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CONTACT STRUCTURES COMING FROM FOLIATIONS ON (2N+1)-CLOSED MANIFOLDS

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ABSTRACT. In this paper we establish a sufficient condition for a contact structure on a closed oriented (2n+1)-dimensional manifold to come from a codimension 1 foliation. Moreover, this condition allow us to generalize a theorem of Etnyre (see [7]) in dimension 3 to (2n+1)-dimensional K-contact manifolds with non-zero first de Rham cohomology group. In conclusion, we give some examples of manifolds possessing such structures.

1. INTRODUCTION

Let M be a differential manifold, TM its tangent bundle and $\xi \subset TM$, a field of hyperplanes on M, that is a C^∞ differentiable sub-bundle of codimension 1 of TM. Locally ξ can always be written as the kernel of a non-vanishing 1-form η. Moreover if the orthogonal complement of ξ in TM is orientable, then ξ is globally defined by a 1-form η. In this study, the manifold M will be assumed to be oriented and all the plane fields considered here are supposed to be coorientable.
There are two classes of hyperplane fields that have received an important attention: the integrable and the non-integrable one. A hyperplane field ξ with the property that through any point $p \in M$ one can find a codimension 1 submanifold S such that $T_xS = \xi_x$ for all $x \in S$, is called integrable and S an integral submanifold of M.
The collection of integral submanifolds of an integrable hyperplane field constitutes what is called a codimension 1 foliation on M. It turns out from the Frobenius integrability condition, that $\xi = \ker \eta$ is integrable if and only if η satisfy

$$\eta \wedge d\eta = 0.$$

Contact structures are in certain sense the exact opposite of integrable hyperplane fields. A contact structure in a $(2n+1)$-dimensional manifold is a maximally non-integrable hyperplane field $\xi = \ker \eta$ where the 1-form η is required to satisfy $\eta \wedge (d\eta)^n \neq 0$. Meaning that $\eta \wedge (d\eta)^n$ is a volume form on M. Such an η is called a contact form and the pair (M, η) a contact manifold and there exist always a unique vector field Z, called the Reeb vector field of the contact manifold M satisfying:

$$i_Z\eta = 1 \quad and \quad i_Zd\eta = 0.$$

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To see the one big difference between these two stuctures, first remark that: if $\xi = \ker \eta$ is a contact stucture defined in $U \subset \mathbb{R}^3$, with coordinates (x, y, z). Suppose that at each point of U, $\frac{\partial}{\partial y} \in \xi$ and $\frac{\partial}{\partial z} \notin \xi$. Then there exists $q : U \longrightarrow \mathbb{R}$ such that

$$\eta = dz - q(x, y, z)dx.$$

Futhermore

$$u = \frac{\partial}{\partial y} \quad and \quad v = q\frac{\partial}{\partial z} + \frac{\partial}{\partial x} \in \xi.$$

We have $\eta \wedge d\eta = -(\frac{\partial q}{\partial y})dx \wedge dy \wedge dz$, then if we fix the standard orientation $-dx \wedge dy \wedge dz$ on U, the contact contion becomes $\dfrac{\partial q}{\partial y} > 0$. So ξ consist of horizontal planes (that is , parallel to the xy-plane) at any point in the xz-plane and as you leave the xz-plane along a ray perpendicular to the xz-plane the plane ξ are always tangent to this ray and twisting a total of 90-degree in a counterclockwise.
Another major difference is given by Gray's Theorem [4]. This theorem says that isotopies of contact forms are equivalent to isotopies of the manifold. We have the converse situation in the case of codimension one foliation: there are deformations of foliations that do not come from diffeomorphisms of the underlying 3-manifold see ([8]).

Despite these differences one has the following local result of Darboux/ Pfaff Theorem, see [4] which gives similarities between foliations and contact structure that is they both have local normal forms.
Other important kind of similarities between theses structures are given by the following results separately proved by Thurston and Eliashberg: Namely they state that if M is a closed oriented irreducible 3-manifold, ξ a 2-plane field and N a closed embedded surface in M, if ξ is a Reebless foliation (Thurston 1986, [12]) or a tight positive contact structure (Eliashberg 1992, [5]), then

$$|e(\xi)[N]| \leqslant -\chi(N) \quad if \ N \neq \mathbb{S}^2$$
$$|e(\xi)[N]| = 0 \quad if \ N = \mathbb{S}^2,$$

where $\chi(N)$ is the Euler characteristic of N *and* $e(\xi)[N] \in H^2(M)$ the Euler class $e(\xi)$ on N.

At the first glance, codimension 1 foliation and contact structures belong entirely to two different worlds. However, the two theories have developed a number of striking similarities and their understanding can produce information about the topology of the underlying manifold. It is then relevant to know how to transport the results from one field to the other. Eliashberg and Thurston are the first who investigate this way and their theory of confoliations (kernel of a non-vanishing 1-form η such that $\eta \wedge (d\eta)^n \geq 0$ holds) is the middle ground of these two field. Theses works of Eliashberg and Thurston open a large field of research, which study the transport from codimension 1 foliation into contact structure in several ways : approximations, deformations, affine deformation etc. Precisely a codimension 1 foliation ξ defined by a non singular 1-form α on M, is called C^r-close to a contact structure if in any C^r-neighborhood of ξ relatively to the C^r-topology of plane field of Withney, there is a contact structure.
ξ is called C^r-deformable into a contact structure, if there is on M a 1-parameter family

hyperplane field $(\xi_t)_{t\geqslant 0}$ of class C^r, defined by the 1-form α_t such that $\xi_0 = \xi$ and for all $t > 0$, α_t is a contact form.
It is well known from [6], through confoliation theory that : Any oriented codimension 1, C^2-foliation on an oriented 3-manifold is C^0-close to a contact structure, except the product foliation of $S^2 \times S^1$ by spheres S^2. But it was then unknown if this approximation can always be done through a deformation. This result brings up the following question in [6] : **Is it always possible to deform a codimension 1 foliation on an oriented $(2n+1)$-manifold into contact structure** ? Even the fact that it is not easy to answer positively to this question, it will be important to find conditions of deformability. The first author get success in this direction by proving in [2] a necessary and sufficient condition to deform an integrable 1-form into contact form. Nonetheless it is very difficult to say that a codimension 1 foliation is not deformable into contact.
In [7], Etnyre study the deformation in the reversed sense and he proved that: every positive and negative contact structure on a closed oriented 3-manifold is a C^∞-deformation of a C^∞-foliation. The point of view of Etnyre gives us the idea to investigate analytic ways, in order to partially generalize his theorem in higher odd dimension.
More precisely we prove that: let (M, β) be a closed, $(2n+1)$-dimensional contact manifold with Reeb vector field Z which contain an integrable 1-form α such that $\beta \wedge (d\alpha) \wedge (d\beta)^{n-1} = 0$ and $\alpha(Z) = 0$. Then the contact structure defined by β converges to the codimension 1 foliation $\{\alpha = 0\}$. This later result guides us to be able to generalize the above theorem in $(2n+1)$-dimensional K-contact manifold M with $H^1(M) \neq 0$. In order to face theses goals, we deal with particular deformations called CB-deformations see [2]. Here C and B are two C^∞-functions : $[0, +\infty[\longrightarrow [0, +\infty[$ with $C(0) = 1$ and $B(0) = 0$.

2. Sufficient condition of convergence of contact structures

In this section unless otherwise stated, M is a $(2n+1)$-dimensional closed oriented manifold.

Definition 2.1. *A contact structure $\zeta = \{\beta = 0\}$ on M comes from (or converges to) a codimension 1 foliation $\xi = \{\alpha = 0\}$ via (CB)-deformations if there exists a 1-parameter family of hyperplane fields $(\xi_t)_{t\geq 0}$ defined by the 1-forms $\alpha_t = C(t)\alpha + B(t)\beta$ such that $\alpha_0 = \alpha$, $\alpha_1 = \beta$ and for all $t > 0$, α_t is contact. One can say also that the contact structure ζ converges to a codimension one foliation ξ.*

Remark 2.1. *If $C(t) = 1 - t$ and $B(t) = t$, we recover the linear deformations introduced by Dathe-Rukimbira in* [3].

Let us recall that a flow $X^t : M \to M$ generated by a vector field X is called Conformally Anosov if there exists a continuous Riemannian metric on M and a continuous splitting $TM^3 = N_+ \oplus N_- \oplus \lambda X$, such that the splitting is invariant under the flow and the differential $dX^t : TM \to TM$ acts by dialations on N_+ and by contractions on N_- after $dX^t|_{N_+ \oplus N_-}$ has been renormalized to have determinant 1. It follows from [6] that, if ξ_+ and ξ_-, generated (X, N_+) et (X, N_-) respectively are C^1-smooth, then they are integrable and are called the unstable and stable foliations of the Conformally-Ansosov flow X^t.

Proposition 2.1. *Let M be a closed oriented 3-manifold, Suppose X^t is a Conformally Anosov flow with C^1-smooth stable and unstable foliations $\xi_+ = ker\alpha_+$ and $\xi_- = ker\alpha_-$. Then $\xi = ker(\alpha_- + \alpha_+)$ is a contact structure coming from ξ_+ or ξ_-.*

Proof. We know from [6] that:

$$\alpha_- \wedge d\alpha_+ + \alpha_+ \wedge d\alpha_- > 0.$$

Then $\beta = \alpha_- + \alpha_+$, is a positive contact form. Consider for all $t \geq 0$, the 1-forms $\alpha_t^+ = C(t)\alpha_+ + B(t)\beta$. One has $\alpha_0^+ = \alpha_+$ and for all $t > 0$:

$$\alpha_t^+ \wedge d\alpha_t^+ = (C(t)B(t) + B^2(t))(\alpha_- \wedge d\alpha_+ + \alpha_+ \wedge d\alpha_-).$$

Since the functions C and B should be chosen more generally, one can, in particular, choose $C(t) = \sin^2[(t+1)\frac{\pi}{2}]$ and $B(t) = t\exp(t-1)$. Then we establish that $\xi = ker(\alpha_- + \alpha_+)$ is a contact structure coming from ξ_+, by using the above computation. The same argument is true for ξ_-. □

The authors of [2] proved the following theorem

Theorem 2.2 ([2]). *Let (V, β, Z) be a closed, $(2n+1)$-dimensional contact manifold with Reeb vector field Z. A 1-form α integrable on V admits a deformation of type CB via β if and only if*

$$\alpha \wedge (d\beta)^n + n\beta \wedge d\alpha \wedge (d\beta)^{n-1} \geq 0. \tag{2.1}$$

The weakness of certain conditions of the theorem 2.2 allow us to give a sufficient condition for a contact structure to converge to a codimension 1 foliation. And this generalize the Ethnyre theorem in some odd dimensional manifold. Precisely we have the following result:

Theorem 2.3. *Let (M, β) be a closed, $(2n+1)$-dimensional contact manifold with Reeb vector field Z, which contain an integrable 1-form α such that :*

i) $\beta \wedge (d\alpha) \wedge (d\beta)^{n-1} = 0$
ii) $\alpha(Z) = 0$

Then the contact sructure ζ defined by β comes from the codimension 1 foliation $\xi = \{\alpha = 0\}$ via CB-deformations.

Proof. The conditions (i) and (ii) give that $\alpha \wedge (d\beta)^n + n\beta \wedge d\alpha \wedge (d\beta)^{n-1} = 0$. So from the theorem 2.2, α admit a deformation of type CB via β.
If we fix $C(t) = \sin^2[(t+1)\frac{\pi}{2}]$, $B(t) = t\exp(t-1)$ and let $\alpha_t = C(t)\alpha + B(t)\beta$, then we have

$$\alpha_t \wedge (d\alpha_t)^n > 0, \quad \forall t > 0; \quad \alpha_0 = \alpha \quad \text{and} \quad \alpha_1 = \beta.$$

Hence the 1-forms $\alpha_t = C(t)\alpha + B(t)\beta$ define a 1-parameter family of hyperplane fields ξ_t which satisfy $\xi_0 = \xi$, $\xi_1 = \zeta$ and ξ_t is a contact structure $\forall t > 0$. □

3. Generalisation of Etnyre result in K-contact manifold

Let (M, β, Z) be a $(2n+1)$-dimensional contact manifold, with contact form β an Reeb vector field Z. A contact metric structure on M is given by the existence of a Riemannian metric g and $(1,1)$-tensor field ϕ such that [1]:

$$\phi^2 X + X = \beta(X)Z, \tag{3.1}$$

$$d\beta(X, Y) = 2g(X, \phi Y), \tag{3.2}$$

$$g(\phi X, \phi Y) = g(X, Y) - \beta(X)\beta(Y), \tag{3.3}$$

for all vector fields X, Y on M, where

$$d\beta(X, Y) = X\beta(Y) - Y\beta(X) - \beta([X, Y]).$$

Notice that identity (3.1) implies that $\phi Z = 0$ and $\beta \circ \phi = 0$. We say that (M, β, Z, g, ϕ) is K-contact manifod if the Reeb field Z is Killing, i.e. $L_Z g = 0$.

Theorem 3.1. *A K-contact structure on a $(2n+1)$-dimensional closed oriented manifold M such that $\dim(H^1(M)) \neq 0$, converges into a codimension 1 foliation.*

Proof. Let (β, Z) be this K-contact form with Reeb vector field Z. Since $\dim(H^1(M)) \neq 0$ there exists a non singular closed 1-form α on M with $[\alpha] \neq 0$. By Hodge's Decomposition Theorem, α is cohomologous to a non singular harmonic 1-form μ. It follows from [11] that $\mu(Z) = 0$ and since μ is closed it satisfies also the statements of Theorem 2.3. This complete the proof. □

4. Some examples where we find this kind of deformations

In certain quotients of a Lie group under a discrete subgroup we can prove

Theorem 4.1. *Let M be a closed 3-manifold diffeomorphic to a quotient of the Lie group G under a discrete subgroup Γ action by left multiplication, where G is one of the following.*

- *$S\tilde{L}_2$, the universal cover of $PSL_2\mathbb{R}$,*
- *$\tilde{E}_2$, the universal cover of group of orientation preserving isometries of the Euclidean plane.*

Then there is on M, a codimension 1 foliation which is CB-deformable into contact structures.

Before proving this theorem, we recall first the following notions due to H. Geiges and J. Gonzolo see [9]

Definition 4.1. *A taut contact circle on a 3-manifold M is a pair of contact forms (β_1, β_2) such that the 1-form $\lambda_1\beta_1 + \lambda_2\beta_2$ is a contact form defining the same volume form for all $(\lambda_1, \lambda_2) \in S^1 \subset \mathbb{R}^2$. Equivalently, we require that the following equations be satisfied:*

$$\begin{aligned} \beta_1 \wedge d\beta_1 &= \beta_2 \wedge d\beta_2 \neq 0 \\ \beta_1 \wedge d\beta_2 &= -\beta_2 \wedge d\beta_1. \end{aligned}$$

If the mixed terms $\beta_1 \wedge d\beta_2$ and $\beta_2 \wedge d\beta_1$ are identically zero rather than just of opposite sign, we speak of a Cartan structure.

If (β_1, β_2) is a taut contact circle, then so is $(f\beta_1, f\beta_2)$ for any positive function f. The conformal class of a taut contact circle (β_1, β_2) is the collection of all pairs $(f\beta_1, f\beta_2)$ obtained from (β_1, β_2) by multiplication by some positive function f.

We will give a detailed proof of the lemma 3.1 in [10] by setting the following proposition which we will use later in the proof the theorem 4.1.

Proposition 4.1. *Let (β_1, β_2) be a Cartan structure on a 3-manifold with, respectively, Z_1 and Z_2 the Reeb vectors field of β_1 and β_2. Then there is a unique 1-form β_3 such that*

$$d\beta_1 = \beta_2 \wedge \beta_3$$

$$\begin{aligned} d\beta_2 &= \beta_3 \wedge \beta_1 \\ \beta_3(Z_1) &= \beta_3(Z_2) = \beta_1 \wedge d\beta_3 = \beta_2 \wedge d\beta_3 = 0. \end{aligned}$$

Proof. The definition 4.1 implies that:

$$\beta_1 \wedge (d\beta_1) = \beta_2 \wedge (d\beta_2), \tag{4.1}$$

$$\beta_1 \wedge (d\beta_2) = 0, \tag{4.2}$$

$$\beta_2 \wedge (d\beta_1) = 0. \tag{4.3}$$

If we compute by i_{Z_2} in (4.3) and by i_{Z_1} in (4.2) one has:

$$d\beta_1 = \beta_2 \wedge i_{Z_2} d\beta_1. \tag{4.4}$$

$$d\beta_2 = \beta_1 \wedge i_{Z_1} d\beta_2. \tag{4.5}$$

Computing also by i_{Z_2} in (4.2) and by i_{Z_1} in (4.3) one has:

$$\beta_1(Z_2) = \beta_2(Z_1) = 0. \tag{4.6}$$

Furthermore computing by i_{Z_2} in (4.1) one has also:

$$\beta_1(Z_2) d\beta_1 - \beta_1 \wedge i_{Z_2} d\beta_1 = d\beta_2 \tag{4.7}$$

Thus (4.5), (4.6) and (4.7) implies that:

$$-\beta_1 \wedge i_{Z_2} d\beta_1 = \beta_1 \wedge i_{Z_1} d\beta_2.$$

Hence by computing this equation by i_{Z_1} we have:

$$-i_{Z_2} d\beta_1 = i_{Z_1} d\beta_2. \tag{4.8}$$

Since $\beta_2 \wedge d\beta_2$ is volume form and $div(Z_1) = 0$ then

$$d(i_{Z_1}(\beta_2 \wedge d\beta_2)) = div(Z_1)\beta_2 \wedge d\beta_2 = 0,$$

which means that:

$$d(\beta_2(Z_1) d\beta_2 - \beta_2 \wedge i_{Z_1} d\beta_2) = 0.$$

Therefore (4.6) and the differentiation implies that:

$$\beta_2 \wedge d(i_{Z_1} d\beta_2) = d\beta_2 \wedge i_{Z_1} d\beta_2 = \frac{1}{2} i_{Z_1}(d\beta_2 \wedge d\beta_2) = 0.$$

Analogously, changing Z_1 by Z_2 and β_2 by β_1 one prouves that:

$$\beta_1 \wedge d(i_{Z_2} d\beta_1) = 0 = \beta_2 \wedge d(i_{Z_1} d\beta_2). \tag{4.9}$$

Hence setting $\beta_3 = i_{Z_1} d\beta_2$, the equalities (4.9), (4.8), (4.4) and (4.5) gives us the existence of a 1-form β_3 such that:

$$\begin{aligned} d\beta_1 &= \beta_2 \wedge \beta_3, \\ d\beta_2 &= \beta_3 \wedge \beta_1, \\ \beta_3(Z_1) &= \beta_3(Z_2) = \beta_1 \wedge (d\beta_3) = \beta_2 \wedge (d\beta_3) = 0. \end{aligned}$$

It is straightforward to check that β_3 is unique: indeed if there exists a 1-form β_4 satisfying the above conditions, then $d\beta_1 = \beta_2 \wedge \beta_3 = \beta_2 \wedge \beta_4$ and computing by i_{Z_2} in this last equality one has $\beta_4 = \beta_3$, because $\beta_4(Z_2) = \beta_3(Z_2) = 0$. □

Definition 4.2. *A Cartan structure (β_1, β_2) is called a K-Cartan structure if the unique β_3 of the proposition 4.1 satisfies*

$$d\beta_3 = K\beta_1 \wedge \beta_2.$$

Here K may be any function on M that is constant along the common kernel of β_1 and β_2 (since $dK \wedge \beta_1 \wedge \beta_2 = d^2\beta_3 = 0$), but the two cases of interest to us will be $K \equiv -1$ and $K \equiv 0$.

Proof. (**of theorem** 4.1)

- If $M = \Gamma \backslash S\tilde{L}_2$, following H. Geiges and J. Gonzalo from [10], in each conformal class of taut contact circles on M there is one and only one 1-Cartan structure. Let $(\beta_1, \beta_2, \beta_3)$ be such a structure in any arbitrary conformal class, following Jacobowitz [10] it is equivalent to the existence on M of a triple of independent 1-forms (η_1, η_2, η_3) called a **projective structure**, satisfying

$$d\eta_1 = -\eta_2 \wedge \eta_3, \quad d\eta_2 = -2\eta_1 \wedge \eta_2, \quad d\eta_3 = 2\eta_1 \wedge \eta_3.$$

  such that $\beta_1 = 2\eta_1$, $\beta_2 = \eta_2 + \eta_3$, $\beta_3 = \eta_2 - \eta_3$.
  Setting $\alpha = \eta_2$ one has

$$\alpha \wedge d\alpha = 0. \tag{4.10}$$

$$\alpha \wedge d\beta_1 = 0. \tag{4.11}$$

  Let Z_1 be the Reeb vector field of β_1 then the proposition 4.1 and its proof imply that $\beta_2(Z_2) = \beta_3(Z_1) = 0$. Futhermore by the equalities $\beta_2 = \eta_2 + \eta_3$, $\beta_3 = \eta_2 - \eta_3$ one has

$$\alpha(Z_1) = \eta_2(Z_1) = \frac{\beta_2(Z_1) + \beta_3(Z_1)}{2} = 0. \tag{4.12}$$

  Hence the theorem 2.3 implies in virtue of (4.10), (4.11) and (4.12) that the foliation defined by α is CB-deformable into contact structures by the way of β_1.
- If $M = \Gamma \backslash \tilde{E}_2$, Also following [10], in each conformal class of taut contact circles on M there is a 0-Cartan structure which is unique up to multiplication by a positive constant. Let $(\beta_1, \beta_2, \beta_3)$ such a structure in any arbitrary conformal class, then $d\beta_3 = 0$ and the proposition 4.1, implies that $\beta_3(Z_1) = \beta_3(Z_2) = 0$. Hence theorem 2.3 implies that the foliation defined by β_3 is (C,B)-deformable into contact structures by the way of β_1 or β_2.

□

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