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THE CYCLIC AND ORTHOCENTRIC CHARACTERISTICS OF QUADRILATERALS

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ABSTRACT. The symbol $\square ABCD$ represents a quadrilateral with vertices A, B, C, and D labelled consecutively. We introduce the cyclic characteristic κ_c and the orthocentric characteristic κ_o of $\square ABCD$. They determine whether $\square ABCD$ is cyclic or orthocentric. Let O_a, O_b, O_c, and O_d (resp., N_a, N_b, N_c, and N_d) be the circumcenters (resp., nine-point centers) of $\triangle BCD$, $\triangle CDA$, $\triangle DAB$, and $\triangle ABC$, respectively. If $\square ABCD$ is neither cyclic nor orthocentric, then the circumcenter quadrilateral $\square O_aO_bO_cO_d$ and the nine-point center quadrilateral $\square N_aN_bN_cN_d$ are similar with $\frac{N_aN_b}{O_aO_b} = \frac{1}{2}\sqrt{\frac{\kappa_o}{\kappa_c}}$; moreover, they have the same normalized cyclic characteristic $\bar{\kappa}_c$ and normalized orthocentric characteristic $\bar{\kappa}_o$ as $\square ABCD$.

1. INTRODUCTION

Given four points A, B, C, and D in a plane such that the line segments $\overline{AB}$, $\overline{BC}$, $\overline{CD}$, and $\overline{DA}$ intersect only at their endpoints, the quadrilateral with vertices A, B, C, and D labelled consecutively, written $\square ABCD$, is defined as $\overline{AB} \cup \overline{BC} \cup \overline{CD} \cup \overline{DA}$ in [5, page 30]. We say that $\square ABCD$ is *cyclic* if its vertices lie on a common circle and that $\square ABCD$ is *orthocentric* if D is the orthocenter of $\triangle ABC$. These two types of quadrilaterals have been well studied, e.g., in [3] and [6]. A quadrilateral cannot be both cyclic and orthocentric. By a *general quadrilateral*, we mean that it is neither cyclic nor orthocentric. The general quadrilaterals are not well studied. Using the canonical vector space of geometric vectors, we introduce the *cyclic characteristic* and *orthocentric characteristic* of a quadrilateral in §3. These characteristics are used to prove Theorems 5.1 and 6.1.

The theorems of the paper are proved using the vector method. Broadly, the vector method is the use of the canonical vector space $\mathcal{V}$ of geometric vectors associated to a plane $\mathcal{E}$ to study problems in a geometrically natural way. By definition, the elements of $\mathcal{V}$ are the equivalence classes of directed line segments defined as follows: given points $A, B, C, D \in \mathcal{E}$ no three of which are collinear, the directed line segments $[A, B]$ and $[C, D]$ are *equivalent*, written $[A, B] \sim [C, D]$, if $\square ABDC$ is a parallelogram. When A, B, C, and D are collinear, we say that $[A, B]$ and $[C, D]$ are *equivalent* if for any directed segment $[X, Y]$ not lying on the line passing through A, B, C, and D, $[A, B] \sim [X, Y]$ if and only if

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$[C, D] \sim [X, Y]$. The vector $\overrightarrow{AB}$ is taken to mean the equivalence class represented by the directed line segment $[A, B]$.
It is well known that $\mathcal{V}$ has a canonical vector addition, a canonical scalar multiplication by real numbers, and has dimension two. If the angle between $\overrightarrow{AB}$ and $\overrightarrow{CD}$ is θ, then following [4] the *dot product* of $\overrightarrow{AB}$ and $\overrightarrow{CD}$ is defined by

$$\overrightarrow{AB} \cdot \overrightarrow{CD} = AB \cdot CD \cdot \cos\theta.$$

Verifying that the dot product is symmetric, bilinear, and positive definite can be done using purely geometric arguments. Additionally, the dot product satisfies a generalization of the Law of Cosines:

$$\overrightarrow{AB} \cdot \overrightarrow{AC} = \tfrac{AB^2+AC^2-BC^2}{2}, \tag{1.1}$$

which is used multiple times in the paper.
In some sense, the use of the vector method originates with Sylvester's characterization of the orthocenter H of a triangle $\triangle ABC$. Sylvester proves that H is determined by the vector equation

$$\overrightarrow{OH} = \overrightarrow{OA} + \overrightarrow{OB} + \overrightarrow{OC}, \tag{1.2}$$

where the point O is the circumcenter of $\triangle ABC$; see [1] and [7, page 251].
In general, the vector method efficiently studies problems in a unified way by avoiding the need to consider multiple cases when using purely geometric methods. In effect, the approach extends purely geometric methods, while avoiding the use of coordinates. The vector method is used in [2] to give a proof of the centroid locus problem posed by N. A. Court. In §2, we illustrate the vector method by reproving three celebrated theorems; see Theorems 2.1, 2.3, and 2.3. In addition, a technical result that characterizes the circumcenter and the nine-point center of $\triangle ABC$ is established; see Lemma 2.1.
Given $\square ABCD$, let R be the intersection point of the diagonal lines ℓ_{AC} and ℓ_{BD}. In §3, the *cyclic characteristic* and *orthocentric characteristic* of $\square ABCD$ are defined by

$$\kappa_c = (\overrightarrow{RA} \cdot \overrightarrow{RC} - \overrightarrow{RB} \cdot \overrightarrow{RD})^2,$$
$$\kappa_o = (\overrightarrow{RA} \cdot \overrightarrow{RC} + \overrightarrow{RB} \cdot \overrightarrow{RD})^2 - 4(\overrightarrow{RA} \cdot \overrightarrow{RB})(\overrightarrow{RC} \cdot \overrightarrow{RD}).$$

The vertices A and D are determined by the vector equations

$$\overrightarrow{RA} = -\alpha\overrightarrow{RC} \quad \text{and} \quad \overrightarrow{RD} = -\beta\overrightarrow{RB},$$

where $\alpha, \beta \in \mathbb{R} \setminus \{0, -1\}$ are unique. In terms of the pair (α, β), the cyclic characteristic and the orthocentric characteristic are

$$\kappa_c = (\beta RB^2 - \alpha RC^2)^2,$$
$$\kappa_o = (\beta RB^2 + \alpha RC^2)^2 - 4\alpha\beta(\overrightarrow{RB} \cdot \overrightarrow{RC})^2.$$

Proposition 3.2 proves that $\square ABCD$ is cyclic (resp. orthocentric) if and only if $\kappa_c = 0$ (resp. $\kappa_o = 0$). The close of §3 addresses the constants α and β. Let $A, B, C, D \in \mathcal{E}$ no three of which are collinear, R be the intersection point of ℓ_{AC} and ℓ_{BD}, and $\overrightarrow{RA} = -\alpha\overrightarrow{RC}$ and $\overrightarrow{RD} = -\beta\overrightarrow{RB}$, where $\alpha, \beta \in \mathbb{R} \setminus \{-1, 0\}$. Lemma 3.1 proves that A, B, C, and D determine a quadrilateral if and only if either $\alpha > 0$ or $\beta > 0$.
In §4, we discuss how κ_c and κ_o determine the convexity of $\square ABCD$. Firstly, we have

$$\kappa_o - \kappa_c = 4\alpha\beta[RB^2 \cdot RC^2 - (\overrightarrow{RB} \cdot \overrightarrow{RC})^2],$$

where

$$\Pi = RB^2 \cdot RC^2 - (\overrightarrow{RB} \cdot \overrightarrow{RC})^2 = 4[\text{Area}(\triangle RBC)]^2. \tag{1.3}$$

Note that if θ is the angle included by $\overrightarrow{RB}$ and $\overrightarrow{RC}$, then

$$\begin{aligned} 4[\text{Area}(\triangle RBC)]^2 &= RB^2 \cdot RC^2 \cdot \sin^2\theta \\ &= RB^2 \cdot RC^2 \cdot (1 - \cos^2\theta) \\ &= RB^2 \cdot RC^2 - (RB \cdot RC \cdot \cos\theta)^2 \\ &= RB^2 \cdot RC^2 - (\overrightarrow{RB} \cdot \overrightarrow{RC})^2. \end{aligned}$$

Hence, κ_c and κ_o do not vanish simultaneously. Proposition 4.1 proves that $\square ABCD$ is convex if and only if $\kappa_o > \kappa_c$. That is, the sign of $\alpha\beta$ determines the convexity of $\square ABCD$. Define

$$\Delta = (\overrightarrow{RA} \cdot \overrightarrow{RC})(\overrightarrow{RB} \cdot \overrightarrow{RD}) - (\overrightarrow{RA} \cdot \overrightarrow{RB})(\overrightarrow{RC} \cdot \overrightarrow{RD}). \tag{1.4}$$

Then $\Delta = \alpha\beta\Pi$. Proposition 4.1 leads naturally to defining "normalized" cyclic and orthocentric characteristics of $\square ABCD$ by

$$\bar{\kappa}_c = \tfrac{\kappa_c}{\Delta} \quad \text{and} \quad \bar{\kappa}_o = \tfrac{\kappa_o}{\Delta};$$

see Definition 4.1. Then $\square ABCD$ is convex if and only if $\bar{\kappa}_c \geqslant 0$ if and only if $\bar{\kappa}_o > 0$.
Note that the "area" term Π in (1.3) appears multiple times in essential calculations involving the basis $\{\overrightarrow{RB}, \overrightarrow{RC}\}$. The paper uses the notation Π since it visually simplifies the important formulas in §5 and §6.
Given a general quadrilateral $\square ABCD$, let O_a, O_b, O_c, and O_d denote the circumcenters of $\triangle BCD$, $\triangle CDA$, $\triangle DAB$, and $\triangle ABC$, respectively; likewise, let N_a, N_b, N_c, and N_d denote the nine-point centers. Theorem 5.1 proves that $\square O_aO_bO_cO_d$ and $\square N_aN_bN_cN_d$ are well-defined quadrilaterals with the same pair (α, β) of constants and the same normalized characteristics as $\square ABCD$.
Theorem 6.1 establishes that for any non-cyclic $\square ABCD$, the six ratios of squared distances are $\frac{N_iN_j^2}{O_iO_j^2} = \frac{\kappa_o}{4\kappa_c}$, where $i, j \in \{a, b, c, d\}$ are distinct indices. Consequently, if $\square ABCD$ is a general quadrilateral, then $\square O_aO_bO_cO_d \sim \square N_aN_bN_cN_d$ The proof is an immediate consequence of Lemma 6.1.

2. Solving Certain Geometry Problems Using The Vector Method

In this section, we shall apply the vector method to reprove several well-known results in Euclidean geometry and derive two formulas in vector form for later use.
Using Sylvester's Law (1.2) and the vector method leads to an efficient proof of the famous Nine-Point Circle Theorem. As a historical note, Brianchon and Poncelet published a proof of the Nine-Point Circle Theorem in the paper *Recherches sur la détermination d'une hyperbole equilatere, au moyen de quatre conditions donnees*, Georgonne's Annales de Mathematiques, Vol XI (1820-1821), pages 205-220. Poncelet called the circle the *nine-point circle*; see [3, page 299], [8], and [9, pages 337-338].

Theorem 2.1 (Nine-Point-Circle)**.** *Let O and H be the circumcenter and orthocenter of $\triangle ABC$, respectively. Let M_a, M_b, and M_c be the midpoints of $\overline{BC}$, $\overline{CA}$, and $\overline{AB}$, respectively. Let D_a, D_b,* *and D_c be the feet of the altitudes at vertices A, B, and C, respectively. Let E_a, E_b, and E_c be the Euler points associated to A, B, and C, respectively. Define the point N by the vector equation*

$$\overrightarrow{ON} = \tfrac{1}{2}(\overrightarrow{OA} + \overrightarrow{OB} + \overrightarrow{OC}).$$

The nine-point circle $\mathcal{N}$ of $\triangle ABC$ obtained by the dilation of the circumcircle of $\triangle ABC$ through H with factor $\frac{1}{2}$ has N as its center and the segments $\overline{E_aM_a}$, $\overline{E_bM_b}$, and $\overline{E_cM_c}$ as diameters; in addition, it contains the points D_a, D_b, and D_c.

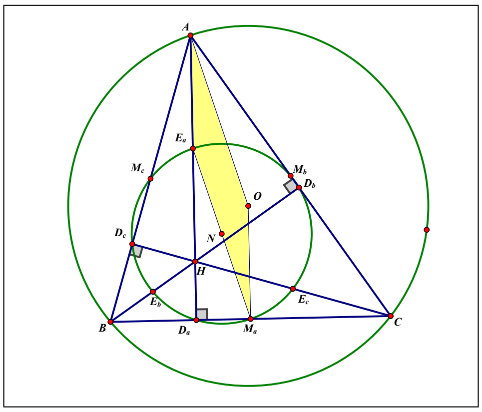

Figure 1. The Nine-Point-Circle of $\triangle ABC$

Proof. Let r be the radius of the circumcircle of $\triangle ABC$. The radius of $\mathcal{N}$ is $\frac{r}{2}$ and $\overrightarrow{ON} = \frac{1}{2}\overrightarrow{OH}$. Note that

$$\overrightarrow{NE_a} = \overrightarrow{OE_a} - \overrightarrow{ON} = \tfrac{1}{2}(\overrightarrow{OA} + \overrightarrow{OH}) - \tfrac{1}{2}\overrightarrow{OH} = \tfrac{1}{2}\overrightarrow{OA}.$$

and

$$\overrightarrow{NM_a} = \overrightarrow{OM_a} - \overrightarrow{ON} = \overrightarrow{OM_a} - \tfrac{1}{2}\overrightarrow{OH} = \tfrac{1}{2}(\overrightarrow{OB} + \overrightarrow{OC}) - \tfrac{1}{2}(\overrightarrow{OA} + \overrightarrow{OB} + \overrightarrow{OC}) = -\tfrac{1}{2}\overrightarrow{OA}.$$

Since N is the midpoint of $\overline{E_aM_a}$ and $NE_a = NM_a = \frac{r}{2}$, $\overline{E_aM_a}$ is a diameter of $\mathcal{N}$.
Since $\ell_{D_aM_a} \perp \ell_{D_aE_a}$, we have $\overrightarrow{D_aM_a} \cdot \overrightarrow{D_aE_a} = 0$. Since $\overrightarrow{E_aM_a} = \overrightarrow{D_aM_a} - \overrightarrow{D_aE_a}$, we get

$$E_aM_a^2 = (\overrightarrow{D_aM_a} - \overrightarrow{D_aE_a}) \cdot (\overrightarrow{D_aM_a} - \overrightarrow{D_aE_a}) = D_aM_a^2 + D_aE_a^2.$$

Note that

$$\overrightarrow{ND_a} + \overrightarrow{D_aM_a} = \overrightarrow{NM_a} = -\overrightarrow{NE_a} = -(\overrightarrow{ND_a} + \overrightarrow{D_aE_a}).$$

Then

$$\overrightarrow{ND_a} = -\tfrac{1}{2}(\overrightarrow{D_aM_a} + \overrightarrow{D_aE_a})$$

and

$$\begin{aligned} ND_a^2 &= \tfrac{1}{4}(\overrightarrow{D_aM_a} + \overrightarrow{D_aE_a}) \cdot (\overrightarrow{D_aM_a} + \overrightarrow{D_aE_a}) \\ &= \tfrac{1}{4}(D_aM_a^2 + D_aE_a^2) \\ &= \tfrac{1}{4}E_aM_a^2. \end{aligned}$$

Since $ND_a = \frac{1}{2}E_aM_a = \frac{r}{2}$ and N is the center of $\mathcal{N}$, we obtain $D_a \in \mathcal{N}$.
By analogy, both $\overline{E_bM_b}$ and $\overline{E_cM_c}$ are diameters of $\mathcal{N}$, and both D_b and D_c lie on $\mathcal{N}$. □

The next two theorems refer to $\square ABCD$. We say that $\square ABCD$ is *cyclic* if all its vertices lie on some circle and that $\square ABCD$ is *orthocentric* if D is the orthocenter of $\triangle ABC$. If $\square ABCD$ is orthocentric, then A is the orthocenter of $\triangle BCD$, B is the orthocenter of $\triangle CDA$, and C is the orthocenter of $\triangle DAB$.

Theorem 2.2. *If $\square ABCD$ is cyclic, then the nine-point circles of $\triangle BCD$, $\triangle CDA$, $\triangle DAB$, and $\triangle ABC$ are congruent.*
Let N_a, N_b, N_c, and N_d denote the nine-point centers of $\triangle BCD$, $\triangle CDA$, $\triangle DAB$, and $\triangle ABC$, respectively. Let P denote the Poncelet point of $\square ABCD$, i.e., the point lying on the four nine-point circles of $\square ABCD$. Then $\square N_aN_bN_cN_d$ is a cyclic quadrilateral with circumcenter P.
Let O denote the circumcenter of $\square ABCD$. Let S be the point that divides the segment $\overline{OP}$ internally in the ratio $\frac{PO}{PS} = 3$. The homothety about S with factor $-\frac{1}{2}$ maps $\square ABCD$ onto $\square N_aN_bN_cN_d$.

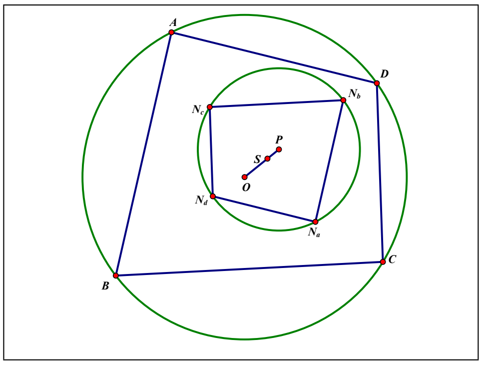

Figure 2. The Nine-Point Quadrilateral of $\square ABCD$

Proof. Since $\triangle BCD$, $\triangle CDA$, $\triangle DAB$, and $\triangle ABC$ have the same circumcircle, their nine-point circles have the same radius. So the four nine-point circles are congruent.
Note that since the Poncelet point P lies on each of the four congruent nine-point circles, $\square N_aN_bN_cN_d$ is a cyclic quadrilateral with center P and radius equal to half the radius of the circumcircle of $\square ABCD$.
To prove that there is a homothety from $\square ABCD$ onto $\square N_aN_bN_cN_d$ with a factor of $-\frac{1}{2}$, it suffices to show that for all distinct $i, j \in \{a, b, c, d\}$,

$$\overrightarrow{IJ} = 2\overrightarrow{N_jN_i}. \tag{2.1}$$

Now $\overrightarrow{ON_a} = \frac{1}{2}(\overrightarrow{OB} + \overrightarrow{OC} + \overrightarrow{OD})$. Hence,

$$\begin{aligned}\overrightarrow{AD} &= \overrightarrow{OD} - \overrightarrow{OA}\\ &= (2\overrightarrow{ON_a} - \overrightarrow{OB} - \overrightarrow{OC}) - \overrightarrow{OA}\\ &= 2\overrightarrow{ON_a} - 2\overrightarrow{ON_d}\\ &= 2\overrightarrow{N_dN_a}.\end{aligned}$$

The other cases of (2.1) are argued similarly.

The homothetic center S is the point that is common to the lines ℓ_{IN_i} for all $i \in \{a, b, c, d\}$. Let $\mathcal{D}_S$ denote the homothety from $\square ABCD$ onto $\square N_aN_bN_cN_d$. Let $\mathcal{C}$ (resp. $\mathcal{N}$) denote the circumcircle of $\square ABCD$ (resp. $\square N_aN_bN_cN_d$). Then $\mathcal{D}_S$ maps $\mathcal{C}$ onto $\mathcal{N}$ and $\mathcal{D}_S(O) = P$. Hence S lies between O and P such that $\frac{OS}{PS} = 2$ or $\frac{PO}{PS} = 3$. □

Theorem 2.3. *If $\square ABCD$ is orthocentric, then the nine-point circles of the four triangles $\triangle BCD$, $\triangle CDA$, $\triangle DAB$, and $\triangle ABC$ are coincident.*
If N is the common nine-point center and O_a, O_b, O_c, and O_d are the circumcenters of $\triangle BCD$, $\triangle CDA$, $\triangle DAB$, and $\triangle ABC$, respectively, then the rotation by $180°$ about N maps $\square ABCD$ onto $\square O_aO_bO_cO_d$. Hence $\square O_aO_bO_cO_d$ is congruent to $\square ABCD$.

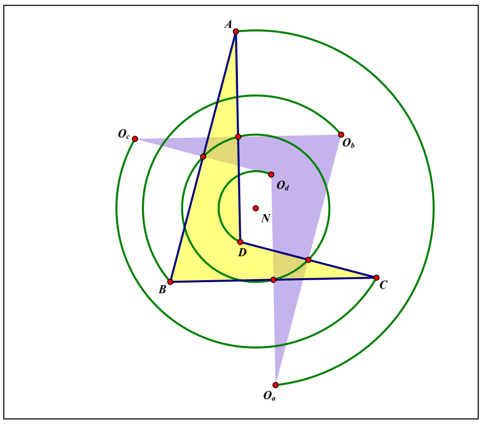

Figure 3. The Circumcenter Quadrilateral of an Orthocentric $\square ABCD$

Proof. Let N_a, N_b, N_c, and N_d be the nine-point centers of $\triangle BCD$, $\triangle CDA$, $\triangle DAB$, and $\triangle ABC$, respectively. Applying Sylvester's Law twice leads to

$$\begin{aligned}
\overrightarrow{AD} &= \tfrac{1}{2}\overrightarrow{AD} + \tfrac{1}{2}\overrightarrow{AD} \\
&= \tfrac{1}{2}(\overrightarrow{O_aD} - \overrightarrow{O_aA}) + \tfrac{1}{2}(\overrightarrow{O_dD} - \overrightarrow{O_dA}) \\
&= \tfrac{1}{2}(-\overrightarrow{O_aB} - \overrightarrow{O_aC}) + \tfrac{1}{2}(\overrightarrow{O_dB} + \overrightarrow{O_dC}) \\
&= \tfrac{1}{2}(\overrightarrow{O_dB} - \overrightarrow{O_aB}) + \tfrac{1}{2}(\overrightarrow{O_dC} - \overrightarrow{O_aC}) \\
&= \tfrac{1}{2}\overrightarrow{O_dO_a} + \tfrac{1}{2}\overrightarrow{O_dO_a} \\
&= \overrightarrow{O_dO_a}.
\end{aligned}$$

Hence $O_aD = O_dA$ so that the circumcircles of $\triangle BCD$ and $\triangle ABC$ are congruent.
Next, since

$$\overrightarrow{O_aO_d} + \overrightarrow{O_dA} = \overrightarrow{O_aA} = 2\overrightarrow{O_aN_a} = 2\overrightarrow{O_aO_d} + 2\overrightarrow{O_dN_a},$$

we get

$$\overrightarrow{O_dN_a} = \tfrac{1}{2}(\overrightarrow{O_dA} + \overrightarrow{O_dO_a}) = \tfrac{1}{2}(\overrightarrow{O_dA} + \overrightarrow{AD}) = \tfrac{1}{2}\overrightarrow{O_dD} = \overrightarrow{O_dN_d}.$$

So $N_a = N_d$. By analogy, the circumcircles of both $\triangle CDA$ and $\triangle DAB$ are congruent to the circumcircle of $\triangle ABC$. We also get that $N_a = N_b = N_c$.

Consequently, since $N_a = N_b = N_c = N_d$ and since the four circumcircles are congruent, the nine-point circles of $\triangle BCD$, $\triangle CDA$, $\triangle DAB$, and $\triangle ABC$ are coincident by the definition of the nine-point circle.
Implicit in the above argument is that for any distinct $i, j \in \{a, b, c, d\}$, $\overrightarrow{IJ} = \overrightarrow{O_jO_i}$ and in addition, N is the midpoint of $\overrightarrow{IO_i}$ for each $i \in \{a, b, c, d\}$. Hence the rotation by 180° about N maps $\square ABCD$ onto $\square O_aO_bO_cO_d$. Figure 3 illustrates the rotation. □

The next lemma illustrates how to characterize the circumcenter and the nine-point center of $\triangle ABC$ by means of the vector method.

Lemma 2.1. *Given $\triangle ABC$, choose a point $R \in \ell_{AC} \setminus \{A, C\}$ and write $\overrightarrow{RA} = -\alpha\overrightarrow{RC}$ with $\alpha \in \mathbb{R} \setminus \{-1, 0\}$. Let O and N be the circumcenter and nine-point center of $\triangle ABC$, respectively. In terms of the basis $\{\overrightarrow{RB}, \overrightarrow{RC}\}$ for the vector space $\mathcal{V}$,*

$$\overrightarrow{RO} = \frac{RC^2[RB^2-\alpha RC^2-(1-\alpha)(\overrightarrow{RB}\cdot\overrightarrow{RC})]}{2\Pi}\overrightarrow{RB} + \frac{(1-\alpha)RB^2\cdot RC^2-(RB^2-\alpha RC^2)(\overrightarrow{RB}\cdot\overrightarrow{RC})}{2\Pi}\overrightarrow{RC} \tag{2.2}$$

and

$$\begin{aligned}\overrightarrow{RN} = &\frac{RB^2\cdot RC^2+\alpha RC^4+(1-\alpha)RC^2(\overrightarrow{RB}\cdot\overrightarrow{RC})-2(\overrightarrow{RB}\cdot\overrightarrow{RC})^2}{4\Pi}\overrightarrow{RB}\\ &+\frac{(1-\alpha)RB^2\cdot RC^2+(RB^2-\alpha RC^2)(\overrightarrow{RB}\cdot\overrightarrow{RC})-2(1-\alpha)(\overrightarrow{RB}\cdot\overrightarrow{RC})^2}{4\Pi}\overrightarrow{RC}\end{aligned} \tag{2.3}$$

where $\Pi = RB^2 \cdot RC^2 - (\overrightarrow{RB} \cdot \overrightarrow{RC})^2 = 4[\,\text{Area}\,(\triangle RBC)]^2$; see (1.3).

Proof. Let L be the foot of the altitude at A. Since $L \in \ell_{BC}$, we write $\overrightarrow{RL} = (1-t)\overrightarrow{RB} + t\overrightarrow{RC}$ for some $t \in \mathbb{R}$. Then

$$\overrightarrow{AL} = \overrightarrow{RL} - \overrightarrow{RA} = t(\overrightarrow{RC} - \overrightarrow{RB}) + (\alpha\overrightarrow{RC} + \overrightarrow{RB}).$$

Since $\overrightarrow{BC} \cdot \overrightarrow{AL} = 0$, we get

$$\begin{aligned}0 &= (\overrightarrow{RC} - \overrightarrow{RB}) \cdot \overrightarrow{AL}\\ &= (\overrightarrow{RC} - \overrightarrow{RB}) \cdot (\overrightarrow{RC} - \overrightarrow{RB})t + (\overrightarrow{RC} - \overrightarrow{RB}) \cdot (\alpha\overrightarrow{RC} + \overrightarrow{RB})\\ &= BC^2 t + \overrightarrow{BC} \cdot (\overrightarrow{RB} + \alpha\overrightarrow{RC})\end{aligned}$$

and hence

$$t = -\frac{\overrightarrow{BC}\cdot(\overrightarrow{RB}+\alpha\overrightarrow{RC})}{BC^2}.$$

So

$$\begin{aligned}\overrightarrow{AL} &= \left[1 + \frac{\overrightarrow{BC}\cdot(\overrightarrow{RB}+\alpha\overrightarrow{RC})}{BC^2}\right]\overrightarrow{RB} + \left[\alpha - \frac{\overrightarrow{BC}\cdot(\overrightarrow{RB}+\alpha\overrightarrow{RC})}{BC^2}\right]\overrightarrow{RC}\\ &= \frac{BC^2+\overrightarrow{BC}\cdot(\overrightarrow{RB}+\alpha\overrightarrow{RC})}{BC^2}\overrightarrow{RB} + \frac{\alpha BC^2-\overrightarrow{BC}\cdot(\overrightarrow{RB}+\alpha\overrightarrow{RC})}{BC^2}\overrightarrow{RC}\\ &= \frac{(1+\alpha)(\overrightarrow{RC}\cdot\overrightarrow{BC})}{BC^2}\overrightarrow{RB} - \frac{(1+\alpha)(\overrightarrow{RB}\cdot\overrightarrow{BC})}{BC^2}\overrightarrow{RC}.\end{aligned}$$

Continuing, since $\overrightarrow{OC} = \overrightarrow{RC} - \overrightarrow{RO}$, $\overrightarrow{OA} = \overrightarrow{RA} - \overrightarrow{RO}$, and $OC^2 = OA^2$, we get

$$\begin{aligned}RC^2 - 2(\overrightarrow{RC} \cdot \overrightarrow{RO}) + RO^2 &= RA^2 - 2(\overrightarrow{RA} \cdot \overrightarrow{RO}) + RO^2\\ &= \alpha^2 RC^2 + 2\alpha(\overrightarrow{RC} \cdot \overrightarrow{RO}) + RO^2\end{aligned}$$

and

$$2(\overrightarrow{RC}\cdot\overrightarrow{RO}) = (1-\alpha)RC^2. \tag{2.4}$$

Let M be the midpoint of $\overline{BC}$. Write $\overrightarrow{OM} = s\overrightarrow{AL}$ for some $s \in \mathbb{R}$. Since $\overrightarrow{OM} = \overrightarrow{RM} - \overrightarrow{RO}$, we get

$$\begin{aligned}\overrightarrow{RO} &= \overrightarrow{RM} - s\overrightarrow{AL}\\ &= \tfrac{1}{2}(\overrightarrow{RB}+\overrightarrow{RC}) - \tfrac{s(1+\alpha)}{BC^2}[(\overrightarrow{RC}\cdot\overrightarrow{BC})\overrightarrow{RB} - (\overrightarrow{RB}\cdot\overrightarrow{BC})\overrightarrow{RC}]\\ &= \tfrac{BC^2-2s(1+\alpha)(\overrightarrow{RC}\cdot\overrightarrow{BC})}{2BC^2}\overrightarrow{RB} + \tfrac{BC^2+2s(1+\alpha)(\overrightarrow{RB}\cdot\overrightarrow{BC})}{2BC^2}\overrightarrow{RC}\end{aligned}$$

and

$$2(\overrightarrow{RC}\cdot\overrightarrow{RO}) = \tfrac{2s(1+\alpha)[(\overrightarrow{RB}\cdot\overrightarrow{RC})^2-RB^2\cdot RC^2]+BC^2\cdot RC^2+BC^2(\overrightarrow{RB}\cdot\overrightarrow{RC})}{BC^2}.$$

Using (2.4) leads to

$$s = \tfrac{BC^2(\alpha RC^2+\overrightarrow{RB}\cdot\overrightarrow{RC})}{2(1+\alpha)\Pi}.$$

So

$$\overrightarrow{RO} = \tfrac{RC^2[RB^2-\alpha RC^2-(1-\alpha)(\overrightarrow{RB}\cdot\overrightarrow{RC})]}{2\Pi}\overrightarrow{RB} + \tfrac{(1-\alpha)RB^2\cdot RC^2-(RB^2-\alpha RC^2)(\overrightarrow{RB}\cdot\overrightarrow{RC})}{2\Pi}\overrightarrow{RC}.$$

Finally, since

$$\begin{aligned}\overrightarrow{RN} &= \overrightarrow{ON} - \overrightarrow{OR}\\ &= \tfrac{1}{2}(\overrightarrow{OA}+\overrightarrow{OB}+\overrightarrow{OC}) - \overrightarrow{OR}\\ &= \tfrac{1}{2}(\overrightarrow{RA}+\overrightarrow{RB}+\overrightarrow{RC}) + \tfrac{1}{2}\overrightarrow{OR}\\ &= \tfrac{1}{2}\overrightarrow{RB} + \tfrac{1-\alpha}{2}\overrightarrow{RC} - \tfrac{1}{2}\overrightarrow{RO},\end{aligned}$$

we get

$$\begin{aligned}\overrightarrow{RN} = &\tfrac{RB^2\cdot RC^2+\alpha RC^4+(1-\alpha)RC^2(\overrightarrow{RB}\cdot\overrightarrow{RC})-2(\overrightarrow{RB}\cdot\overrightarrow{RC})^2}{4\Pi}\overrightarrow{RB}\\ &+ \tfrac{(1-\alpha)RB^2\cdot RC^2-(\alpha RC^2-RB^2)(\overrightarrow{RB}\cdot\overrightarrow{RC})-2(1-\alpha)(\overrightarrow{RB}\cdot\overrightarrow{RC})^2}{4\Pi}\overrightarrow{RC}.\end{aligned}$$ □

3. Cyclic and Orthocentric Characteristics of a Quadrilateral

Definition 3.1. Given $\square ABCD$, let R be the intersection point of ℓ_{AC} and ℓ_{BD}.

(1) The *cyclic characteristic* of $\square ABCD$ is defined by

$$\kappa_c = (\overrightarrow{RA}\cdot\overrightarrow{RC} - \overrightarrow{RB}\cdot\overrightarrow{RD})^2. \tag{3.1}$$

(2) The *orthocentric characteristic* of $\square ABCD$ is defined by

$$\kappa_o = (\overrightarrow{RA}\cdot\overrightarrow{RC} + \overrightarrow{RB}\cdot\overrightarrow{RD})^2 - 4(\overrightarrow{RA}\cdot\overrightarrow{RB})(\overrightarrow{RC}\cdot\overrightarrow{RD}). \tag{3.2}$$

Write $\overrightarrow{RA} = -\alpha\overrightarrow{RC}$ and $\overrightarrow{RD} = -\beta\overrightarrow{RB}$ for some $\alpha, \beta \in \mathbb{R}\setminus\{-1, 0\}$. We have

$$\kappa_c = (\beta RB^2 - \alpha RC^2)^2$$

and

$$\begin{aligned}\kappa_o &= (\beta RB^2 + \alpha RC^2)^2 - 4\alpha\beta(\overrightarrow{RB}\cdot\overrightarrow{RC})^2 \\ &= (\beta RB^2 - \alpha RC^2)^2 + 4\alpha\beta[RB^2\cdot RC^2 - (\overrightarrow{RB}\cdot\overrightarrow{RC})^2] \\ &= \kappa_c + 4\alpha\beta[RB^2\cdot RC^2 - (\overrightarrow{RB}\cdot\overrightarrow{RC})^2].\end{aligned} \tag{3.3}$$

If $\alpha\beta < 0$, then the first equation in (3.3) implies $\kappa_o \geqslant 0$. On the other hand, $\kappa_c \geqslant 0$ by definition. Hence if $\alpha\beta > 0$, then the bottom equation of (3.3) implies $\kappa_o \geqslant 0$. So both κ_c and κ_o are always non-negative.

Proposition 3.1. *$\kappa_o = 0$ if and only if $\beta RB^2 + \alpha RC^2 = 0$ and $\overrightarrow{RB}\cdot\overrightarrow{RC} = 0$.*

Proof. $\Rightarrow$) When $\kappa_o = 0$, we have $(\beta RB^2 + \alpha RC^2)^2 = 4\alpha\beta(\overrightarrow{RB}\cdot\overrightarrow{RC})^2$. We will prove that $\beta RB^2 + \alpha RC^2 = 0$ which also implies $\overrightarrow{RB}\cdot\overrightarrow{RC} = 0$.
Suppose that $\beta RB^2 + \alpha RC^2 \neq 0$. Then $4\alpha\beta(\overrightarrow{RB}\cdot\overrightarrow{RC})^2 = (\beta RB^2 + \alpha RC^2)^2 > 0$, so $\alpha\beta > 0$. Since $(\beta RB^2 + \alpha RC^2)^2 - 4\alpha\beta RB^2\cdot RC^2 = (\beta RB^2 - \alpha RC^2)^2 \geqslant 0$, we have

$$(\beta RB^2 + \alpha RC^2)^2 \geqslant 4\alpha\beta RB^2\cdot RC^2.$$

By the Cauchy-Schwartz inequality,

$$4\alpha\beta RB^2\cdot RC^2 \leqslant (\beta RB^2 + \alpha RC^2)^2 = 4\alpha\beta(\overrightarrow{RB}\cdot\overrightarrow{RC})^2 \leqslant 4\alpha\beta RB^2\cdot RC^2.$$

This leads to $|\overrightarrow{RB}\cdot\overrightarrow{RC}| = RB\cdot RC$, or equivalently, $\overrightarrow{RB}$ and $\overrightarrow{RC}$ are collinear. Since B, R, and C are collinear and A, R, and C are collinear, A, B, and C are collinear, a contradiction. So $\beta RB^2 + \alpha RC^2 = 0$.
$\Leftarrow$) The converse follows from the definition of κ_o. □

Proposition 3.2. *□$ABCD$ is cyclic (resp. orthocentric) if and only if $\kappa_c = 0$ (resp. $\kappa_o = 0$).*

Proof. Denote the circumcenter of $\triangle ABC$ by O. By equation (2.2) in Lemma 2.1, we have $2\overrightarrow{RB}\cdot\overrightarrow{RO} = RB^2 - \alpha RC^2$ and $2\overrightarrow{RC}\cdot\overrightarrow{RO} = (1-\alpha)RC^2$. By definition, □$ABCD$ is cyclic (resp. orthocentric) if and only if $OB = OD$ (resp. $\overrightarrow{OD} = \overrightarrow{OA} + \overrightarrow{OB} + \overrightarrow{OC}$).
(1) Note that $\overrightarrow{OB} = \overrightarrow{RB} - \overrightarrow{RO}$ and $\overrightarrow{OD} = \overrightarrow{RD} - \overrightarrow{RO} = -\beta\overrightarrow{RB} - \overrightarrow{RO}$. Then

$$\begin{aligned} OB = OD &\iff (1-\beta)RB^2 = 2\overrightarrow{RB}\cdot\overrightarrow{RO} \\ &\iff (1-\beta)RB^2 = RB^2 - \alpha RC^2 \\ &\iff \beta RB^2 - \alpha RC^2 = 0 \\ &\iff \kappa_c = 0.\end{aligned}$$

(2) Note that $\{\overrightarrow{RB}, \overrightarrow{RC}\}$ is a basis for $\mathcal{V}$.

$$\begin{aligned}
\overrightarrow{OD} = \overrightarrow{OA} + \overrightarrow{OB} + \overrightarrow{OC} &\iff \overrightarrow{RD} = \overrightarrow{RA} + \overrightarrow{RB} + \overrightarrow{RC} - 2\overrightarrow{RO} \\
&\iff (1+\beta)\overrightarrow{RB} + (1-\alpha)\overrightarrow{RC} = 2\overrightarrow{RO} \\
&\iff \begin{cases} [(1+\beta)\overrightarrow{RB} + (1-\alpha)\overrightarrow{RC}] \cdot \overrightarrow{RB} &= 2\overrightarrow{RO} \cdot \overrightarrow{RB} \\ [(1+\beta)\overrightarrow{RB} + (1-\alpha)\overrightarrow{RC}] \cdot \overrightarrow{RC} &= 2\overrightarrow{RO} \cdot \overrightarrow{RC} \end{cases} \\
&\iff \begin{cases} \beta RB^2 + \alpha RC^2 &= (1-\alpha)(\overrightarrow{RB} \cdot \overrightarrow{RC}) \\ (1+\beta)(\overrightarrow{RB} \cdot \overrightarrow{RC}) &= 0 \end{cases} \\
&\iff \begin{cases} \beta RB^2 + \alpha RC^2 &= 0 \\ \overrightarrow{RB} \cdot \overrightarrow{RC} &= 0 \end{cases}
\end{aligned}$$

By Proposition 3.1, $\square ABCD$ is orthocentric if and only if $\kappa_o = 0$. $\square$

The following lemma is used in the proof of Theorem 5.1.

Lemma 3.1. *Let A, B, C, and D be points in a plane with the following properties:*

(1) no three of these points are collinear.
(2) ℓ_{AC} and ℓ_{BD} intersect at a point R.
(3) $\overrightarrow{RA} = -\alpha\overrightarrow{RC}$ and $\overrightarrow{RD} = -\beta\overrightarrow{RB}$ for some $\alpha, \beta \in \mathbb{R} \setminus \{-1, 0\}$.

Then A, B, C, and D determine a quadrilateral if and only if either $\alpha > 0$ or $\beta > 0$.

Proof. By definition, A, B, C, and D do not determine a quadrilateral if and only if either $\overline{AB} \cap \overline{CD} \neq \emptyset$ or $\overline{AD} \cap \overline{BC} \neq \emptyset$. Note that $\ell_{AB} \cap \ell_{CD} \neq \emptyset$ if and only if there exist $s, t \in \mathbb{R}$ satisfying

$$\begin{aligned}
&(1-s)\overrightarrow{RA} + s\overrightarrow{RB} = (1-t)\overrightarrow{RC} + t\overrightarrow{RD} \\
\iff\ & s\overrightarrow{RB} - \alpha(1-s)\overrightarrow{RC} = -\beta t\overrightarrow{RB} + (1-t)\overrightarrow{RC} \\
\iff\ & s = -\beta t \quad \text{and} \quad -\alpha(1-s) = 1-t \\
\iff\ & \begin{pmatrix} 1 & \beta \\ \alpha & 1 \end{pmatrix} \begin{pmatrix} s \\ t \end{pmatrix} = \begin{pmatrix} 0 \\ 1+\alpha \end{pmatrix} \\
\iff\ & \alpha\beta \neq 1, \quad s = \tfrac{-(1+\alpha)\beta}{1-\alpha\beta}, \quad \text{and} \quad t = \tfrac{1+\alpha}{1-\alpha\beta}.
\end{aligned}$$

Then

$$\begin{aligned}
\overline{AB} \cap \overline{CD} \neq \emptyset &\iff \alpha\beta \neq 1, \quad 0 < \tfrac{-(1+\alpha)\beta}{1-\alpha\beta} < 1, \quad \text{and} \quad 0 < \tfrac{1+\alpha}{1-\alpha\beta} < 1 \\
&\iff \text{either } (-1 < \alpha < 0 \text{ and } -1 < \beta < 0) \text{ or } (\alpha < -1 \text{ and } \beta < -1).
\end{aligned}$$

Note that $\ell_{AD} \cap \ell_{BC} \neq \emptyset$ if and only if there are $s, t \in \mathbb{R}$ satisfying

$$\begin{aligned} & (1-s)\overrightarrow{RA} + s\overrightarrow{RD} = (1-t)\overrightarrow{RB} + t\overrightarrow{RC} \\ \iff & -\beta s\overrightarrow{RB} - \alpha(1-s)\overrightarrow{RC} = t\overrightarrow{RB} + (1-t)\overrightarrow{RC} \\ \iff & -\beta s = 1-t \quad \text{and} \quad -\alpha(1-s) = t \\ \iff & \begin{pmatrix} -\beta & 1 \\ \alpha & -1 \end{pmatrix} \begin{pmatrix} s \\ t \end{pmatrix} = \begin{pmatrix} 1 \\ \alpha \end{pmatrix} \\ \iff & \alpha \neq \beta, \quad s = \tfrac{\alpha+1}{\alpha-\beta}, \quad \text{and} \quad t = \tfrac{\alpha+\alpha\beta}{\alpha-\beta}. \end{aligned}$$

Then

$$\begin{aligned} \overline{AD} \cap \overline{BC} \neq \emptyset \iff & \alpha \neq \beta, \quad 0 < \tfrac{\alpha+1}{\alpha-\beta} < 1, \quad \text{and} \quad 0 < \tfrac{\alpha+\alpha\beta}{\alpha-\beta} < 1 \\ \iff & \text{either } (-1 < \alpha < 0 \text{ and } \beta < -1) \text{ or } (\alpha < -1 \text{ and } -1 < \beta < 0). \end{aligned}$$

So A, B, C, and D do not determine a quadrilateral if and only if both $\alpha < 0$ and $\beta < 0$. □

Remark 3.1. A purely geometric proof of Lemma 3.1 can be obtained using the Crossbar Theorem. A thorough discussion of the Crossbar Theorem is given in [5].

4. Convex and Non-Convex Quadrilaterals

We say that $\square ABCD$ is *convex* if the region enclosed by $\square ABCD$ is a convex set, that is, given any two points X, Y in the enclosed region, the line segment $\overline{XY}$ lies in the region. Given $\square ABCD$, denote the intersection point of ℓ_{AC} and ℓ_{BD} by R. Write $\overrightarrow{RA} = -\alpha\overrightarrow{RC}$ and $\overrightarrow{RD} = -\beta\overrightarrow{RB}$, where $\alpha, \beta \in \mathbb{R} \setminus \{-1, 0\}$.

Note that $\overline{AB} \cap \overline{CD} = \emptyset$ and $\overline{AD} \cap \overline{BC} = \emptyset$. If $\alpha > 0$ and $\beta > 0$, then $\square ABCD$ is convex. If $\alpha\beta < 0$, then $\square ABCD$ is non-convex; more precisely, we have

α	β	Non-Convexity of $\square ABCD$
$\alpha > 0$	$-1 < \beta < 0$	D is an interior point of $\triangle ABC$.
$\alpha > 0$	$\beta < -1$	B is an interior point of $\triangle ACD$.
$-1 < \alpha < 0$	$\beta > 0$	A is an interior point of $\triangle BCD$.
$\alpha < -1$	$\beta > 0$	C is an interior point of $\triangle ABD$.

The five figures shown in Figure 4 illustrate the above table.

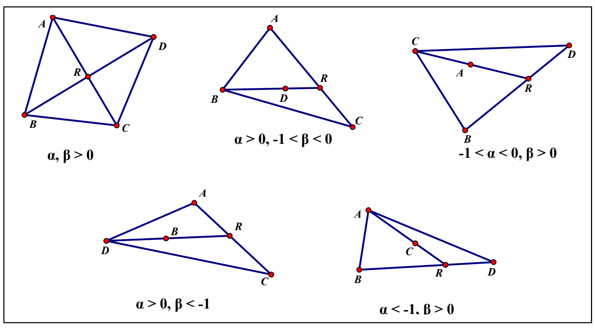

Figure 4. Five Cases of Quadrilaterals

The following proposition characterizes the convexity of $\square ABCD$ in terms of the cyclic and orthocentric characteristics; see Definition 3.1.

Proposition 4.1. *A quadrilateral $\square ABCD$ is convex if $\kappa_o > \kappa_c$ or non-convex if $\kappa_o < \kappa_c$.*

Proof. By the definitions of κ_o and κ_c, we have

$$\begin{aligned}\kappa_o - \kappa_c &= (\beta RB^2 + \alpha RC^2)^2 - 4\alpha\beta(\overrightarrow{RB}\cdot\overrightarrow{RC})^2 - (\beta RB^2 - \alpha RC^2)^2 \\ &= 4\alpha\beta[RB^2 \cdot RC^2 - (\overrightarrow{RB}\cdot\overrightarrow{RC})^2] \\ &= 16\alpha\beta[\text{Area}(\triangle RBC)]^2 \\ &\neq 0\end{aligned} \tag{4.1}$$

Hence κ_c and κ_o cannot vanish simultaneously.

(1) If $\kappa_o > \kappa_c$, then $\alpha\beta > 0$, i.e., $\alpha > 0$ and $\beta > 0$, so $\square ABCD$ is convex.
(2) If $\kappa_o < \kappa_c$, then $\alpha\beta < 0$, so $\square ABCD$ is non-convex. □

Remark 4.1. As observed by equation (4.1) in the proof of Proposition 4.1, κ_o and κ_c do not vanish simultaneously. Hence $0 \leqslant \frac{\kappa_o}{4\kappa_c} \leqslant \infty$. The orthocentric and cyclic quadrilaterals are the limiting cases of the ratio with the general quadrilateral satisfying the strict inequality $0 < \frac{\kappa_o}{4\kappa_c} < \infty$.

Definition 4.1. Following (1.4), set $\Delta = (\overrightarrow{RA}\cdot\overrightarrow{RC})(\overrightarrow{RB}\cdot\overrightarrow{RD}) - (\overrightarrow{RA}\cdot\overrightarrow{RB})(\overrightarrow{RC}\cdot\overrightarrow{RD})$.

(1) The *normalized cyclic characteristic* of $\square ABCD$ is defined by

$$\bar{\kappa}_c = \frac{\kappa_c}{\Delta}. \tag{4.2}$$

(2) The *normalized orthocentric characteristic* of $\square ABCD$ is defined by

$$\bar{\kappa}_o = \frac{\kappa_o}{\Delta}. \tag{4.3}$$

Note that $\bar{\kappa}_c = \frac{\kappa_c}{\alpha\beta\Pi}$ and $\bar{\kappa}_o = \frac{\kappa_o}{\alpha\beta\Pi}$; see (1.3). Then $\bar{\kappa}_o = \bar{\kappa}_c + 4$ by equation (3.3). Note that the sign of each normalized characteristic is the sign of $\alpha\beta$. Moreover, $\square ABCD$ is convex if and only if $\bar{\kappa}_c \geqslant 0$ if and only if $\bar{\kappa}_o > 0$.

5. Circumcenter and Nine-Point Center Quadrilaterals

The four lemmas in this section are used to prove Theorem 5.1. All of them depend upon Lemma 2.1 in §2. Together, these four lemmas and Lemma 6.1 form the proof of Theorem 6.1.

First, we use Lemma 2.1 to express the eight vectors

$$\overrightarrow{RO_a},\quad \overrightarrow{RO_b},\quad \overrightarrow{RO_c},\quad \overrightarrow{RO_d};\quad \overrightarrow{RN_a},\quad \overrightarrow{RN_b},\quad \overrightarrow{RN_c},\quad \overrightarrow{RN_d}$$

in terms of the basis $\{\overrightarrow{RB}, \overrightarrow{RC}\}$ for $\mathcal{V}$. The details are given below for two of the eight vectors. The proofs for the remaining six vectors are similar. All eight are summarized in Lemma 5.1 just ahead.

To begin, recall from (1.3) the notation $\Pi = RB^2 \cdot RC^2 - (\overrightarrow{RB}\cdot\overrightarrow{RC})^2$ as well as $x = \overrightarrow{RB}\cdot\overrightarrow{RC}$. Consider $\triangle BCD$, where $R \in \ell_{BD}$. By Lemma 2.1, the $\overrightarrow{RC}$-coefficient of $\overrightarrow{RO_a}$ (see (2.2)) is

$$\frac{RB^2[RC^2 - \beta RB^2 + (\beta-1)x]}{2\Pi}.$$

On the other hand, because $\overrightarrow{RD} = -\beta\overrightarrow{RB}$, the $\overrightarrow{RB}$-coefficient of $\overrightarrow{RO_a}$ is

$$\frac{(1-\beta)RB^2 \cdot RC^2 + (\beta RB^2 - RC^2)x]}{2\Pi}.$$

Next, apply Lemma 2.1 to $\triangle CDA$, where $R \in \ell_{AC}$, i.e., setting $C = A$, $D = B$, and $A = C$. So α is replaced by $\frac{1}{\alpha}$, i.e., $\overrightarrow{RC} = -\frac{1}{\alpha}\overrightarrow{RA}$. Since $\overrightarrow{RD} = -\beta\overrightarrow{RB}$, we get $RD^2 = \beta^2 RB^2$ and $\overrightarrow{RD} \cdot \overrightarrow{RA} = \alpha\beta(\overrightarrow{RB} \cdot \overrightarrow{RC})$. Then the $\overrightarrow{RD} = -\beta\overrightarrow{RB}$ summand of $\overrightarrow{RO_b}$ is

$$\begin{aligned}\frac{RA^2[RD^2 - \frac{1}{\alpha}RA^2 - (1-\frac{1}{\alpha})(\overrightarrow{RD}\cdot\overrightarrow{RA})]}{2[RD^2 \cdot RA^2 - (\overrightarrow{RD}\cdot\overrightarrow{RA})^2]}\overrightarrow{RD} &= \frac{-\beta\alpha^2 RC^2[\beta^2 RB^2 - \alpha RC^2 - \beta(\alpha-1)(\overrightarrow{RB}\cdot\overrightarrow{RC})]}{2\alpha^2\beta^2\Pi}\overrightarrow{RB} \\ &= \frac{RC^2[\alpha RC^2 - \beta^2 RB^2 + \beta(\alpha-1)x]}{2\beta\Pi}\overrightarrow{RB}.\end{aligned}$$

Similarly, the $\overrightarrow{RC}$-summand of $\overrightarrow{RO_b}$ is

$$\frac{(\beta^2 RB^2 - \alpha RC^2)x - \beta(\alpha-1)RB^2 \cdot RC^2}{2\beta\Pi}\overrightarrow{RC}.$$

The vector $\overrightarrow{RN_a}$ is determined in terms of $\overrightarrow{RB}$ and $\overrightarrow{RC}$ using equation (2.3) in Lemma 2.1. In summary, the four circumcenters and four nine-point centers are given by the vector equations listed below.

Lemma 5.1. *Let $x = \overrightarrow{RB} \cdot \overrightarrow{RC}$. The four circumcenters of $\square ABCD$ are given by*

$$\begin{aligned}\overrightarrow{RO_a} &= \frac{(1-\beta)RB^2 \cdot RC^2 + (\beta RB^2 - RC^2)x}{2\Pi}\overrightarrow{RB} + \frac{RB^2[RC^2 - \beta RB^2 - (1-\beta)x]}{2\Pi}\overrightarrow{RC}, \\ \overrightarrow{RO_b} &= \frac{RC^2[\beta(\alpha-1)x + \alpha RC^2 - \beta^2 RB^2]}{2\beta\Pi}\overrightarrow{RB} + \frac{(\beta^2 RB^2 - \alpha RC^2)x - \beta(\alpha-1)RB^2 \cdot RC^2}{2\beta\Pi}\overrightarrow{RC}, \\ \overrightarrow{RO_c} &= \frac{(\alpha^2 RC^2 - \beta RB^2)x - \alpha(\beta-1)RB^2 \cdot RC^2}{2\alpha\Pi}\overrightarrow{RB} + \frac{RB^2[\alpha(\beta-1)x + \beta RB^2 - \alpha^2 RC^2]}{2\alpha\Pi}\overrightarrow{RC}, \\ \overrightarrow{RO_d} &= \frac{RC^2[RB^2 - \alpha RC^2 - (1-\alpha)x]}{2\Pi}\overrightarrow{RB} + \frac{(1-\alpha)RB^2 \cdot RC^2 + (\alpha RC^2 - RB^2)x}{2\Pi}\overrightarrow{RC}.\end{aligned}$$

Additionally, the four nine-point centers are given by

$$\begin{aligned}\overrightarrow{RN_a} &= \frac{-2(1-\beta)x^2 - (\beta RB^2 - RC^2)x + (1-\beta)RB^2 \cdot RC^2}{4\Pi}\overrightarrow{RB} \\ &\quad + \frac{-2x^2 + (1-\beta)RB^2 x + RB^2 \cdot RC^2 + \beta RB^4}{4\Pi}\overrightarrow{RC}, \\ \overrightarrow{RN_b} &= \frac{2\beta^2 x^2 + \beta(1-\alpha)RC^2 x - \beta^2 RB^2 \cdot RC^2 - \alpha RC^4}{4\beta\Pi}\overrightarrow{RB} \\ &\quad + \frac{2\beta(\alpha-1)x^2 + (\alpha RC^2 - \beta^2 RB^2)x + \beta(1-\alpha)RB^2 \cdot RC^2}{4\beta\Pi}\overrightarrow{RC}, \\ \overrightarrow{RN_c} &= \frac{2\alpha(\beta-1)x^2 + (\beta RB^2 - \alpha^2 RC^2)x + \alpha(1-\beta)RB^2 \cdot RC^2}{4\alpha\Pi}\overrightarrow{RB} \\ &\quad + \frac{2\alpha^2 x^2 + \alpha(1-\beta)RB^2 x - \alpha^2 RB^2 \cdot RC^2 - \beta RB^4}{4\alpha\Pi}\overrightarrow{RC}, \\ \overrightarrow{RN_d} &= \frac{-2x^2 + (1-\alpha)RC^2 x + RB^2 \cdot RC^2 + \alpha RC^4}{4\Pi}\overrightarrow{RB} \\ &\quad + \frac{-2(1-\alpha)x^2 - (\alpha RC^2 - RB^2)x + (1-\alpha)RB^2 \cdot RC^2}{4\Pi}\overrightarrow{RC}.\end{aligned}$$

Using Lemma 5.1, the vectors between the circumcenters as well as the vectors between the nine-point centers are expressed in terms of the basis $\{\overrightarrow{RB}, \overrightarrow{RC}\}$ as follows:

Lemma 5.2. *Let $x = \overrightarrow{RB} \cdot \overrightarrow{RC}$ and $\Pi = RB^2 \cdot RC^2 - x^2$. The six circumcenter vectors are*

$$\begin{aligned}
\overrightarrow{O_aO_b} &= \tfrac{-(\beta RB^2-\alpha RC^2)[RC^2+\beta x]}{2\beta\Pi}\overrightarrow{RB} + \tfrac{(\beta RB^2-\alpha RC^2)[\beta RB^2+x]}{2\beta\Pi}\overrightarrow{RC},\\
\overrightarrow{O_aO_c} &= \tfrac{-(1+\alpha)(\beta RB^2-\alpha RC^2)x}{2\alpha\Pi}\overrightarrow{RB} + \tfrac{(1+\alpha)(\beta RB^2-\alpha RC^2)RB^2}{2\alpha\Pi}\overrightarrow{RC},\\
\overrightarrow{O_aO_d} &= \tfrac{(\beta RB^2-\alpha RC^2)[RC^2-x]}{2\Pi}\overrightarrow{RB} + \tfrac{(\beta RB^2-\alpha RC^2)[RB^2-x]}{2\Pi}\overrightarrow{RC},\\
\overrightarrow{O_bO_c} &= \tfrac{(\beta RB^2-\alpha RC^2)[\alpha RC^2-\beta x]}{2\alpha\beta\Pi}\overrightarrow{RB} + \tfrac{(\beta RB^2-\alpha RC^2)[\beta RB^2-\alpha x]}{2\alpha\beta\Pi}\overrightarrow{RC},\\
\overrightarrow{O_bO_d} &= \tfrac{(1+\beta)(\beta RB^2-\alpha RC^2)RC^2}{2\beta\Pi}\overrightarrow{RB} + \tfrac{-(1+\beta)(\beta RB^2-\alpha RC^2)x}{2\beta\Pi}\overrightarrow{RC},\\
\overrightarrow{O_cO_d} &= \tfrac{(\beta RB^2-\alpha RC^2)[\alpha RC^2+x]}{2\alpha\Pi}\overrightarrow{RB} + \tfrac{-(\beta RB^2-\alpha RC^2)[RB^2+\alpha x]}{2\alpha\Pi}\overrightarrow{RC}.
\end{aligned}$$

Additionally, the six nine-point center vectors are

$$\begin{aligned}
\overrightarrow{N_aN_b} &= \tfrac{2\beta x^2+\beta(\beta RB^2-\alpha RC^2)x-(\beta RB^2+\alpha RC^2)RC^2}{4\beta\Pi}\overrightarrow{RB}\\
&\quad + \tfrac{2\alpha\beta x^2-(\beta RB^2-\alpha RC^2)x-\beta(\beta RB^2+\alpha RC^2)RB^2}{4\beta\Pi}\overrightarrow{RC},\\
\overrightarrow{N_aN_c} &= \tfrac{(1+\alpha)(\beta RB^2-\alpha RC^2)x}{4\alpha\Pi}\overrightarrow{RB}\\
&\quad + \tfrac{(1+\alpha)[2\alpha x^2-(\beta RB^2+\alpha RC^2)RB^2]}{4\alpha\Pi}\overrightarrow{RC},\\
\overrightarrow{N_aN_d} &= \tfrac{-2\beta x^2+(\beta RB^2-\alpha RC^2)x+(\beta RB^2+\alpha RC^2)RC^2}{4\Pi}\overrightarrow{RB}\\
&\quad + \tfrac{2\alpha x^2+(\beta RB^2-\alpha RC^2)x-(\beta RB^2+\alpha RC^2)RB^2}{4\Pi}\overrightarrow{RC},\\
\overrightarrow{N_bN_c} &= \tfrac{-2\alpha\beta x^2+\beta(\beta RB^2-\alpha RC^2)x+\alpha(\beta RB^2+\alpha RC^2)RC^2}{4\alpha\beta\Pi}\overrightarrow{RB}\\
&\quad + \tfrac{2\alpha\beta x^2+\alpha(\beta RB^2-\alpha RC^2)x-\beta(\beta RB^2+\alpha RC^2)RB^2}{4\alpha\beta\Pi}\overrightarrow{RC},\\
\overrightarrow{N_bN_d} &= \tfrac{(1+\beta)[-2\beta x^2+(\beta RB^2+\alpha RC^2)RC^2]}{4\beta\Pi}\overrightarrow{RB}\\
&\quad + \tfrac{(1+\beta)(\beta RB^2-\alpha RC^2)x}{4\beta\Pi}\overrightarrow{RC},\\
\overrightarrow{N_cN_d} &= \tfrac{-2\alpha\beta x^2-(\beta RB^2-\alpha RC^2)x+\alpha(\beta RB^2+\alpha RC^2)RC^2}{4\alpha\Pi}\overrightarrow{RB}\\
&\quad + \tfrac{-2\alpha x^2+\alpha(\beta RB^2-\alpha RC^2)x+(\beta RB^2+\alpha RC^2)RB^2}{4\alpha\Pi}\overrightarrow{RC}.
\end{aligned}$$

Lemma 5.3. *Assume that $\square ABCD$ is not cyclic. Let R_o be the intersection point of $\ell_{O_aO_c}$ and $\ell_{O_bO_d}$. Then*

$$\begin{aligned}
\overrightarrow{R_oO_a} &= -\alpha\overrightarrow{R_oO_c}, & \overrightarrow{R_oO_d} &= -\beta\overrightarrow{R_oO_b},\\
R_oO_b^2 &= \tfrac{\alpha}{\beta}\cdot\tfrac{\bar{\kappa}_c RC^2}{4}, & R_oO_c^2 &= \tfrac{\beta}{\alpha}\cdot\tfrac{\bar{\kappa}_c RB^2}{4},\\
\overrightarrow{R_oO_b}\cdot\overrightarrow{R_oO_c} &= \tfrac{\bar{\kappa}_c}{4}(\overrightarrow{RB}\cdot\overrightarrow{RC}), & \tfrac{\overrightarrow{R_oO_b}\cdot\overrightarrow{R_oO_c}}{R_oO_b\cdot R_oO_c} &= \operatorname{sgn}(\alpha\beta)\tfrac{\overrightarrow{RB}\cdot\overrightarrow{RC}}{RB\cdot RC}.
\end{aligned}$$

Consequently, if $\bar{\kappa}_c'$ and $\bar{\kappa}_o'$ are the normalized cyclic and normalized orthocentric characteristics of $\square O_aO_bO_cO_d$, respectively, then $\bar{\kappa}_c' = \bar{\kappa}_c$ and $\bar{\kappa}_o' = \bar{\kappa}_o$.

Proof. Choose a point R' on $\ell_{O_aO_c}$ such that $\overrightarrow{R'O_a} = -\alpha\overrightarrow{R'O_c}$, i.e.,

$$\begin{aligned}\overrightarrow{RR'} &= \tfrac{1}{1+\alpha}\overrightarrow{RO_a} + \tfrac{\alpha}{1+\alpha}\overrightarrow{RO_c}\\ &= \tfrac{(1-\beta)RB^2\cdot RC^2-(1-\alpha)RC^2x}{2\Pi}\overrightarrow{RB} + \tfrac{(1-\alpha)RB^2\cdot RC^2-(1-\beta)RB^2x}{2\Pi}\overrightarrow{RC}.\end{aligned} \tag{5.1}$$

Using Lemma 5.1 along with $\overrightarrow{R'O_a} = \overrightarrow{RO_a} - \overrightarrow{RR'}$, etc., we get

$$\begin{aligned}\overrightarrow{R'O_a} &= \tfrac{\beta RB^2-\alpha RC^2}{2\Pi}(x\overrightarrow{RB} - RB^2\overrightarrow{RC}),\\ \overrightarrow{R'O_b} &= \tfrac{\beta RB^2-\alpha RC^2}{2\beta\Pi}(-RC^2\overrightarrow{RB} + x\overrightarrow{RC}),\\ \overrightarrow{R'O_c} &= \tfrac{\beta RB^2-\alpha RC^2}{2\alpha\Pi}(-x\overrightarrow{RB} + RB^2\overrightarrow{RC}),\\ \overrightarrow{R'O_d} &= \tfrac{\beta RB^2-\alpha RC^2}{2\Pi}(RC^2\overrightarrow{RB} - x\overrightarrow{RC}).\end{aligned} \tag{5.2}$$

Consequently, $\overrightarrow{R'O_d} = -\beta\overrightarrow{R'O_b}$ and hence R' is the intersection point of $\ell_{O_aO_c}$ and $\ell_{O_bO_d}$, i.e., $R_o = R'$. So $\overrightarrow{R_oO_a} = -\alpha\overrightarrow{R_oO_c}$, $\overrightarrow{R_oO_d} = -\beta\overrightarrow{R_oO_b}$. Then, we have

$$R_oO_b^2 = \tfrac{RC^2(\beta^2RB^2-\alpha RC^2)^2}{4\beta^2\Pi} = \tfrac{\alpha}{\beta}\cdot\tfrac{\bar{\kappa}_cRC^2}{4}, \tag{5.3}$$

$$R_oO_c^2 = \tfrac{RB^2(\beta^2RB^2-\alpha RC^2)^2}{4\alpha^2\Pi} = \tfrac{\beta}{\alpha}\cdot\tfrac{\bar{\kappa}_cRB^2}{4}, \tag{5.4}$$

$$\overrightarrow{R_oO_b}\cdot\overrightarrow{R_oO_c} = \tfrac{(\overrightarrow{RB}\cdot\overrightarrow{RC})(\beta RB^2-\alpha RC^2)^2}{4\alpha\beta\Pi} = \tfrac{\bar{\kappa}_c(\overrightarrow{RB}\cdot\overrightarrow{RC})}{4}, \tag{5.5}$$

and

$$\tfrac{\overrightarrow{R_oO_b}\cdot\overrightarrow{R_oO_c}}{R_oO_b\cdot R_oO_c} = \operatorname{sgn}(\alpha\beta)\tfrac{\overrightarrow{RB}\cdot\overrightarrow{RC}}{RB\cdot RC}.$$

Finally, using (5.3), (5.4), and (5.5), the normalized cyclic characteristic is

$$\begin{aligned}\bar{\kappa}'_c &= \frac{(\beta R_oO_b^2-\alpha R_oO_c^2)^2}{\alpha\beta[R_oO_b^2\cdot R_oO_c^2-(\overrightarrow{R_oO_b}\cdot\overrightarrow{R_oO_c})^2]}\\ &= \frac{[\frac{\bar{\kappa}_c}{4}(\alpha RC^2-\beta RB^2)]^2}{\alpha\beta\{\frac{\bar{\kappa}_c^2}{16}[RB^2\cdot RC^2-(\overrightarrow{RB}\cdot\overrightarrow{RC})^2]\}}\\ &= \bar{\kappa}_c.\end{aligned}$$

In the same way, we show $\bar{\kappa}'_o = \bar{\kappa}_o$. ☐

Lemma 5.4. *Assume that ☐ABCD is not orthocentric. Let R_n be the intersection point of $\ell_{N_aN_c}$ and $\ell_{N_bN_d}$. Then*

$$\begin{aligned}\overrightarrow{R_nN_a} &= -\alpha\overrightarrow{R_nN_c}, & \overrightarrow{R_nN_d} &= -\beta\overrightarrow{R_nN_b},\\ R_nN_b^2 &= \tfrac{\alpha}{\beta}\cdot\tfrac{\bar{\kappa}_oRC^2}{16}, & R_nN_c^2 &= \tfrac{\beta}{\alpha}\cdot\tfrac{\bar{\kappa}_oRB^2}{16},\\ \overrightarrow{R_nN_b}\cdot\overrightarrow{R_nN_c} &= \tfrac{\bar{\kappa}_o}{16}(\overrightarrow{RB}\cdot\overrightarrow{RC}), & \tfrac{\overrightarrow{R_nN_b}\cdot\overrightarrow{R_nN_c}}{R_nN_b\cdot R_nN_c} &= \operatorname{sgn}(\alpha\beta)\tfrac{\overrightarrow{RB}\cdot\overrightarrow{RC}}{RB\cdot RC}.\end{aligned}$$

Consequently, if $\bar{\kappa}''_c$ and $\bar{\kappa}''_o$ are the normalized cyclic and normalized orthocentric characteristics of ☐$N_aN_bN_cN_d$, respectively, then $\bar{\kappa}''_c = \bar{\kappa}_c$ and $\bar{\kappa}''_o = \bar{\kappa}_o$.

Proof. Choose a point S' on $\ell_{N_aN_c}$ such that $\overrightarrow{S'N_a} = -\alpha\overrightarrow{S''N_c}$, i.e.,

$$\begin{aligned}\overrightarrow{RS'} &= \tfrac{1}{1+\alpha}\overrightarrow{RN_a} + \tfrac{\alpha}{1+\alpha}\overrightarrow{RN_c} \\ &= \tfrac{(1-\beta)RB^2\cdot RC^2+(1-\alpha)RC^2x-2(1-\beta)x^2}{4\Pi}\overrightarrow{RB} + \tfrac{(1-\alpha)RB^2\cdot RC^2+(1-\beta)RB^2x-2(1-\alpha)x^2}{4\Pi}\overrightarrow{RC}.\end{aligned} \tag{5.6}$$

Using Lemma 5.1 along with $\overrightarrow{S'N_a} = \overrightarrow{RN_a} - \overrightarrow{RS'}$, etc., we get

$$\begin{aligned}\overrightarrow{S'N_a} &= \tfrac{-(\beta RB^2-\alpha RC^2)x}{4\Pi}\overrightarrow{RB} + \tfrac{[(\beta RB^2+\alpha RC^2)RB^2-2\alpha x^2]}{4\Pi}\overrightarrow{RC}, \\ \overrightarrow{S'N_b} &= \tfrac{[-(\beta RB^2+\alpha RC^2)RC^2+2\beta x^2]}{4\beta\Pi}\overrightarrow{RB} + \tfrac{-(\beta RB^2-\alpha RC^2)x}{4\beta\Pi}\overrightarrow{RC}, \\ \overrightarrow{S'N_c} &= \tfrac{(\beta RB^2-\alpha RC^2)x}{4\alpha\Pi}\overrightarrow{RB} + \tfrac{[-(\beta RB^2+\alpha RC^2)RB^2+2\alpha x^2]}{4\alpha\Pi}\overrightarrow{RC}, \\ \overrightarrow{S'N_d} &= \tfrac{[(\beta RB^2+\alpha RC^2)RC^2-2\beta x^2]}{4\Pi}\overrightarrow{RB} + \tfrac{(\beta RB^2-\alpha RC^2)x}{4\Pi}\overrightarrow{RC}.\end{aligned}$$

Consequently, $\overrightarrow{S'N_d} = -\beta\overrightarrow{S'N_b}$, and hence S' is the intersection point of the lines $\ell_{N_aN_c}$ and $\ell_{N_bN_d}$, i.e., $R_n = S'$. So $\overrightarrow{R_nN_a} = -\alpha\overrightarrow{R_nN_c}$ and $\overrightarrow{R_nN_d} = -\beta\overrightarrow{R_nN_b}$. Moreover, we have

$$R_nN_b^2 = \tfrac{RC^2[(\beta^2RB^2+\alpha RC^2)^2-4\alpha\beta x^2]}{16\beta^2\Pi} = \tfrac{\alpha}{\beta}\cdot\tfrac{\bar\kappa_o RC^2}{16}, \tag{5.7}$$

$$R_nN_c^2 = \tfrac{RB^2[(\beta^2RB^2+\alpha RC^2)^2-4\alpha\beta x^2]}{16\alpha^2\Pi} = \tfrac{\beta}{\alpha}\cdot\tfrac{\bar\kappa_o RB^2}{16}, \tag{5.8}$$

$$\overrightarrow{R_nN_b}\cdot\overrightarrow{R_nN_c} = \tfrac{(\overrightarrow{RB}\cdot\overrightarrow{RC})[(\beta^2RB^2+\alpha RC^2)^2-4\alpha\beta(\overrightarrow{RB}\cdot\overrightarrow{RC})^2]}{16\alpha\beta\Pi} = \tfrac{\bar\kappa_o}{16}(\overrightarrow{RB}\cdot\overrightarrow{RC}), \tag{5.9}$$

and

$$\tfrac{\overrightarrow{R_nN_b}\cdot\overrightarrow{R_nN_c}}{R_nN_b\cdot R_nN_c} = \operatorname{sgn}(\alpha\beta)\tfrac{\overrightarrow{RB}\cdot\overrightarrow{RC}}{RB\cdot RC}.$$

Finally, using the (5.7), (5.8), and (5.9) equations $\bar\kappa_c'' = \bar\kappa_c$ and $\bar\kappa_o'' = \bar\kappa_o$ follow routinely. □

In summary, Lemmas 3.1, 5.3, 5.4, and Definition 4.1 prove the following theorem.

Theorem 5.1. *(1) If $\square ABCD$ is not cyclic, then $\square O_aO_bO_cO_d$ is a well-defined quadrilateral and its normalized cyclic and normalized orthocentric characteristics are equal to $\bar\kappa_c$ and $\bar\kappa_o$, respectively.*

(2) If $\square ABCD$ is not orthocentric, then $\square N_aN_bN_cN_d$ is a well-defined quadrilateral and its normalized cyclic and normalized orthocentric characteristics are equal to $\bar\kappa_c$ and $\bar\kappa_o$, respectively.

6. Two Similar Quadrilaterals Associated to a General Quadrilateral

Lemma 6.1. *Given $\square ABCD$ and any pairwise distinct $I, J, K, L \in \{A, B, C, D\}$, we have*

$$O_iO_j^2 = \tfrac{c_{IJ}\bar\kappa_c}{4}KL^2 \quad \textit{and} \quad N_iN_j^2 = \tfrac{c_{IJ}\bar\kappa_o}{16}KL^2, \tag{6.1}$$

where

$$\begin{aligned} c_{AB} &= \tfrac{\alpha}{\beta}, & c_{BC} &= \tfrac{1}{\alpha\beta}, & c_{AC} &= \tfrac{\beta(1+\alpha)^2}{\alpha(1+\beta)^2}, \\ c_{CD} &= \tfrac{\beta}{\alpha}, & c_{AD} &= \alpha\beta, & c_{BD} &= \tfrac{\alpha(1+\beta)^2}{\beta(1+\alpha)^2}.\end{aligned}$$

Proof. The proof to follow uses Lemma 5.2 repeatedly. First of all,

$$\begin{aligned}-\beta\overrightarrow{RB}\cdot\overrightarrow{RC} &= \overrightarrow{RD}\cdot\overrightarrow{RC}\\ &= \tfrac{1}{2}(RD^2+RC^2-CD^2)\\ &= \tfrac{1}{2}(\beta^2RB^2+RC^2-CD^2)\end{aligned}$$

so that

$$\begin{aligned}CD^2 &= \beta^2RB^2+RC^2+2\beta\overrightarrow{RB}\cdot\overrightarrow{RC}\\ &= 2\beta x+\beta^2RB^2+RC^2.\end{aligned}\tag{6.2}$$

Equation (6.2) will be used in the proof of the equation for $O_aO_b^2$.
Recall $x = \overrightarrow{RB}\cdot\overrightarrow{RC}$, $\Pi = RB^2\cdot RC^2 - x^2$ from (1.3), and $\Delta = \alpha\beta\Pi$ from (1.4). Then the vector equation for $\overrightarrow{O_aO_b}$ in Lemma 5.2 and the algebraic properties of the dot product imply

$$\begin{aligned}4\beta^2\Pi^2O_aO_b^2 &= (\beta RB^2-\alpha RC^2)^2(RC^2+\beta x)^2RB^2\\ &\quad -2x(\beta RB^2-\alpha RC^2)^2(RC^2+\beta x)(\beta RB^2+x)\\ &\qquad +(\beta RB^2-\alpha RC^2)^2(\beta RB^2+x)^2RC^2\\ &= \kappa_c[(RC^2+\beta x)^2RB^2-2x(RC^2+\beta x)(\beta RB^2+x)+(\beta RB^2+\mu)^2RC^2]\\ &= \kappa_c[RB^2\cdot RC^2(2\beta x+\beta^2RB^2+RC^2)-x^2(2\beta x+\beta^2RB^2+RC^2)]\\ &= \kappa_c(RB^2\cdot RC^2-x^2)(2\beta x+\beta^2RB^2+RC^2)\\ &= \kappa_c\Pi CD^2.\end{aligned}$$

Hence, $O_aO_b^2 = \frac{c_{AB}\bar{\kappa}_c}{4}CD^2$ holds.
To prove $N_aN_b^2 = \frac{\alpha}{\beta}\cdot\frac{\bar{\kappa}_o\cdot CD^2}{16}$, let $Y = RB^2$ and $Z = RC^2$. Define

$$\begin{aligned}f(x) &= 2\beta x^2+\beta(\beta Y-\alpha Z)x-(\beta Y+\alpha Z),\\ g(x) &= 2\alpha\beta x^2-(\beta Y-\alpha Z)x-\beta(\beta Y+\alpha Z).\end{aligned}$$

By Lemma 5.2,

$$4\beta\Pi\overrightarrow{N_aN_b} = f(x)\overrightarrow{RB}+g(x)\overrightarrow{RC}.$$

By the algebraic properties of the dot product

$$16\beta^2\Pi^2N_aN_b^2 = [f(x)]^2Y+2xf(x)g(x)+[g(x)]^2Z.\tag{6.3}$$

The right side of (6.3) is the degree-five polynomial

$$h(x) = a_5x^5+a_4x^4+a_3x^3+a_2x^2+a_1x+a_0,$$

where

$$\begin{aligned}
a_5 &= 8\alpha\beta,\\
a_4 &= 4\alpha\beta^3 Y + 4\alpha\beta Z,\\
a_3 &= -2\beta^3 Y^2 - 12\alpha\beta^2 YZ - 2\alpha^2\beta Z^2,\\
a_2 &= -\beta^4 Y^3 - \beta^2 Y^2 Z - 6\alpha\beta^3 Y^2 Z - 6\alpha\beta YZ^2 - \alpha^2\beta^2 YZ^2 - \alpha^2 Z^3,\\
a_1 &= 2\beta^3 Y^3 Z + 4\alpha\beta^2 Y^2 Z^2 + 2\alpha^2\beta YZ^3,\\
a_0 &= \beta^4 Y^4 Z + \beta^2 Y^3 Z^2 + 2\alpha\beta^3 Y^3 Z^2 + 2\alpha\beta Y^2 Z^3 + \alpha^2\beta^2 Y^2 Z^3 + \alpha^2 YZ^4.
\end{aligned}$$

By equations (3.2) and (6.2),

$$\begin{aligned}
h(x) &= 8\alpha\beta^2 x^3(x^2 - YZ) + (4\alpha\beta^3 Y + 4\alpha\beta Z)x^2(x^2 - YZ)\\
&\quad - 2\beta(\beta Y + \alpha Z)^2 x(x^2 - YZ) - (\beta Y + \alpha Z)^2(\beta^2 Y + Z)(x^2 - YZ)\\
&= [8\alpha\beta^2 x^3 + (4\alpha\beta^3 Y + 4\alpha\beta Z)x^2 - 2\beta(\beta Y + \alpha Z)^2 x\\
&\quad - (\beta Y + \alpha Z)^2(\beta^2 Y + Z)](x^2 - YZ)\\
&= \{2\beta[4\alpha\beta x^2 - (\beta Y + \alpha Z)^2]x + (\beta^2 Y + Z)[4\alpha\beta x^2 - (\beta Y + \alpha Z)^2]\}(x^2 - YZ)\\
&= [(\beta Y + \alpha Z)^2 - 4\alpha\beta x^2](2\beta x + \beta^2 Y + Z)(YZ - x^2)\\
&= \kappa_o \Pi CD^2.
\end{aligned}$$

Hence $16\beta^2\Pi^2 N_a N_b^2 = \kappa_o \Pi CD^2$ so that

$$N_a N_b^2 = \tfrac{\kappa_o \cdot CD^2}{16\beta^2\Pi} = \left(\tfrac{\alpha}{\beta} \cdot \tfrac{\bar{\kappa}_o}{16}\right) CD^2.$$

Next, we verify (6.1) for $O_a O_c^2$ and $N_a N_c^2$. First, write

$$2\alpha\Pi\overrightarrow{O_a O_c} = f(x)\overrightarrow{RB} + g(x)\overrightarrow{RC}$$

where

$$\begin{aligned}
f(x) &= -(1+\alpha)(\beta RB^2 - \alpha RC^2)x,\\
g(x) &= (1+\alpha)(\beta RB^2 - \alpha RC^2)RB^2.
\end{aligned}$$

Then

$$\begin{aligned}
4\alpha^2\Pi^2 O_a O_c^2 &= [f(x)]^2 RB^2 + 2xf(x)g(x) + [g(x)]^2 RC^2\\
&= (1+\alpha)^2\kappa_c RB^2(RB^2 \cdot RC^2 - x^2)\\
&= (1+\alpha)^2\kappa_c \Pi RB^2.
\end{aligned}$$

However, since $\overrightarrow{BD} = \overrightarrow{RD} - \overrightarrow{RB} = -(1+\beta)\overrightarrow{RB}$, we get $BD^2 = (1+\beta)^2 RB^2$. Hence

$$O_a O_c^2 = \left[\tfrac{\beta(1+\alpha)^2}{\alpha(1+\beta)^2} \cdot \tfrac{\bar{\kappa}_c}{4}\right] BD^2.$$

To obtain the second equation, proceed in the same way by setting $Y = RB^2$, $Z = RC^2$, and $x = \overrightarrow{RB} \cdot \overrightarrow{RC}$. Next, define

$$\begin{aligned}
f(x) &= (\beta Y - \alpha Z)x,\\
g(x) &= 2\alpha x^2 - (\beta Y + \alpha Z)Y
\end{aligned}$$

so that

$$\tfrac{4\alpha\Pi}{1+\alpha}\overrightarrow{N_aN_c} = f(x)\overrightarrow{RB} + g(x)\overrightarrow{RC}.$$

Then

$$\begin{aligned}\tfrac{16\alpha^2\Pi^2}{(1+\alpha)^2}N_aN_c^2 &= [f(x)]^2Y + 2xf(x)g(x) + [g(x)]^2Z,\\ &= 4\alpha\beta x^4 + (-\beta^2Y^3 - 6\alpha\beta Y^2Z - \alpha^2YZ^2)x^2\\ &\quad + \beta^2Y^4Z + 2\alpha Y^3Z^2 + \alpha^2Y^2Z^3\\ &= 4\alpha\beta Y(x^2 - YZ)x^2 + Y(\beta Y + \alpha Z)^2(YZ - x^2)\\ &= Y[(\beta Y + \alpha Z)^2 - 4\alpha\beta x^2](YZ - x^2)\\ &= \tfrac{\kappa_o\Pi}{(1+\beta)^2}BD^2.\end{aligned}$$

Hence $N_aN_c^2 = \left[\frac{\beta(1+\alpha)^2}{\alpha(1+\beta)^2}\cdot\frac{\bar{\kappa}_o}{16}\right]BD^2$.
The remaining four pairs of squared distances are argued in similar fashion. □

We say that $\square A_1A_2A_3A_4$ and $\square B_1B_2B_3B_4$ are *similar*, written $\square A_1A_2A_3A_4 \sim \square B_1B_2B_3B_4$, if $\angle A_i = \angle B_i$ for all $i \in \{1,2,3,4\}$ and the six ratios $\frac{A_iA_j}{B_iB_j}$ are equal to each other for all distinct $i, j \in \{1,2,3,4\}$,. Note that if the six ratios are equal to each other, then the four angle equations must hold.

Theorem 6.1. *Assume that $\square ABCD$ is not cyclic. Let R be the intersection point of ℓ_{AC} and ℓ_{BD}. Let κ_c and κ_o be the cyclic and orthocentric characteristics of $\square ABCD$, respectively; see (3.1) and (3.2). Following the earlier notation, let O_a, O_b, O_c, and O_d denote the circumcenters and N_a, N_b, N_c, and N_d denote the nine-point centers of $\triangle BCD$, $\triangle CDA$, $\triangle DAB$, and $\triangle ABC$, respectively. Then for any distinct $i, j \in \{a,b,c,d\}$, $\frac{N_iN_j^2}{O_iO_j^2} = \frac{\kappa_o}{4\kappa_c}$. Consequently, if $\square ABCD$ is a general quadrilateral, then $\square O_aO_bO_cO_d \sim \square N_aN_bN_cN_d$.*

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