$

given by $lim_{n\to\infty}R_{sn}$, where $R_{s,n+1} = R_{sn} - \frac{f_s(R_{sn})}{f_s'(R_{sn})}$, with

$$R_{s0} = \frac{2}{\sqrt{3}bL}\left[1 + \left(1 + 3bL^2c\left(s+\frac{1}{2}\right)\right)^{\frac{1}{2}}\right], \tag{5.12}$$

would characterize metastable states with spin s in a locally anti-de Sitter geometry.

6. **Conclusion**

The quantization of the worldsheet action in anti-de Sitter space requires physical states which satisfy modified Virasoro constraints with extra terms proportional to the normal ordering constant. The expectation value of $L_0 + \bar{L}_0$ between spin-two states in a vacuum with $\langle m^2\alpha' \rangle = 0$ yields the critical dimension 25. It is well known that this constant is non-zero by Lorentz invariance for strings propagating in ten-dimensional space-time. Nevertheless, the expectation value of the gravitino mass would be zero from the spectrum of strin states. Consistency in the zero-curvature limit is restored by identifying an additional contribution to the expectation valu if the term dependent on the time coordinate δx^0 is included in the worldsheet action. Furthermore, it is established that the masslessness of the graviton may be established in an anti-de Sitter space-time of the same number of dimensions necessary for the vanishing of the conformal anomaly.
The selection of the anti-de Sitter background for the string theory also induces corrections to the masses as a result of the Luscher effect in finite-size regions. The geometry may be transformed to a compactified space-time for conformally invariant theories with a quantization of energies derived from reflective boundary conditions for the fields. The energy spectrum of conformally coupled scalar modes in the compactified space-time begins at begins at a positive value, which is equivalent to the positivity of the squared mass of these modes relative to an overall negative shift of the eigenvalues of the Casimir operator in anti-de Sitter space. The interpretation of the energy of a conformally coupled mode differs from that of the expectation of the squared mass of the string, because the latter requires all of the vibrational modes and zeta function regularization in a complete sum of states to determine the dimension. The eigenvalues of the energy operator have an angular momentum $\ell \geq s$ which is positive only for non-zero momentum. This momentum generally would contribute to the Hamiltonian separate from the expectation value of the squared mass except when $\ell = s$, which is the intrinsic spin. The higher ℓ modes occur in a momentum space expansion of the spin-s field. Therefore, the expectation value of the energy of a spin-s field at rest can be evaluated in a linearized supergravity theory with N supersymmetries through a summation over the eigenvalue spectrum. The vanishing of the vacuum energy for $N \geq 5$ [1] is consistent with zero expectation value of the squared mass in the superstring effective field actions reduced from ten to four dimensions.
The finiteness of the scattering region in the quantum theory, which may be deduced from the nonvanishing of the probability of an interaction [9] and the finite range of the coordinates in the Einstein static universe representing a single copy of the compactified anti-de Sitter space, requires the existence of a mass gap, independent of exponentially decreasing terms and necessary for cluster decomposition to be valid in the infinite-volume limit. It is demonstrated that spin is introduced by enclosing the particles within a local anti-de Sitter region, removing the reflecting wall in the Einstein static universe and then translating in the conformally related Minkowski space-time. Extremization of the energy of the Green-Schwarz superstring and the Luscher term in anti-de Sitter space yields a value for the radius of the interaction region which provides the a non-zero mass of the particle state independent of the infinite-volume limit.

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RESEARCH FOUNDATION OF SOUTHERN CALIFORNIA
8861 VILLA LA JOLLA DRIVE #13595
LA JOLLA, CA 92037.
Email address: sbvdavis@outlook.com