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# CLOSING THEOREMS FOR PERSPECTIVITIES IN SPACE

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ABSTRACT. We consider perspectivities $\varphi_P : E_1 \to E_2$ with fixed point $P$, mapping a plane $E_1$ in the projective space $\mathbb{RP}^3$ to another plane $E_2$. This map has a unique projective extension to $\mathbb{RP}^3$. If one chooses points $P_1, \ldots, P_n$ properly, closing theorems result, namely so that the composition $\varphi_{P_n} \circ \varphi_{P_{n-1}} \circ \ldots \circ \varphi_{P_1}$ is the identity on $E_1$ or even on the whole $\mathbb{RP}^3$. We examine the conditions on the positions of the points $P_i$ so that such theorems apply. This results in theorems for coplanar points $P_i$ and in general position. The findings are also extended to perspectivities between more than two planes. We also prove similar results for closing theorems for perspectivities between lines in $\mathbb{RP}^3$.

## 1. INTRODUCTION

Closing theorems, also called porisms, of reversion maps have recently been studied quite intensively. However, their history goes back a long way. In fact, one can interpret for example Pappus's hexagon theorem as a closing theorem in the following way: Let $A_1, A_2, \ldots, A_6$ be a Pappus hexagon on the lines $\ell_1, \ell_2$ with Pappus points $P_1, P_2, P_3$ on the Pappus line $\ell$. Then the hexagon with the same Pappus points closes for any choice of the initial point $A_1'$ on $\ell_1$ (see Figure 1).

The mapping which maps the point $A_1$ on $\ell_1$ to the point $A_2$ on $\ell_2$ via the point $P_1$ is a perspectivity (here also called a reversion) from $\ell_1$ to $\ell_2$. Exchanging the degenerate conic $\ell_1 \cup \ell_2$ by a non-degenerate conic $C$, one can in the same way interpret Pascal's hexagon theorem as a closing theorem, and define a reversion map on $C$. Closing theorems of this type have been studied in [6], [1], [5], [10], [9], [7] and [4]. We refer to [2] and [3] for a detailed account of such closing theorems and their generalizations. The aim of this article is to study closing theorems of perspectivities in the three dimensional projective space.

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*Key words and phrases.* Closing theorems, porisms, perspectivities, projective maps.

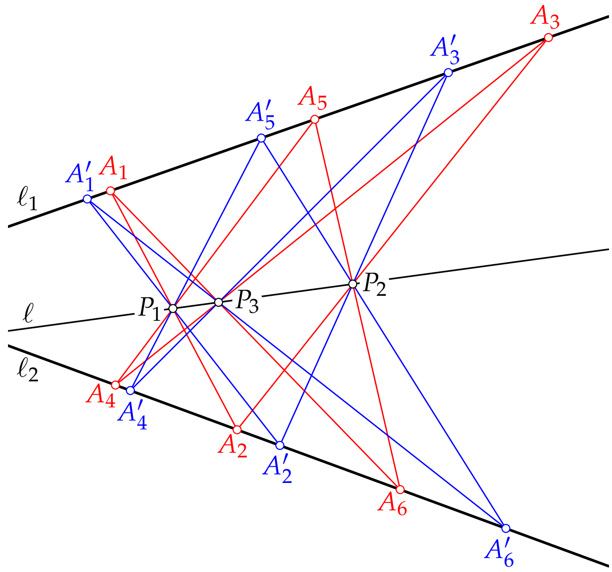


**Figure 1.** Pappus hexagons $A_1, A_2, \ldots, A_6$ and $A_1', A_2', \ldots, A_6'$.

## 2. Setup and Notation

We will work in the standard model of the real projective space. The set of points $\mathbb{P}$ is given by $\mathbb{RP}^3 = \mathbb{R}^4 \setminus \{0\}/\sim$, where $X \sim Y \in \mathbb{R}^4 \setminus \{0\}$ are equivalent if $X = \lambda Y$ for some $\lambda \in \mathbb{R}$. Similarly, the set of planes $\mathbb{B}$ is also $\mathbb{R}^4 \setminus \{0\}/\sim$, where again $E \sim F \in \mathbb{R}^4 \setminus \{0\}$ are equivalent if $E = \lambda F$ for some $\lambda \in \mathbb{R}$. A point $[X]$ and a plane $[E]$ are incident if $\langle X, E\rangle = 0$, where we denoted equivalence classes by square brackets and the standard inner product in $\mathbb{R}^4$ by $\langle \cdot, \cdot \rangle$. Since we mostly work with representatives we will omit the square brackets in the notation of equivalence classes. Vectors will be written as rows or, for better readability, as columns. Lines are intersections of two different planes, or equivalently the linear span of two different points.

We will work with perspectivities between planes. Let $E_1$ be a plane given by the equation $\langle e_1, X\rangle = 0$, $E_2$ be a plane given by $\langle e_2, X\rangle = 0$, and $P$ be a point incident neither with $E_1$ nor with $E_2$. Then, the perspectivity $\varphi_P : E_1 \to E_2$ with respect to $P$ maps the point $X$ on $E_1$ to the point $\varphi_P(X)$ on $E_2$ which is the intersection of the line through $X$ and $P$ with $E_2$ (see Figure 2).

Throughout this paper we will use the notation

$$X \xrightarrow[E_1 \quad E_2]{P} Y$$

for the situation shown in Figure 2. If it is clear from the context which planes are involved, we will omit them in the notation.

The map $\varphi_P : E_1 \to E_2$ defined in this way has a projective extension to the whole space $\mathbb{RP}^4$, which we still denote by $\varphi_P$. The first lemma gives an explicit formula for this extension.

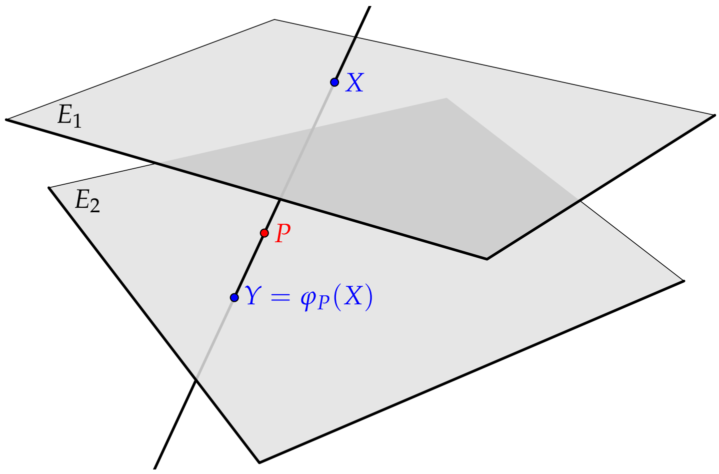


**Figure 2.** The perspectivitiy $\varphi_P : E_1 \to E_2, X \mapsto Y = \varphi_P(X)$.

**Lemma 1.** *The map*

$$\varphi_P : \mathbb{RP}^4 \to \mathbb{RP}^4, \quad X \mapsto \big(\langle X, e_1\rangle\langle P, e_2\rangle + \langle P, e_1\rangle\langle X, e_2\rangle\big)P - \langle P, e_1\rangle\langle P, e_2\rangle X \qquad (2.1)$$

*is the unique projective involution which extends the perspectivity* $\varphi_P : E_1 \to E_2$.

*Proof.* Clearly, the map $\varphi_P$ given by (2.1) is linear. Furthermore the points $X, P$ and $\varphi_P(X)$ are collinear. It is easy to check that $\varphi_P(X)$ is a point on $E_2$ if $X$ is a point on $E_1$. Finally, we have that $\varphi_P \circ \varphi_P = \langle e_1, P\rangle^2\langle e_2, P\rangle^2 \,\mathrm{id}$, where id denotes the identity on $\mathbb{RP}^4$.

To show uniqueness, take two projective extensions $\varphi'_P$ and $\varphi''_P$. Then, the composition $\varphi''_P \circ {\varphi'_P}^{-1}$ has $P$ and all points on $E_1 \cup E_2$ as fixed points and is hence the identity. □

Notice that the image of a point $X$ not on $E_1 \cup E_2$ under the projective extension of $\varphi_P$ can easily be constructed as follows. Take two lines $\ell_1 \neq \ell_2$ passing through $X$. The four intersection points of $\ell_1$ and $\ell_2$ with $E_1$ and $E_2$ have well defined images by $\varphi_P$ on the other plane. Hence the images of $\ell_1$ and $\ell_2$ are determined, and their intersection is $\varphi_P(X)$.

## 3. Closing theorems for two planes

We come to a first result.

**Proposition 2.** *Let*

$$E_1 : \langle X, e_1\rangle = 0, \quad E_2 : \langle X, e_2\rangle = 0, \quad F : \langle X, f\rangle = 0$$

*be three different planes with common intersection line* $\ell$, *i.e.,* $f = \lambda_1 e_1 + \lambda_2 e_2$, *and let* $P_1, P_2, \ldots, P_{2n}$ *be points on* $F \setminus \ell$. *Then the composition*

$$\varphi_{P_{2n}} \circ \varphi_{P_{2n-1}} \circ \ldots \circ \varphi_{P_2} \circ \varphi_{P_1} = \mathrm{id}$$

*is the identity on* $\mathbb{RP}^4$ *if and only if we have*

$$\sum_{k=1}^{2n} \frac{(-1)^k P_k}{\langle P_k, e\rangle} = 0 \tag{3.1}$$

*for* $e = \lambda_1 e_1 - \lambda_2 e_2$.

*Proof.* Consider the map $\alpha : \mathbb{RP}^4 \to \mathbb{RP}^4, X \mapsto AX$, for a regular $4 \times 4$-matrix $A$ such that $E_1$ is mapped to $E_1' : \langle X, e_1' \rangle = 0$, $E_2$ is mapped to $E_2' : \langle X, e_2' \rangle = 0$, and $F$ is mapped to $F' : \langle X, f' \rangle = 0$, with

$$e_1' = 2\lambda_1 A^{-\top} e_1 = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 1 \end{pmatrix}, \quad e_2' = 2\lambda_2 A^{-\top} e_2 = \begin{pmatrix} 0 \\ 0 \\ 1 \\ -1 \end{pmatrix}, \quad f' = A^{-\top} f = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}.$$

This is achieved by choosing the third row of $A$ as $f$ and the fourth row as $e$. Then the points $P_k' = AP_k$ lie on $F'$ and have normalized coordinates $(p_{k1}, p_{k2}, 0, 1)$. By using (2.1) the map $\varphi_{P_k'}$ can be written as

$$\varphi_{P_k'}(X) = \begin{pmatrix} 1 & 0 & 0 & -2p_{k1} \\ 0 & 1 & 0 & -2p_{k2} \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix}.$$

It is then easy to see that the map $\varphi_{P_{2n}} \circ \varphi_{P_{2n-1}} \circ \ldots \circ \varphi_{P_2} \circ \varphi_{P_1}$ is given by the matrix

$$\begin{pmatrix} 1 & 0 & 0 & 2\sum_{k=1}^{2n}(-1)^{k+1} p_{k1} \\ 0 & 1 & 0 & 2\sum_{k=1}^{2n}(-1)^{k+1} p_{k2} \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

This is the identity matrix iff

$$\sum_{k=1}^{2n} \frac{(-1)^k P_k'}{\langle P_k', e_1' - e_2' \rangle} = 0.$$

For the original points $P_k$, this translates to (3.1). □

**Remarks.**

- If the points $P_1, \ldots, P_{2n-1}$ on $F \setminus \ell$ are given, then there exists a unique point $P_{2n}$ on $F \setminus \ell$ such that (3.1) is satisfied.
- The sum in (3.1) is invariant under permutations of the points with odd indices and under permutations of the the points with even indices.

As a consequence we get the following result.

**Corollary 3.** *Let* $E_1, E_2$ *and* $F$ *be three different planes with common intersection line* $\ell$, *and let* $P_1, P_2, \ldots, P_{2n}$ *be points on* $F \setminus \ell$. *Assume that*

$$\varphi_{P_{2n}} \circ \varphi_{P_{2n-1}} \circ \ldots \circ \varphi_{P_2} \circ \varphi_{P_1}(X_1) = X_1$$

*for just one point* $X_1$ *on* $E_1$. *Then,*

$$\varphi_{P_{2n}} \circ \varphi_{P_{2n-1}} \circ \ldots \circ \varphi_{P_2} \circ \varphi_{P_1} = \mathrm{id}$$

*is the identity on $\mathbb{R}P^4$. In particular, if for just one $X_1$ on $E_1$ the chain of points*

$$X_1 \xrightarrow[E_1 \quad E_2]{P_1} X_2 \xrightarrow[E_2 \quad E_1]{P_2} X_3 \xrightarrow[E_1 \quad E_2]{P_3} \ldots X_{2n} \xrightarrow[E_2 \quad E_1]{P_{2n}} X_1 \tag{3.2}$$

*closes, then it closes for any other $X_1$ on $E_1$, and also the chain*

$$X_1 \xrightarrow[E_2 \quad E_1]{P_1} X_2 \xrightarrow[E_1 \quad E_2]{P_2} X_3 \xrightarrow[E_2 \quad E_1]{P_3} \ldots X_{2n} \xrightarrow[E_1 \quad E_2]{P_{2n}} X_1 \tag{3.3}$$

*closes for every $X_1$ on $E_2$. See Figure 3.*

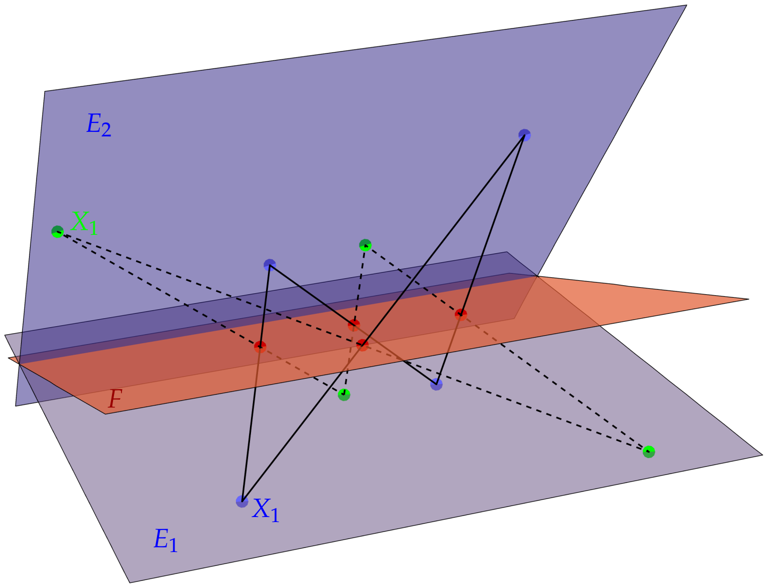


**Figure 3.** Illustration Corollary 3. The red points $P_1, \ldots, P_4$ on the red plane $F$ satisfy (3.1). Then the solid chain (3.2) closes for every starting point $X_1$ on the plane $E_1$, and the dashed chain (3.3) closes for every starting point $X_1$ on the plane $E_2$.

*Proof.* Let $e$ be the vector from Proposition 2, and $Q$ be the unique point on $F$ such that

$$\frac{Q}{\langle Q, e\rangle} = -\sum_{k=1}^{2n-1} \frac{(-1)^k P_k}{\langle P_k, e\rangle}.$$

Then, according to Proposition 2, the map

$$\varphi_Q \circ \varphi_{P_{2n-1}} \circ \varphi_{P_{2n-2}} \circ \ldots \circ \varphi_{P_1} = \mathrm{id}$$

on $\mathbb{R}P^4$. On the other hand, for a point $X_1$ on $E_1$, the point $X_{2n}$ on $E_2$ is determined by $X_1$ and the points $P_1, \ldots, P_{2n-1}$. Hence the point $P_{2n} = Q$ is the intersection of the plane $F$ with the line through $X_{2n}$ and $X_1$. $\square$

The last point of Corollary 3, namely that the chain closes for every starting point on $E_1$ and also for every starting point on $E_2$ is in sharp contrast to our next result.

**Proposition 4.** *Let $E_1, E_2$ and $F$ be three different planes which do not share a common intersection line, and let $P_1, P_2, \ldots, P_{2n}$ be points on $F$ but outside $E_1$ and $E_2$. Assume that*

$$\varphi_{P_{2n}} \circ \varphi_{P_{2n-1}} \circ \ldots \circ \varphi_{P_2} \circ \varphi_{P_1}(X_1) = X_1 \tag{3.4}$$

*for just one point $X_1$ on $E_1 \setminus F$. Then,*

$$\varphi_{P_{2n}} \circ \varphi_{P_{2n-1}} \circ \ldots \circ \varphi_{P_2} \circ \varphi_{P_1}(X) = X \tag{3.5}$$

*holds for all $X \in E_1$.*

Notice that if (3.5) holds for all $X \in E_1$ then (3.5) is in general not true for $X \in E_2$.

*Proof.* By applying a suitable projective transformation $\alpha : \mathbb{RP}^4 \to \mathbb{RP}^4, X \mapsto AX$, we may assume that $E_1$ is mapped to $E_1' : \langle X, e_1' \rangle = 0$, $E_2$ is mapped to $E_2' : \langle X, e_2' \rangle = 0$, and $F$ is mapped to $F' : \langle X, f' \rangle = 0$, with

$$e_1' = A^{-\top} e_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \end{pmatrix}, \quad e_2' = A^{-\top} e_2 = \begin{pmatrix} 1 \\ 0 \\ 0 \\ -1 \end{pmatrix}, \quad f' = A^{-\top} f = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}.$$

Then the points $P_k' = AP_k$ lie on $F'$ and have normalized coordinates $(p_{k1}, p_{k2}, 0, p_{k4})$. By using (2.1) the map $\varphi_{P_k'}$ can be written as

$$\varphi_{P_k'}(X) = \begin{pmatrix} p_{k1}^2 + p_{k4}^2 & 0 & 0 & -2p_{k1}p_{k4} \\ 2p_{k1}p_{k2} & -p_{k1}^2 + p_{k4}^2 & 0 & -2p_{k2}p_{k4} \\ 0 & 0 & -p_{k1}^2 + p_{k4}^2 & 0 \\ 2p_{k1}p_{k4} & 0 & 0 & -p_{k1}^2 - p_{k4}^2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix}.$$

Using induction, we find for $X = (1, x_2, x_3, 1)$

$$\varphi_{P_{2n}} \circ \varphi_{P_{2n-1}} \circ \ldots \circ \varphi_{P_2} \circ \varphi_{P_1}(X) = \begin{pmatrix} \lambda \\ \mu + \nu x_2 \\ \nu x_3 \\ \lambda \end{pmatrix}$$

with

$$\begin{aligned}
\lambda &= \prod_{k=1}^{2n} (p_{k1} + (-1)^k p_{k4})^2, \\
\mu &= 2 \prod_{k=1}^{2n} (p_{k1} + (-1)^k p_{k4}) \sum_{k=1}^{2n} (-1)^k p_{k2} \prod_{i=1}^{k-1} (p_{i1} + (-1)^i p_{i4}) \prod_{i=k+1}^{2n} (p_{i1} - (-1)^i p_{i4}), \\
\nu &= \prod_{k=1}^{2n} (p_{k1}^2 - p_{k4}^2).
\end{aligned}$$

Hence, if we have (3.4) for some $X_1$ on $E_1 \setminus F$ it follows that $\nu = \lambda$ and $\mu = 0$. (Here we used the assumption that $X_1 \notin F$, i.e., $x_3 \neq 0$.) But then, we have (3.5) for all $X$ on $E_1$. □

**Remarks.**

- If (3.4) holds for some $X_1$ on $E_1 \cap F$, then (3.5) is in general not true for all $X$ on $E_2$.
- Notice that for given $P_1, \dots, P_{2n-1}$ on $F \setminus (E_1 \cup E_2)$ we always find a unique $P_{2n}$ such that (3.4) holds for a given $X_1$ on $E_1 \setminus F$: The point $P_{2n}$ is the intersection of the line through $X_1$ and $\varphi_{P_{2n-1}} \circ \dots \circ \varphi_{P_1}(X_1)$ with the plane $F$.

So far we have considered reversion points which are coplanar. Next, we investigate the case of arbitrary reversion points.

**Proposition 5.** *Let $E_1, E_2$ be two planes in the projective space and $P_1, \dots, P_{2n-1}$ points neither on $E_1$ nor on $E_2$. Then there is a unique point $P_{2n}$ such that*

$$\varphi_{P_{2n}} \circ \varphi_{P_{2n-1}} \circ \dots \circ \varphi_{P_1}(X) = X \tag{3.6}$$

*for all $X$ on $E_1$.*

*Proof.* We may assume that $n > 1$. According to Proposition 4 there is a point $P_1'$ on the plane spanned by $P_1, P_2, P_3$ such that $\varphi_{P_1'} \circ \varphi_{P_3} \circ \varphi_{P_2} \circ \varphi_{P_1}(X) = X$ for all $X$ on $E_1$. Hence we can replace $\varphi_{P_3} \circ \varphi_{P_2} \circ \varphi_{P_1}$ by $\varphi_{P_1'}$ in (3.6). In the same way, we can replace $\varphi_{P_5} \circ \varphi_{P_4} \circ \varphi_{P_1'}$ by $\varphi_{P_2'}$ for a point on the plane spanned by $P_1', P_4, P_5$. Continuing this way, we find $P_{2n} = P_{n-1}'$.

To see that $P_{2n}$ is unique with this property, observe that $P_{2n}$ must be the intersection of the line trough $X_1$ and $\varphi_{P_{2n-1}} \circ \dots \circ \varphi_{P_1}(X_1)$, and $Y_1$ and $\varphi_{P_{2n-1}} \circ \dots \circ \varphi_{P_1}(Y_1)$ for two points $X_1, Y_1$ on $E_1$. ☐

The following theorem corresponds to [3, Theorem 12] in two dimensions. We thus obtain a statement for arbitrary reversion points, so that a corresponding closure figure even applies for all starting points $A_1$ on $E_1$ *and* $E_2$.

**Theorem 6.** *Let $E_1$ and $E_2$ be two arbitrary planes. Let the points $X_1, X_2, \dots, X_{2n-1}$ lie alternately on $E_1$ and $E_2$, and let $P_1, P_2, \dots, P_{2n-2}$ be reversion points that lie arbitrarily, but not on $E_1$ nor on $E_2$. If the composition*

$$X_1 \xrightarrow{P_1} X_2 \xrightarrow{P_2} X_3 \xrightarrow{P_3} \dots X_{2n-2} \xrightarrow{P_{2n-2}} X_{2n-1} \tag{3.7}$$

*closes neither for $X_1$ on $E_1$ nor for $X_1$ on $E_2$, then there exists a unique straight line $\ell$ on which a reversion point $P_{2n-1}$ can be chosen arbitrarily and a corresponding unique point $P_{2n}$ on $\ell$, so that the closing figure*

$$X_1 \xrightarrow{P_1} X_2 \xrightarrow{P_2} X_3 \xrightarrow{P_3} \dots X_{2n-1} \xrightarrow{P_{2n-1}} X_{2n} \xrightarrow{P_{2n}} X_1 \tag{3.8}$$

*holds for any initial point $X_1$ on $E_1$ and also for all $X_1$ on $E_2$.*

*Proof.* We consider an affine embedding of $\mathbb{R}^3$ in $\mathbb{RP}^3$ such that $E_1$ and $E_2$ are parallel planes. We can assume that planes $E_1$ and $E_2$ are orthogonal to the ground plane $x_3 = 0$ and to the elevation plane $x_1 = 0$. We draw the situation as depicted in Figure 4 according to the rules of descriptive geometry (see [8]). A point $P$ in the ground plane is labeled $P'$, and in the elevation plane $P''$. Now we can apply [3, Theorem 12] separately in the ground plane and in the elevation plane.

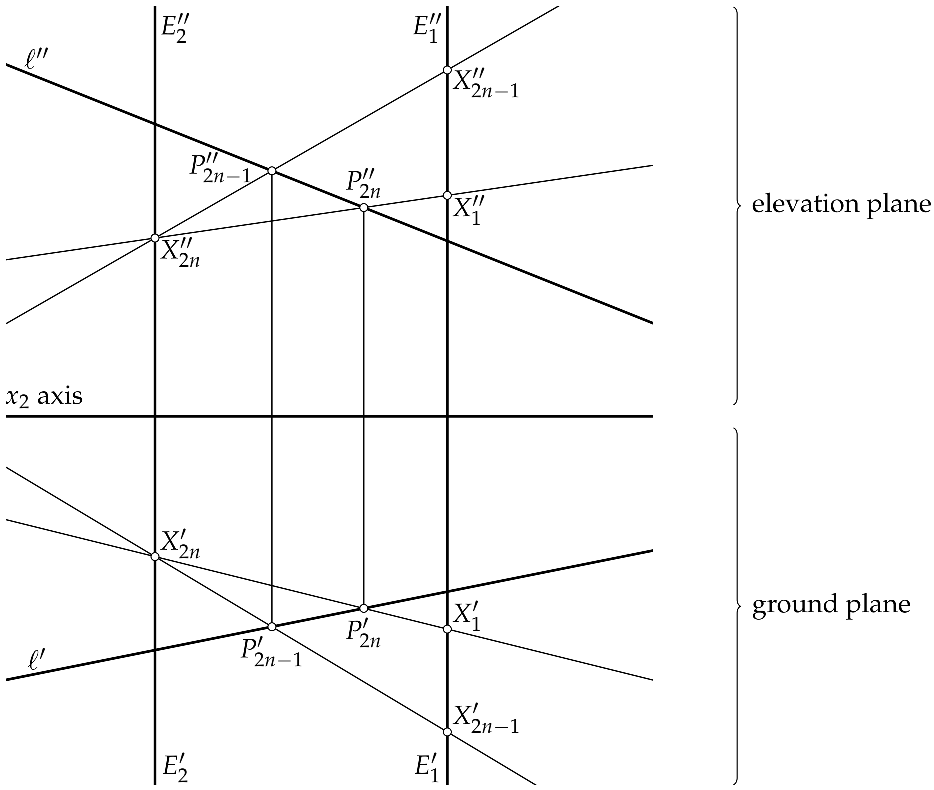


**Figure 4.** Theorem 6 in the ground plane and the elevation plane.

According to Theorem [3, Theorem 12] there is a unique line $\ell'$ in the ground plane with the following property. For an arbitrary point $P'_{2n-1}$ on $\ell'$ there is a unique point $P_{2n'}$ on $\ell'$ such that the porism

$$X_1' \xrightarrow{P_1'} X_2' \xrightarrow{P_2'} X_3' \xrightarrow{P_3'} \dots X'_{2n-1} \xrightarrow{P'_{2n-1}} X'_{2n} \xrightarrow{P'_{2n}} X_1'$$

closes for any $X_1'$ on $E_1'$. Similarly, there is a line $\ell''$ in the elevation plane with the analogous property. Observe that $\ell'$ and $\ell''$ are the ground and elevation projection of a line $\ell$ in space, and we will see now that this line has the desired property.

Let us choose a point $P_{2n-1}$ on $\ell$ with projections $P'_{2n-1}$ on $\ell'$ and $P''_{2n-1}$ on $\ell''$. Then there are points $P'_{2n}$ on $\ell'$ and $P''_{2n}$ on $\ell''$ with the closing property in the ground plane and in the elevation plane respectively. We still need to show, that these points are the projections of the same point on $\ell$.

According to Proposition 5, there is a unique point $\tilde{P}_{2n}$ with the closing property for all $X_1$ on $E_1$. But then, the corresponding projections in the ground plane and the elevation plane also close. Hence $\tilde{P}'_{2n} = P'_{2n}$ and $\tilde{P}''_{2n} = P''_{2n}$. $\square$

With the following lemma we can replace arbitrary reversion points except for the last one by corresponding reversion points on a common plane. This will be useful later on.

**Lemma 7.** *Let $E_1$ and $E_2$ be two distinct planes and let $P_1, P_2, \ldots, P_n$ be arbitrary reversion points acting between $E_1$ and $E_2$. We can then choose any plane $\Pi$ (different from $E_1$ and $E_2$) and find points $Q_1, Q_2, \ldots, Q_{n-1}$ on $\Pi$ and a point $Q'_n$ such that*

$$\varphi_{P_n} \circ \varphi_{P_{n-1}} \cdots \varphi_{P_1}(X_1) = \varphi_{Q'_n} \circ \varphi_{Q_{n-1}} \cdots \varphi_{Q_1}(X_1) \tag{3.9}$$

*for any $X_1$ on $E_1$ or $E_2$.*

*Proof.* Let $l_{12}$ be the straight line through $P_1$ and $P_2$. We choose $Q_1$ to be the intersection of $l_{12}$ with $\Pi$. Let $X_1$ be an arbitrary point on $E_1$ and $X_2$ on $E_2$ and $X_3$ on $E_1$ be such that

$$X_1 \xrightarrow{P_1} X_2 \xrightarrow{P_2} X_3.$$

Then, let $X'_2$ be the intersection of the line through $X_1$ and $Q_1$ with $E_2$, and define $Q'_2$ as the intersection of the line through $X_3$ and $X'_2$ with $l_{12}$. Hence, we have

$$X_1 \xrightarrow{P_1} X_2 \xrightarrow{P_2} X_3 \xrightarrow{Q'_2} X'_2 \xrightarrow{Q_1} X_1. \tag{3.10}$$

The points $X_1, X_2, X_3, X'_2, P_1, P_2, Q_1, Q'_1$ all lie on a plane $J$. Because of the Scissors Theorem (see, e.g., [3, Theorem 2]), the closing property (3.10) holds for all $X_1$ on $E_1 \cap J$ and on $E_2 \cap J$. Hence we have

$$\varphi_{P_2} \circ \varphi_{P_1} = \varphi_{Q'_2} \circ \varphi_{Q_1} \tag{3.11}$$

on $(E_1 \cup E_2) \cap J$. Because of Corollary 3 and Proposition 4 the relation actually holds for all points $X_1$ on $E_1 \cup E_2$.

Now let $l_{23}$ be the straight line through $P_3$ and $Q'_2$. As before, we choose $Q_2$ to be the intersection of $l_{23}$ and $\Pi$. Repeating the above arguments, we find a uniquely determined point $Q'_3$ on $l_{23}$ such that

$$\varphi_{P_3} \circ \varphi_{Q'_2} = \varphi_{Q'_3} \circ \varphi_{Q_2} \tag{3.12}$$

holds on $E_1 \cup E_2$. Together we infer from (3.11) and (3.12)

$$\varphi_{P_3} \circ \varphi_{P_2} \circ \varphi_{P_1} = \varphi_{Q'_3} \circ \varphi_{Q_2} \circ \varphi_{Q_1}.$$

Continuing in the same way, we arrive at (3.9). □

The next result shows how to construct a closing theorem when an odd number of prescribed reversion points is given.

**Theorem 8.** *Let $E_1$ and $E_2$ be two different planes, let $n$ be an odd number of points $P_1, P_2, \ldots, P_n$ that lie neither on $E_1$ nor $E_2$. Then there exists a plane $\Sigma$ with the following property: Each point $P_{n+1}$ on $\Sigma$ defines a unique straight line $\ell$ on $\Sigma$, so that for each point $P_{n+2}$ on $\ell$ there is a unique point $P_{n+3}$ on $\ell$, so that the closure figure*

$$X_1 \xrightarrow{P_1} X_2 \xrightarrow{P_2} X_3 \xrightarrow{P_3} \ldots \xrightarrow{P_{n+2}} X_{n+3} \xrightarrow{P_{n+3}} X_1.$$

*is valid for every $X_1$ on $E_1$ or $E_2$.*

*Proof.* As in the proof of Lemma 7 we choose a plane $\Pi$, different from $E_1$ and $E_2$ and find points $Q_1, Q_2, \ldots, Q_{n-1}$ on $\Pi$, and an additional point $Q'_n$ such that

$$\varphi_{P_n} \circ \varphi_{P_{n-1}} \circ \ldots \circ \varphi_{P_2} \circ \varphi_{P_1} = \varphi_{Q'_n} \circ \varphi_{Q_{n-1}} \circ \ldots \circ \varphi_{Q_2} \circ \varphi_{Q_1}. \tag{3.13}$$

Then, we apply Theorem 12 in [3] to construct points $X, Y$ on $\Pi$ such that

$$\varphi_X \circ \varphi_Y \circ \varphi_{Q_{n-1}} \circ \ldots \circ \varphi_{Q_2} \circ \varphi_{Q_1} = \mathrm{id} \tag{3.14}$$

on $(E_1 \cup E_2) \cap \Pi$. Because of Corollary 3 and Proposition 4 this actually holds on $E_1 \cup E_2$. It follows from (3.13) and (3.14) that we have

$$\varphi_{P_n} \circ \ldots \circ \varphi_{P_2} \circ \varphi_{P_1} = \varphi_{Q'_n} \circ \varphi_Y \circ \varphi_X \tag{3.15}$$

on $E_1 \cup E_2$. Now we consider the plane $\Sigma$ through the points $Q'_n, X$ and $Y$. Let $e_1$ and $e_2$ be the intersection lines of $X$ with $E_1$ and $E_2$ respectively. Then we choose $P_{n+1}$ on $\Sigma \setminus (e_1 \cup e_2)$. Like in the proof of Theorem 6 we find a line $\ell$ in $\Sigma$ on which we can choose a point $P_{n+2}$, which in turn determines the point $P_{n+3}$ on $\ell$ such that

$$\varphi_{P_{n+3}} \circ \varphi_{P_{n+2}} \circ \varphi_{P_{n+1}} \circ \varphi_{Q'_n} \circ \varphi_Y \circ \varphi_X = \mathrm{id} \tag{3.16}$$

on $e_1 \cup e_2$. Then, because of Proposition 4, this actually holds on $E_1 \cup E_2$. The claim now follows from (3.15) and (3.16). □

## 4. Closing theorems for more than two planes

In this section we consider reversions between more than two planes and ask for closing theorems. We start with the situation, when the reversion points are coplanar.

**Theorem 9.** *Let $E_1, E_2, \ldots, E_n$ be planes in $\mathbb{RP}^3$ which share a common line $g$, and let $F$ be another plane, different from the planes $E_i$ such that $E_i \neq E_{i+1}$ and $E_n \neq E_1$. Let $P_1, P_2, \ldots, P_n$ be points on $F$ such that $\varphi_{P_i} : E_i \to E_{i+1}$ is well defined. If*

$$X_1 \xrightarrow[E_1 \quad E_2]{P_1} X_2 \xrightarrow[E_2 \quad E_3]{P_2} X_3 \xrightarrow[E_3 \quad E_4]{P_3} \ldots X_{n-1} \xrightarrow[E_{n-1} \quad E_n]{P_{n-1}} X_n \xrightarrow[E_n \quad E_1]{P_n} X_1 \tag{4.1}$$

*for one point $X_1$ on $E_1 \setminus F$, then this porism holds for all $X$ on $E_1$. Moreover, for all $X_1$ on $E_i$ we have*

$$X_1 \xrightarrow[E_i \quad E_{i+1}]{P_i} X_2 \xrightarrow[E_{i+1} \quad E_{i+2}]{P_{i+1}} X_3 \xrightarrow[E_{i+2} \quad E_{i+3}]{P_{i+2}} \ldots X_{n-1} \xrightarrow[E_{i-2} \quad E_{i-1}]{P_{i-2}} X_n \xrightarrow[E_{i-1} \quad E_i]{P_{i-1}} X_1 \tag{4.2}$$

*and*

$$X_1 \xrightarrow[E_i \quad E_{i-1}]{P_{i-1}} X_2 \xrightarrow[E_{i-1} \quad E_{i-2}]{P_{i-2}} X_3 \xrightarrow[E_{i-2} \quad E_{i-3}]{P_{i-3}} \ldots X_{n-1} \xrightarrow[E_{i+2} \quad E_{i+1}]{P_{i+1}} X_n \xrightarrow[E_{i+1} \quad E_i]{P_i} X_1. \tag{4.3}$$

Notice that we do not require the planes $E_i$ to be different.

*Proof.* We consider an affine embedding of $\mathbb{R}^3$ in $\mathbb{RP}^3$ such that the planes $E_i$ are parallel. Observe that $\varphi_{P_i} : E_i \to E_{i+1}$ maps a line on $E_i$ to a parallel line on $E_{i+1}$. In particular, the map $\varphi := \varphi_{P_n} \circ \varphi_{P_{n-1}} \circ \ldots \circ \varphi_{P_1} : E_1 \to E_1$ is either a proper translation or a homothety. A proper translation does not have a fixed point, hence $\varphi$ is a homothety with ratio $\lambda \neq 0$. We have to show that $\lambda = 1$.

*1. case:* $F$ not parallel to the planes $E_i$. In this case, the line $\ell := F \cap E_1$ is a fixed line of $\varphi$. If $\lambda \neq 1$, then the center of the homothety lies on $\ell$ since all fixed lines of a homothety pass through its center. Since we have a second fixed point $X_1$ not on $\ell$, we conclude that $\lambda = 1$, and hence $\varphi$ is the identity map on $E_1$.

*2. case:* $F$ parallel to the planes $E_i$. In this case the factor $\lambda_i$ of the homothety $\varphi_i$ is given by the oriented ratio $\varepsilon_i \operatorname{dist}(E_{i+1}, F)/\operatorname{dist}(E_i, F)$, where $\varepsilon_i = -1$ if $F$ lies between $E_i$ and $E_{i+1}$, and $\varepsilon_i = 1$ otherwise. The ratio $\lambda$ of $\varphi$ is the product $\lambda_1\lambda_2 \ldots \lambda_n = 1$. Indeed, it is trivial to see that this product must be 1 or $-1$ since all numerators and denominators cancel. To see that the product of all $\varepsilon_i$ is 1, we may argue as follows: If $F$ lies on one side of all $E_i$, then all $\varepsilon_i = 1$. If $F$ moves from one side of an $E_i$ to the other side, an even number of the $\varepsilon_i$ changes sign.

This finishes the proof of (4.1), and (4.2) and (4.3) follow immediately. □

We consider again an affine embedding of $\mathbb{R}^3$ in $\mathbb{RP}^3$ such that the planes $E_i$ are parallel. Notice that we can always find a point $P_n$ on $F$ satisfying (4.1) for given $P_1, P_2, \ldots, P_{n-1}$ on $F$. Indeed, choose $P_n$ as the intersection of $F$ with the line trough a point $X_1$ on $E_1 \setminus F$ and the point $X_n := \varphi_{P_{n-1}} \circ \varphi_{P_{n-2}} \circ \ldots \circ \varphi_{P_1}(X_1)$ on $E_n$. This leads to the following more general result.

**Theorem 10.** *Let $E_1, E_2, \ldots, E_n$ be planes in $\mathbb{RP}^3$ which share a common line $g$, such that $E_i \neq E_{i+1}$ and $E_n \neq E_1$. Let $P_1, P_2, \ldots, P_{n-1}$ be points such that $\varphi_{P_i} : E_i \to E_{i+1}$ is well defined. Then there is a unique point $P_n$ such that*

$$X_1 \xrightarrow[E_1 \quad E_2]{P_1} X_2 \xrightarrow[E_2 \quad E_3]{P_2} X_3 \xrightarrow[E_3 \quad E_4]{P_3} \ldots X_{n-1} \xrightarrow[E_{n-1} \quad E_n]{P_n} X_n \xrightarrow[E_n \quad E_1]{P_n} X_1 \tag{4.4}$$

*holds for each point $X_1$ on $E_1 \setminus g$.*

Notice that we do not require the planes $E_i$ to be different.

*Proof.* We consider again an affine embedding of $\mathbb{R}^3$ in $\mathbb{RP}^3$ such that the planes $E_i$ are parallel. Let $X_1 \neq Y_1$ be points in $E_1$, $X_n := \varphi_{P_{n-1}} \circ \varphi_{P_{n-2}} \circ \ldots \circ \varphi_{P_1}(X_1)$ and $Y_n := \varphi_{P_{n-1}} \circ \varphi_{P_{n-2}} \circ \ldots \circ \varphi_{P_1}(Y_1)$ on $E_n$. Observe that the line segments $X_1Y_1$ and $X_nY_n$ are parallel. Hence the lines $X_1X_n$ and $Y_1Y_n$ intersect in a point $P_n$. Then, $\varphi := \varphi_{P_n} \circ \varphi_{P_{n-1}} \circ \ldots \circ \varphi_{P_1} : E_1 \to E_1$ is a translation or a homothety with two different fixed points $X_1$ and $Y_1$, and is hence the identity map. □

The next theorem shows another way to generate a porism.

**Theorem 11.** *Let $E_1, E_2, \ldots, E_n$ be planes in $\mathbb{RP}^3$ which share a common line $g$, such that $E_i \neq E_{i+1}$ and $E_n \neq E_1$. Let $P_1, P_2, \ldots, P_n$ be points such that $\varphi_{P_i} : E_i \to E_{i+1}$ is well defined. Then there exists a line $\ell$ with the following property. For each point $P_{n+1}$ on $\ell$, $P_{n+1}$ not on the planes $E_1, E_2, \ldots, E_n$, there is a plane $E_{n+1}$ such that*

$$X_1 \xrightarrow[E_1 \quad E_2]{P_1} X_2 \xrightarrow[E_2 \quad E_3]{P_2} X_3 \xrightarrow[E_3 \quad E_4]{P_3} \ldots X_n \xrightarrow[E_n \quad E_{n+1}]{P_n} X_{n+1} \xrightarrow[E_{n+1} \quad E_1]{P_{n+1}} X_1 \tag{4.5}$$

*holds for each point $X_1$ on $E_1 \setminus g$.*

*Proof.* We consider an affine embedding of $\mathbb{R}^3$ in $\mathbb{RP}^3$ such that the planes $E_i$ are perpendicular to the $x_3$ axis. Let $X_1 \neq Y_1$ be points on $E_1$, $X_n := \varphi_{P_{n-1}} \circ \varphi_{P_{n-2}} \circ \ldots \circ \varphi_{P_1}(X_1)$ and $Y_n := \varphi_{P_{n-1}} \circ \varphi_{P_{n-2}} \circ \ldots \circ \varphi_{P_1}(Y_1)$ on $E_n$. Then, according to Theorem 10, the lines $X_1X_n$ and $Y_1Y_n$ meet in a point $P$, and $\varphi_{P_{n-1}} \circ \varphi_{P_{n-2}} \circ \ldots \circ \varphi_{P_1} = \varphi_P$. Let $\ell$ be the line $PP_n$. Then the claim follows from [3, Theorem 14], applied to the front elevation plane and the side elevation plane. ☐

## 5. Porisms between straight lines in space

In this section, we consider porisms that arise when the reversion maps act between straight lines. First of all, we note that two corresponding straight lines should not be skewed, otherwise the reversion is only defined for one point. But if two straight lines $\ell_1, \ell_2$ intersect in $\mathbb{R}P^3$, they span a plane, and if $P$ is a point not on $\ell_1$ and $\ell_2$, then the reversion map $\varphi_P : \ell_1 \to \ell_2$ is well defined. With this setting, we find similar porisms as for reversions between planes.

**Proposition 12.** *Let $\ell_1, \ell_2, \ldots, \ell_n$ be straight lines in $\mathbb{R}P^3$ which intersect in a point $O$ such that $\ell_i \neq \ell_{i+1}$ and $\ell_n \neq \ell_1$. Let $P_1, P_2, \ldots, P_{n-1}$ be points such that $\varphi_{P_i} : \ell_i \to \ell_{i+1}$ are well defined. Then, there exists a unique point $P_n$ such that*

$$X_1 \xrightarrow[\ell_1 \quad \ell_2]{P_1} X_2 \xrightarrow[\ell_2 \quad \ell_3]{P_2} X_3 \xrightarrow[\ell_3 \quad \ell_4]{P_3} \ldots X_n \xrightarrow[\ell_n \quad \ell_1]{P_n} X_1 \tag{5.1}$$

*holds for all $X_1$ on $\ell_1 \setminus O$.*

Notice that we do not require the lines $\ell_i$ to be different.

*Proof.* We choose planes $E_1, E_2, \ldots, E_n$ which share a common line, such that $E_i \neq E_{i+1}$, $E_n \neq E_1$, and $\ell_i$ lies in $E_i$. Then, by Theorem 10 there exists a unique $P_n$ such that (5.1) holds with $\ell_i$ replaced by $E_i$ for all $X_1$ on $E_1$. It follows that (5.1) holds for all $X_1$ on $\ell_1$. We still need to check that there is no other point $P_n$ which works for this restricted case. But this follows easily by considering two different starting points $X_1, X_1'$ on $\ell_1$ in (5.1). ☐

We conclude this discussion with the following theorem, which contains a more general statement.

**Theorem 13.** *Let $\ell_1, \ell_2, \ldots, \ell_n$ be straight lines in $\mathbb{R}P^3$ with $\ell_i \neq \ell_{i+1}, \ell_n \neq \ell_1$, and suppose that $\ell_i$ and $\ell_{i+1}$ and $\ell_n$ and $\ell_1$ are concurrent. Let $P_1, P_2, \ldots, P_{n-2}$ be given points such that $\varphi_{P_i} : \ell_i \to \ell_{i+1}$ is well defined. Then, there exists a straight line $\ell$ with the following property. For any point $P_{n-1}$ on $\ell$ there exists a unique point $P_n$ such that*

$$X_1 \xrightarrow[\ell_1 \quad \ell_2]{P_1} X_2 \xrightarrow[\ell_2 \quad \ell_3]{P_2} X_3 \xrightarrow[\ell_3 \quad \ell_4]{P_3} \ldots X_n \xrightarrow[\ell_n \quad \ell_1]{P_n} X_1 \tag{5.2}$$

*holds for all points $X_1$ on $\ell_1$.*

*Proof.* We consider an affine embedding of $\mathbb{R}^3$ in $\mathbb{R}P^3$ such that the lines $\ell_i$ are not orthogonal to the ground plane and to the elevation plane. Then, we may apply [3, Theorem 16] in the ground plane and in the elevation plane to find the projections of the line $\ell$ in these planes. This determines $\ell$.

More concretely, following the proof of [3, Theorem 16], we can construct $\ell$, $P_{n-1}$ and $P_n$ as follows. Let $O_1$ be the intersection of $\ell_1$ and $\ell_2$, and $X_1, X_1', X_1''$ points on $\ell_1$. Consider the points

$$\begin{aligned}
&X_1 \xrightarrow[\ell_1 \quad \ell_2]{P_1} X_2 \xrightarrow[\ell_2 \quad \ell_3]{P_2} \dots \xrightarrow[\ell_{n-2} \quad \ell_{n-1}]{P_{n-2}} X_{n-1}\\
&X_1' \xrightarrow[\ell_1 \quad \ell_2]{P_1} X_2' \xrightarrow[\ell_2 \quad \ell_3]{P_2} \dots \xrightarrow[\ell_{n-2} \quad \ell_{n-1}]{P_{n-2}} X_{n-1}'\\
&X_1'' \xrightarrow[\ell_1 \quad \ell_2]{P_1} X_2'' \xrightarrow[\ell_2 \quad \ell_3]{P_2} \dots \xrightarrow[\ell_{n-2} \quad \ell_{n-1}]{P_{n-2}} X_{n-1}''\\
&O_1 \xrightarrow[\ell_1 \quad \ell_2]{P_1} O_2 \xrightarrow[\ell_2 \quad \ell_3]{P_2} \dots \xrightarrow[\ell_{n-2} \quad \ell_{n-1}]{P_{n-2}} O_{n-1}.
\end{aligned}$$

For the cross ratios we have $(O_1, X_1, X_1', X_1'') = (O_{n-1}, X_{n-1}, X_{n-1}', X_{n-1}'')$. Let $X$ denote the intersection of $\ell_1$ and $\ell_n$. Then there is a unique point $\tilde{X}$ on $\ell_{n-1}$ with the property

$$(X, X_1, X_1', X_1'') = (\tilde{X}, X_{n-1}, X_{n-1}', X_{n-1}'').$$

Now let $\ell$ be the line joining $X$ and $\tilde{X}$ and choose $P_{n-1}$ on $\ell$, not incident with $\ell_1$ and $\ell_{n-1}$. In particular, we have $\tilde{X} \xrightarrow[\ell_{n-1} \quad \ell_n]{P_{n-1}} X$. Consider the points

$$X_{n-1} \xrightarrow[\ell_{n-1} \quad \ell_n]{P_{n-1}} X_n, \quad X_{n-1}' \xrightarrow[\ell_{n-1} \quad \ell_n]{P_{n-1}} X_n', \quad X_{n-1}'' \xrightarrow[\ell_{n-1} \quad \ell_n]{P_{n-1}} X_n'',$$

$$\text{and } O_{n-1} \xrightarrow[\ell_{n-1} \quad \ell_n]{P_{n-1}} O_n.$$

The cross ratio of four of the points $X_{n-1}, X_{n-1}', X_{n-1}'', O_{n-1}, \tilde{X}$ equals the cross ratio of the four corresponding image points $X_n, X_n', X_n'', O_n, X$. In particular, the lines $X_1X_n, X_1'X_n'$, $X_1''X_n'', O_1O_n$ are concurrent in a point $P_n$. □

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