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DISTANCES OF NAPOLEON POINTS AT OTHER NOTABLE POINTS

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ABSTRACT. The purpose of this paper is to find formulas for distances from Napoleon points N_+, N_- to other notable points in triangle geometry. These distances are expressed by Δ, l_+, l_-, f and finally by a, b, c.

Consider a triangle ABC and denote by N_+, N_- the inner and outer Napoleon points associated with it. We also denote by $F_\pm$ the Fermat points, by $J_\pm$ the isodynamic points, and by E the nine-point center (the Euler point) of the given triangle. We consider known the meanings of the notations O, H, G, K. The purpose of this note is to find a lot of nineteen formulas for the distances of points N_+, N_- at points $F_\pm$, $J_\pm$, O, H, G, E, K and between themselves, all these formulas expressed by a, b, c. We will need some formulas of [3], where distances from $F_\pm$, $J_\pm$ to other points of the triangle ABC are found.
For historical notes and properties of these points, the reader is referred to the following treatises: [1], [4], [5], [6], [7].

1. INTRODUCTION AND PRELIMINARIES

Let A_+ and A_- be the vertices of the equilateral triangles built on the BC outside and inside the triangle ABC, respectively; similar for B_+, B_- and C_+, C_- (Fig. 1). It is known that $F_+ = AA_+ \cap BB_+ \cap CC_+$ and $F_- = AA_- \cap BB_- \cap CC_-$ and that $AA_+ = BB_+ = CC_+$ and $AA_- = BB_- = CC_-$ For the common lengths of these segments, denoted l_+ and l_- respectively, we have:

$$l_+^2 = \frac{1}{2}\left(a^2 + b^2 + c^2 + 4\sqrt{3}\Delta\right), \quad l_-^2 = \frac{1}{2}\left(a^2 + b^2 + c^2 - 4\sqrt{3}\Delta\right) \tag{1.1}$$

(see, for ex., [4], p. 220). Note that l_+ and l_- will frequently appear in the formulas for distances to be obtained below. In the calculations that follow we will routinely use the following simple relations:

$$16\Delta^2 = 2a^2b^2 + 2a^2c^2 + 2b^2c^2 - a^4 - b^4 - c^4 \text{ (Heron)}, \tag{1.2}$$

$$l_+^2 + l_-^2 = a^2 + b^2 + c^2, \qquad l_+^2 - l_-^2 = 4\sqrt{3}\Delta, \tag{1.3}$$

$$4l_+^2 l_-^2 = \left(a^2 + b^2 + c^2\right)^2 - 3 \cdot 16\Delta^2, \tag{1.4}$$

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$$9a^2b^2c^2 - 16\Delta^2\left(a^2+b^2+c^2\right) = f\left(a,b,c\right), \tag{1.5}$$

where

$$f\left(a,b,c\right) = a^6+b^6+c^6+3a^2b^2c^2-a^4b^2-a^2b^4-a^4c^2-a^2c^4-b^4c^2-b^2c^4. \tag{1.6}$$

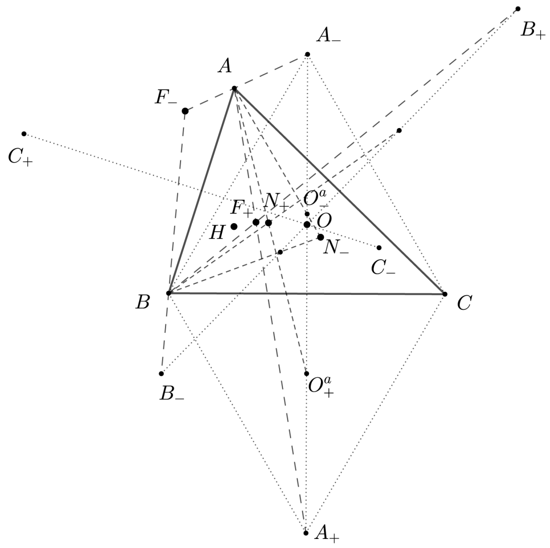

Figure 1

To achieve the proposed goal we will use some geometric properties of the points that are in our attention. Thus, it is known that $F_+J_+ \parallel F_-J_- \parallel OH$ (Fig. 2; [5], Table 5.3 or, for a synthetic proof, [2], p.15). The following collineations relative to the specified points are also known (Fig. 2):

1) O, H, G, E, M (midpoint of HG) in order $O-G-E-M-H$ and $2OH = 3HG = 6OG = 6HM = 4OE = 12GE$ (Euler line),

2) F_+, F_-, K, M in order $M - F_+ - K - F_-$ (K and M are harmonic conjugates with respect to F_+ and F_-),

3) O, F_+, N_+ and O, F_-, N_-,

4) H, J_+, N_+ and H, J_-, N_-,

5) E, F_+, N_- and E, F_-, N_+,

6) K, N_+, N_-.

The collineations in the last four statements appear in [5], Table 5.1 (they can be easily verified using barycentric coordinates).

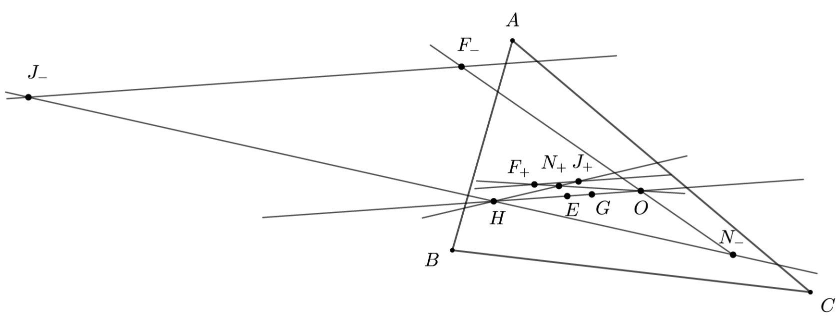

Figure 2

In connection with statements 3)-6), we are interested in the order in which the points are located.

Lemma 1.1. *The positions of point N_+ on lines F_+O and F_-E are given by*
(i) $F_+ - N_+ - O$, and
(ii) $F_- - N_+ - E$.

Proof. We consider the triangle MOF_- (Fig. 3). According to 3) and 5) above, we have $N_+ = OF_+ \cap F_-E$. According to 1) and 2), we have $F_+ \in MF_-$ and $E \in MO$. It follows from this that (i) and (ii) hold. □

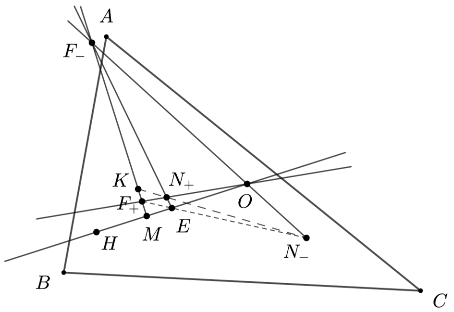

Figure 3

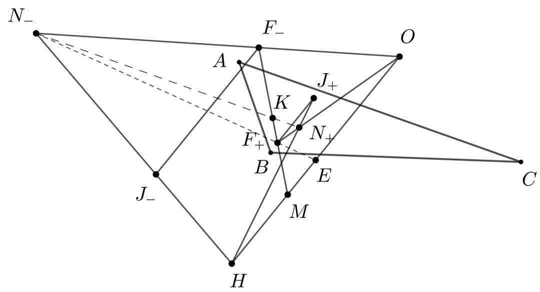

Figure 4

Lemma 1.2. *The position of point N_- on line F_-O is specified by the following statements:*
(i) $N_- \notin F_-O$ (i.e. N_- is not between F_- and O),
(ii) $F_- - O - N_-$ if and only if $4l_-^2 - l_+^2 < 0$,
(iii) $N_- - F_- - O$ if and only if $4l_-^2 - l_+^2 > 0$.

Proof. (i) it is inferred from the collinearity of points F_+, E, N_- and the fact that $E \in OM$ (see 5) and 1) above) (Fig. 3, 4).
Let's denote with O_-^a, O_-^b, O_-^c the circumcenters of equilateral triangles BCA_-, CAB_-, ABC_- respectively. By definition, $N_- = AO_-^a \cap BO_-^b \cap CO_-^c$ if this intersection is non-empty; otherwise, N_- is the infinite point in the direction of the line F_-O. The second case occurs if and only if $AO_-^a \parallel BO_-^b \parallel CO_-^c$.
We will show that the condition $AO_-^a \parallel BO_-^b \parallel CO_-^c$ is equivalent to $4l_-^2 - l_+^2 = 0$ using barycentric coordinates. The barycentric coordinates of points O_-^a, O_-^b, O_-^c are:

$$\left(a, 2b\sin\left(C - \frac{\pi}{6}\right), 2c\sin\left(B - \frac{\pi}{6}\right)\right),$$

$$\left(2a\sin\left(C - \frac{\pi}{6}\right), b, 2c\sin\left(A - \frac{\pi}{6}\right)\right),$$

$$\left(2a\sin\left(B - \frac{\pi}{6}\right), 2b\sin\left(A - \frac{\pi}{6}\right), c\right).$$

The equations of lines BO_-^b and CO_-^c are:

$$c\sin\left(A - \frac{\pi}{6}\right)\alpha = a\sin\left(C - \frac{\pi}{6}\right)\gamma \quad \text{and} \quad b\sin\left(A - \frac{\pi}{6}\right)\alpha = a\sin\left(B - \frac{\pi}{6}\right)\beta.$$

Writing that these lines and the line at infinity, $\alpha+\beta+\gamma=0$, are concurrent, we find the condition that the lines BO_-^b and CO_-^c (therefore also AO_-^a) are parallel:

$$a\sin\left(B-\frac{\pi}{6}\right)\sin\left(C-\frac{\pi}{6}\right)+b\sin\left(C-\frac{\pi}{6}\right)\sin\left(A-\frac{\pi}{6}\right)+c\sin\left(A-\frac{\pi}{6}\right)\sin\left(B-\frac{\pi}{6}\right)=0.$$

Using formulas like $\sin A=\dfrac{2\Delta}{bc}, \cos A=\dfrac{b^2+c^2-a^2}{2bc}$, etc. and performing routine calculations, we will express this condition by the sidelenghts of the given triangle:

$$3\left(a^2+b^2+c^2\right)-20\sqrt{3}\Delta=0 \quad \text{or} \quad 4l_-^2-l_+^2=0,$$

and the claim is proved.
The sign of the expression $4l_-^2-l_+^2$ in relation to the position of the point N_- outside the segment F_-O (Fig. 3, 4) is easily established (for example, taking ABC isosceles triangle). Consequently, the statements (ii) and (iii) are true. This completes the proof. □

Lemma 1.3. *We have:*
(i) $N_- - E - F_+$ *if and only if* $4l_-^2-l_+^2<0$, *and*
(ii) $N_- - F_+ - E$ *if and only if* $4l_-^2-l_+^2>0$.

Proof. (i) From Lemma 2, the condition $4l_-^2-l_+^2<0$ is equivalent to $F_- - O - N_-$, which in turn is equivalent to $N_- - E - F_+$ (Fig. 3).
(ii) Similar arguments are used (Fig. 4). □

Lemma 1.4. *Relative to point* H, *we have:*
(i) $H - N_+ - J_+$,
(ii) $J_- - H - N_-$ *if and only if* $4l_-^2-l_+^2<0$, *and*
(iii) $N_- - J_- - H$ *if and only if* $4l_-^2-l_+^2>0$.

Proof. The statements are direct consequences of Lemmas 1 and 2 and the fact that $F_+J_+ \parallel F_-J_- \parallel OH$ (Fig. 2, 4). □

Lemma 1.5. *We have:*
(i) $K - N_+ - N_-$ *if and only if* $4l_-^2-l_+^2<0$, *and*
(ii) $N_- - K - N_+$ *if and only if* $4l_-^2-l_+^2>0$.

Proof. (i) From Lemma 2, condition $4l_-^2-l_+^2<0$ is equivalent to $F_- - O - N_-$ (Fig. 3). Because we always have $F_+ - K - F_-$ and $F_+ - N_+ - O$ (Lemma 1), it folows that $F_- - O - N_-$ is equivalent to $K - N_+ - N_-$. That is, statement (i) is true.
(ii) It is shown reasoning similarly (Fig. 4). □

The preceding lemmas establish the order of points in the sequences of collinear points 3)-5) indicated above. We will need these results in the next sections.

2. Distances from $N_\pm$ to $F_\pm, J_\pm, O, H$

We will find eight formulas for distances of this type using the fact that $F_+J_+ \parallel F_-J_- \parallel OH$.

Proposition 2.1. *The distances from* N_+ *to points* F_+, O, J_+, H *are given by the formulas:*

$$N_+F_+=\frac{\sqrt{3}}{3l_+\left(4l_+^2-l_-^2\right)}\sqrt{32\Delta^2l_+^2l_-^2+\left(2l_+^2-l_-^2\right)f}, \tag{2.1}$$

$$N_+O = \frac{l_+}{4\Delta\left(4l_+^2 - l_-^2\right)}\sqrt{32\Delta^2 l_+^2 l_-^2 + \left(2l_+^2 - l_-^2\right)f}, \tag{2.2}$$

$$N_+J_+ = \frac{2\sqrt{2}}{2l_+\left(4l_+^2 - l_-^2\right)}\sqrt{8\Delta^2 l_-^4 + l_+^2 f - 3a^2b^2c^2l_-^2}, \tag{2.3}$$

$$N_+H = \frac{\sqrt{6}l_+}{4\Delta\left(4l_+^2 - l_-^2\right)}\sqrt{8\Delta^2 l_-^4 + l_+^2 f - 3a^2b^2c^2l_-^2}. \tag{2.4}$$

Proof. Since $F_+J_+ \parallel OH$, it follows that $\Delta N_+F_+J_+ \sim \Delta N_+OH$ (Fig. 2 or 4). Hence,

$$\frac{N_+F_+}{N_+O} = \frac{N_+J_+}{N_+H} = \frac{F_+J_+}{OH}.$$

From the fact that

$$OH = \frac{1}{4\Delta}\sqrt{f} \tag{2.5}$$

([3], (2.10)) and

$$F_+J_+ = \frac{\sqrt{3}}{3l_+^2}\sqrt{f}$$

([3], (4.3)), it follows that

$$\frac{N_+F_+}{N_+O} = \frac{N_+J_+}{N_+H} = \frac{4\sqrt{3}\Delta}{3l_+^2}.$$

Taking into account the relations $F_+N_+ + N_+O = F_+O$ and $J_+N_+ + N_+H = J_+H$, we infer that

$$N_+F_+ = \frac{4\sqrt{3}\Delta}{4l_+^2 - l_-^2}F_+O \quad \text{and} \quad N_+O = \frac{3l_+^2}{4l_+^2 - l_-^2}F_+O,$$

$$N_+J_+ = \frac{4\sqrt{3}\Delta}{4l_+^2 - l_-^2}J_+H \quad \text{and} \quad N_+H = \frac{3l_+^2}{4l_+^2 - l_-^2}J_+H.$$

Now, using formulas

$$F_+O^2 = \frac{1}{144\Delta^2 l_+^2}\left[32\Delta^2 l_+^2 l_-^2 + \left(2l_+^2 - l_-^2\right)f\right] \tag{2.6}$$

([3], (2.6)) and

$$J_+H^2 = \frac{1}{24\Delta^2 l_+^2}\left(8\Delta^2 l_-^4 + l_+^2 f - 3a^2b^2c^2l_-^2\right)$$

([3], (3.11)), we immediately obtain the required formulas (2.1)-(2.4). This concludes the proof. □

Proposition 2.2. *The distances from N_- to points F_-, O, J_-, H are given by the formulas:*

$$N_-F_- = \frac{\sqrt{3}}{3l_-\left|4l_-^2 - l_+^2\right|}\sqrt{32\Delta^2 l_+^2 l_-^2 + \left(2l_-^2 - l_+^2\right)f}, \tag{2.7}$$

$$N_-O = \frac{l_-}{4\Delta\left|4l_-^2 - l_+^2\right|}\sqrt{32\Delta^2 l_+^2 l_-^2 + \left(2l_-^2 - l_+^2\right)f}, \tag{2.8}$$

$$N_-J_- = \frac{2\sqrt{2}}{2l_-\left|4l_-^2 - l_+^2\right|}\sqrt{8\Delta^2 l_+^4 + l_-^2 f - 3a^2b^2c^2l_+^2}, \tag{2.9}$$

$$N_-H = \frac{\sqrt{6}l_-}{4\Delta\left|4l_-^2 - l_+^2\right|}\sqrt{8\Delta^2 l_+^4 + l_-^2 f - 3a^2b^2c^2 l_+^2}. \tag{2.10}$$

Proof. We will proceed in the same way, but, according to Lemma 2, we consider two cases: I. $4l_-^2 - l_+^2 < 0$, i.e. $F_- - O - N_-$, and II. $4l_-^2 - l_+^2 > 0$, i.e. $N_- - F_- - O$ (Fig. 5). In both cases we have $\Delta N_-F_-J_- \sim \Delta N_-OH$ and therefore

$$\frac{N_-F_-}{N_-O} = \frac{N_-J_-}{N_-H} = \frac{F_-J_-}{OH} = \frac{4\sqrt{3}\Delta}{3l_-^2}$$

(for the last tie we used (2.5) and the formula $F_-J_- = \dfrac{\sqrt{3}}{3l_-^2}\sqrt{f}$ ([3], (4.4))).

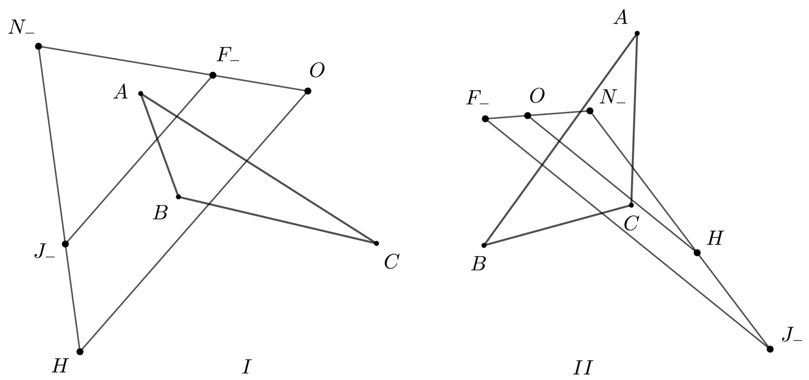

Figure 5

Because in case I we have $N_-F_- = N_-O + OF_-$ and $N_-J_- = N_-H + HJ_-$, we get:

$$N_-F_- = \frac{4\sqrt{3}\Delta}{-\left(4l_-^2 - l_+^2\right)}F_-O \quad \text{and} \quad N_-O = \frac{3l_-^2}{-\left(4l_-^2 - l_+^2\right)}F_-O,$$

$$N_-J_- = \frac{4\sqrt{3}\Delta}{-\left(4l_-^2 - l_+^2\right)}J_-H \quad \text{and} \quad N_-H = \frac{3l_-^2}{-\left(4l_-^2 - l_+^2\right)}J_-H.$$

On the other hand, in case II we have $N_-F_- = N_-O - F_-O$ and $N_-J_- = N_-H - J_-H$, and so we get:

$$N_-F_- = \frac{4\sqrt{3}\Delta}{4l_-^2 - l_+^2}F_-O \quad \text{and} \quad N_-O = \frac{3l_-^2}{4l_-^2 - l_+^2}F_-O,$$

$$N_-J_- = \frac{4\sqrt{3}\Delta}{4l_-^2 - l_+^2}J_-H \quad \text{and} \quad N_-H = \frac{3l_-^2}{4l_-^2 - l_+^2}J_-H.$$

It remains to use the formulas

$$F_-O^2 = \frac{1}{144\Delta^2 l_-^2}\left[32\Delta^2 l_+^2 l_-^2 + \left(2l_-^2 - l_+^2\right)f\right] \tag{2.11}$$

([3], (2.7)) and

$$J_-H^2 = \frac{1}{24\Delta^2 l_-^2}\left(8\Delta^2 l_+^4 + l_-^2 f - 3a^2b^2c^2 l_+^2\right)$$

([3], (3.12)) to infer formulas (2.7)-(2.10), valid in both cases. The proof is complete. □

Remark 2.1. *Since $4l_+^2 - l_-^2 > 0$, we can switch from the formulas in Proposition 6 to the corresponding formulas in Proposition 7 and vice versa by replacing l_+ with l_- and l_- with l_+.*

The formulas for distances N_+F_-, N_+J_- and N_-F_+, N_-J_+ will be established in Section 5.

3. Distances from $N_\pm$ to some points of Euler line

We will only consider points G and E, but we would do the same for point M or another point on the Euler line.

Proposition 3.1. *The distances between points N_+, N_- and E are given by*

$$N_+E = \frac{l_-}{8\Delta\left(4l_+^2 - l_-^2\right)}\sqrt{32\Delta^2 l_+^2 l_-^2 + \left(2l_+^2 - l_-^2\right)f}, \tag{3.1}$$

and

$$N_-E = \frac{l_+}{8\Delta\left|4l_-^2 - l_+^2\right|}\sqrt{32\Delta^2 l_+^2 l_-^2 + \left(2l_-^2 - l_+^2\right)f}. \tag{3.2}$$

Proof. Applying the median theorem to the triangle N_+OH (Fig. 2), we have:

$$4N_+E^2 = 2\left(N_+O^2 + N_+H^2\right) - OH^2.$$

Taking into account (2.2), (2.4) and (2.5), this equality is written:

$$64\Delta^2 N_+E^2 = \frac{2l_+^2}{\left(4l_+^2 - l_-^2\right)^2}\left[32\Delta^2 l_+^2 l_-^2 + \left(2l_+^2 - l_-^2\right)f + 6\left(8\Delta^2 l_-^4 + l_+^2 f - 3a^2b^2c^2 l_-^2\right)\right] - f$$

from where

$$\begin{aligned} 64\Delta^2\left(4l_+^2 - l_-^2\right)^2 N_+E^2 &= l_-^2\left[32\Delta^2 l_+^2\left(2l_+^2 + 3l_-^2\right) - 36a^2b^2c^2 l_+^2 + \left(6l_+^2 - l_-^2\right)f\right] \\ &= l_-^2\left[32\Delta^2 l_+^2\left(2l_+^2 + 3l_-^2\right) - 4l_+^2\left(9a^2b^2c^2 - f\right) + \left(2l_+^2 - l_-^2\right)f\right]. \end{aligned}$$

Now, using (1.5), we get:

$$64\Delta^2\left(4l_+^2 - l_-^2\right)^2 N_+E^2 = l_-^2\left[32\Delta^2 l_+^2 l_-^2 + \left(2l_+^2 - l_-^2\right)f\right],$$

hence the formula (3.1).
Obviously, considering the triangle N_-OH (Fig. 2) and performing similar calculations, we find the formula (3.2). The proof is finished. □

Remark 3.1. *The formulas (3.1), (3.2) pass into each other if we substitute l_+ by l_- and l_- by l_+. This remark is valuable for all subsequent propositions.*

Corollary 3.1. *We have*

$$N_+E = \frac{3l_+l_-}{2\left(4l_+^2 - l_-^2\right)}F_+O$$

and

$$N_-E = \frac{3l_+l_-}{2\left|4l_-^2 - l_+^2\right|}F_-O.$$

Proof. It immediately follows from (3.1), (3.2) and (2.6), (2.11).

We will establish the formulas for distances N_+G and N_-G by applying Stewart's theorem to triangles N_+OH and N_-OH (same as above) and point G (Fig. 2). □

Proposition 3.2. *The distances between N_+, N_- and G are given by*

$$N_+G = \frac{\sqrt{3}}{3\left(4l_+^2 - l_-^2\right)}\sqrt{l_+^2 l_-^2\left(2l_+^2 + l_-^2\right) - 2f} \tag{3.3}$$

and

$$N_-G = \frac{\sqrt{3}}{3\left|4l_+^2 - l_-^2\right|}\sqrt{l_+^2 l_-^2\left(2l_-^2 + l_+^2\right) - 2f}. \tag{3.4}$$

Proof. We will establish only the first formula. According to Stewart's theorem, we have

$$N_+G^2 \cdot OH = N_+O^2 \cdot GH + N_+H^2 \cdot GO - OH \cdot GO \cdot GH.$$

But, it is well known that $GO = \frac{1}{3}OH$ and $GH = \frac{2}{3}OH$. Then,

$$9N_+G^2 = 6N_+O^2 + 3N_+H^2 - 2OH^2.$$

Using (2.2) and (2.4), we find $f + 16\Delta^2\left(l_+^2 + l_-^2\right) +$

$$\begin{aligned} 72\Delta^2 N_+G^2 &= \frac{3l_+^2}{\left(4l_+^2 - l_-^2\right)^2}\left[32\Delta^2 l_+^2 l_-^2 + \left(2l_+^2 - l_-^2\right)f + 3\left(8\Delta^2 l_-^4 + l_+^2 f - 3a^2b^2c^2 l_-^2\right)\right] - f \\ &= \frac{3l_+^2}{\left(4l_+^2 - l_-^2\right)^2}\left[32\Delta^2 l_+^2 l_-^2 + 24\Delta^2 l_-^4 - 9a^2b^2c^2 l_-^2 + \left(5l_+^2 - l_-^2\right)f\right] - f \end{aligned}$$

and thus, replacing $9a^2b^2c^2$ according to (1.5) and (1.3),

$$72\Delta^2 N_+G^2 = \frac{3l_+^2}{\left(4l_+^2 - l_-^2\right)^2}\left[16\Delta^2 l_+^2 l_-^2 + 8\Delta^2 l_-^4 + \left(5l_+^2 - 2l_-^2\right)f\right] - f.$$

Hence,

$$\begin{aligned} 72\Delta^2\left(4l_+^2 - l_-^2\right)^2 N_+G^2 &= 24\Delta^2 l_+^2 l_-^2\left(2l_+^2 + l_-^2\right) - \left[\left(4l_+^2 - l_-^2\right)^2 - 3l_+^2\left(5l_+^2 - 2l_-^2\right)\right]f \\ &= 24\Delta^2 l_+^2 l_-^2\left(2l_+^2 + l_-^2\right) - \left(l_+^4 - 2l_+^2 l_-^2 + l_+^4\right)f \\ &= 24\Delta^2 l_+^2 l_-^2\left(2l_+^2 + l_-^2\right) - 3\cdot 16\Delta^2 f. \end{aligned}$$

Consequently

$$3\left(4l_+^2 - l_-^2\right)^2 N_+G^2 = l_+^2 l_-^2\left(2l_+^2 + l_-^2\right) - 2f,$$

from which we deduce the formula (3.3). The proof is complete. □

4. Distances from $N_\pm$ to K

The points F_+, F_-, K are known to be collinear in the order $F_+ - K - F_-$. We will apply Stewart's theorem again. We will need the formulas (2.1), (2.2), (2.6); (2.7), (2.8), (2.11) above, but also by the following three:

$$OK = \frac{abc}{2\Delta}\cdot\frac{l_+ l_-}{l_+^2 + l_-^2}, \tag{4.1}$$

$$F_+K = \frac{\sqrt{f}}{\sqrt{3}\left(l_+^2 + l_-^2\right)}\frac{l_-}{l_+} \quad \text{and} \quad F_-K = \frac{\sqrt{f}}{\sqrt{3}\left(l_+^2 + l_-^2\right)}\frac{l_+}{l_-}. \tag{4.2}$$

([3]; (3.1), (2.13), (2.14)).

Proposition 4.1. *The distances between Napoleon points $N_\pm$ and K are given by*

$$N_+K = \frac{1}{\sqrt{3}\left(l_+^2 + l_-^2\right)\left(4l_+^2 - l_-^2\right)}\sqrt{160\Delta^2 l_+^2 l_-^2\left(l_+^2 + l_-^2\right) + \left(3l_+^2 - 2l_-^2\right)\left(3l_-^2 - 2l_+^2\right)f}, \quad (4.3)$$

$$N_-K = \frac{1}{\sqrt{3}\left(l_+^2 + l_-^2\right)\left|4l_-^2 - l_+^2\right|}\sqrt{160\Delta^2 l_+^2 l_-^2\left(l_+^2 + l_-^2\right) + \left(3l_+^2 - 2l_-^2\right)\left(3l_-^2 - 2l_+^2\right)f}. \quad (4.4)$$

Hint. The calculation of these distances is done following the same procedure as above. To calculate the distance N_+K we apply Stewart's theorem to the triangle KOF_+ and to the point N_+ (we have $O - N_+ - F$ according to Lema 1):

$$N_+K^2 \cdot F_+O = OK^2 \cdot N_+F_+ + F_+K^2 \cdot N_+O - F_+O \cdot N_+F_+ \cdot N_+O$$

(Fig. 6). The formula for distance N_+K is obtained by substituting the factors N_+F_+, N_+O, F_+O, OK, F_+K by their expressions given by (2.1), (2.2), (2.6), (4.1), (4.2) respectively. Next, we arrange the resulting equality by separating and associating terms that have the factor f from the other terms. A special statement relative to the expression of OK^2: its factor $a^2b^2c^2$ is replaced by $f + 16\Delta^2\left(l_+^2 + l_-^2\right)$ (according to (1.5) and (1.3)) before grouping terms. Finally, the formula (4.3) is found.

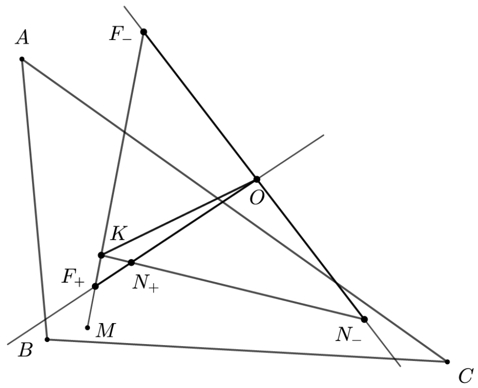

Figure 6

For the second formula, we apply Stewart's theorem to the triangle KOF_- and to the point N_- (Fig. 6). Taking into account the position of the point N_- on the line OF_- (Lemma 2), we have two cases to consider. But we are led to the same calculations in both cases. (This is equivalent to using Stewart's theorem variant that takes into account the orientation of segments.) Next, we perform the calculations with the same precautions as above and finally obtain the formula (4.4). □

Now, taking into account (1.1), through a routine calculation we find

$$160\Delta^2 l_+^2 l_-^2\left(l_+^2 + l_-^2\right) + \left(3l_+^2 - 2l_-^2\right)\left(3l_-^2 - 2l_+^2\right)f = \varphi(a,b,c),$$

where

$$\begin{aligned}\varphi(a,b,c) \;=\; & a^{10} + b^{10} + c^{10} - 4a^8b^2 - 4a^2b^8 - 4a^8c^2 - 4a^2c^8 - 4b^8c^2 - 4b^2c^8 \\ & +3a^6b^4 + 3a^4b^6 + 3a^6c^4 + 3a^4c^6 + 3b^6c^4 + 3b^4c^6 \\ & +16a^6b^2c^2 + 16a^2b^6c^2 + 16a^2b^2c^6 \\ & -15a^4b^4c^2 - 15a^4b^2c4 - 15a^2b^4c^4. \qquad (4.5)\end{aligned}$$

From (1.1) it readily follows that

$$4l_+^2 - l_-^2 = \frac{1}{2}\left[3\left(a^2+b^2+c^2\right)+20\sqrt{3}\Delta\right],$$

$$4l_-^2 - l_+^2 = \frac{1}{2}\left[3\left(a^2+b^2+c^2\right)-20\sqrt{3}\Delta\right].$$

Then the distances N_+K and N_-K are expressed by sidelenghts a, b, c as follows:

$$N_+K = \frac{2\sqrt{\varphi}}{\sqrt{3}\left(a^2+b^2+c^2\right)\left[3\left(a^2+b^2+c^2\right)+20\sqrt{3}\Delta\right]},$$

$$N_-K = \frac{2\sqrt{\varphi}}{\sqrt{3}\left(a^2+b^2+c^2\right)\left|3\left(a^2+b^2+c^2\right)-20\sqrt{3}\Delta\right|}.$$

Obviously, we have

$$\frac{N_+K}{N_-K} = \frac{\left|4l_-^2 - l_+^2\right|}{4l_+^2 - l_-^2} = \frac{\left|3\left(a^2+b^2+c^2\right)-20\sqrt{3}\Delta\right|}{3\left(a^2+b^2+c^2\right)+20\sqrt{3}\Delta}.$$

5. Distances N_+F_-, N_-F_+, N_+J_-, N_-J_+

Thanks to Lemmas 1 and 3, we have a simpler way to calculate distances N_+F_- and N_-F_+.

Proposition 5.1. *We have the formulas:*

$$N_+F_- = \frac{2\sqrt{3}}{3l_-\left(4l_+^2 - l_-^2\right)}\sqrt{32\Delta^2 l_+^2 l_-^2 + \left(2l_+^2 - l_-^2\right)f} \tag{5.1}$$

and

$$N_-F_+ = \frac{2\sqrt{3}}{3l_+\left|4l_-^2 - l_+^2\right|}\sqrt{32\Delta^2 l_+^2 l_-^2 + \left(2l_-^2 - l_+^2\right)f}. \tag{5.2}$$

Proof. Indeed, from Lemma 1 the points N_+, F_-, E are collinear and located in the order $F_- - N_+ - E$ (Fig. 3). Hence, $N_+F_- = F_-E - N_+E$. By the formulas (3.1) and

$$F_-E = \frac{1}{24\Delta l_-}\sqrt{32\Delta^2 l_+^2 l_-^2 + \left(2l_+^2 - l_-^2\right)f}$$

([3], (2.12)), this equality is written in the form

$$N_+F_- = \left(\frac{1}{24\Delta l_-} - \frac{l_-}{8\Delta\left(4l_+^2 - l_-^2\right)}\right)\sqrt{32\Delta^2 l_+^2 l_-^2 + \left(2l_+^2 - l_-^2\right)f},$$

from where we immediately get the formula (5.1).
Similarly, it follows from Lemma 3 that $N_-F_+ = N_-E + F_+E$ if $4l_-^2 - l_+^2 < 0$ (Fig. 3), and $N_-F_+ = N_-E - F_+E$ if $4l_-^2 - l_+^2 > 0$ (Fig. 4). In both cases, using the formulas (3.2) and

$$F_+E = \frac{1}{24\Delta l_+}\sqrt{32\Delta^2 l_+^2 l_-^2 + \left(2l_-^2 - l_+^2\right)f}$$

([3], (2.11)), and then performing the calculations, we obtain the required formula (5.2). Hence the proposition. □

The distances N_+J_- and N_-J_+ are not as easy to obtain. Again we will use Stewart's theorem. We will need the following formulas:

$$J_+O = \frac{abc}{4\Delta}\frac{l_-}{l_+}, \quad J_-O = \frac{abc}{4\Delta}\frac{l_+}{l_-} \tag{5.3}$$

([3]; (3.3), (3.4)) and

$$J_+F_- = \frac{4\sqrt{3}\Delta}{3l_+} = \frac{l_+^2 - l_-^2}{3l_+}, \quad J_-F_+ = \frac{4\sqrt{3}\Delta}{3l_-} = \frac{l_+^2 - l_-^2}{3l_-} \tag{5.4}$$

([3]; (4.6), (4.5)).

Proposition 5.2. *We have*

$$N_+J_- = \frac{1}{3l_-\left(4l_+^2 - l_-^2\right)}\sqrt{192\Delta^2 l_+^2\left(4l_+^2 + l_-^2\right) + 3\left(4l_+^2 - 3l_-^2\right)f} \tag{5.5}$$

and

$$N_-J_+ = \frac{1}{3l_+\left|4l_-^2 - l_+^2\right|}\sqrt{192\Delta^2 l_-^2\left(l_+^2 + 4l_-^2\right) + 3\left(4l_-^2 - 3l_+^2\right)f}. \tag{5.6}$$

Proof. For the first formula we consider the triangle J_-F_+O and the point $N_+ \in F_+O$. By Lemma 1, $F_+ - N_+ - O$. Then,

$$N_+J_-^2 \cdot F_+O = J_-F_+^2 \cdot N_+O + J_-O^2 \cdot N_+F_+ - F_+O \cdot N_+F_+ \cdot N_+O$$

holds (Fig. 7). Except for N_+J_-, we replace the factors in this equality with their expressions given by the preceding formulas. Performing the calculations as above we will obtain the formula (5.5).

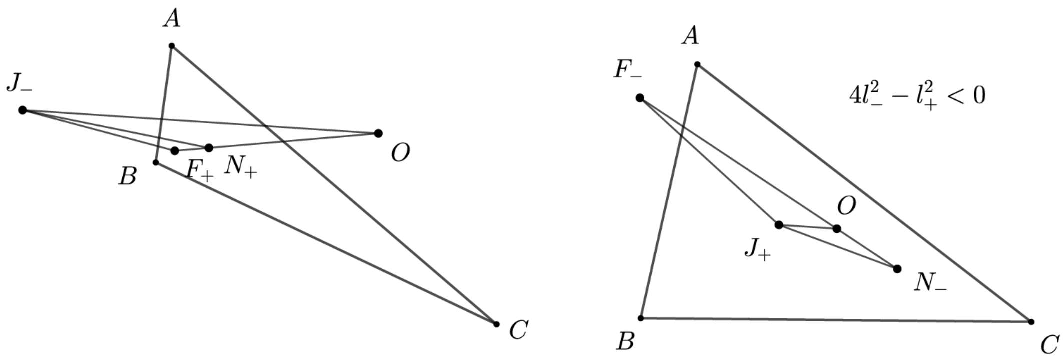

Figure 7

For the second formula we consider the triangle J_+F_-O and the point $N_- \in F_-O$ (Fig. 7). We omit the details of the routine calculation. Thus, the desired result is proven. □

6. Distance between points N_+ and N_-

We will pay special attention to the distance N_+N_-.

Proposition 6.1. *The distance N_+N_- is given by*

$$N_+N_- = \frac{\sqrt{3}}{\left(4l_+^2 - l_-^2\right)\left|4l_-^2 - l_+^2\right|}\sqrt{160\Delta^2 l_+^2 l_-^2\left(l_+^2 + l_-^2\right) + \left(3l_+^2 - 2l_-^2\right)\left(3l_-^2 - 2l_+^2\right)f}. \tag{6.1}$$

Proof. First solution. It is based on Lemma 5. If $4l_-^2 - l_+^2 < 0$, the collinear points K, N_+, N_- are in the order $K - N_+ - N_-$ and therefore $N_+N_- = KN_- - KN_+$. Using (4.3) and (4.4), it follows that

$$N_+N_- = \frac{\sqrt{160\Delta^2 l_+^2 l_-^2 \left(l_+^2 + l_-^2\right) + \left(3l_+^2 - 2l_-^2\right)\left(3l_-^2 - 2l_+^2\right) f}}{\sqrt{3}\left(l_+^2 + l_-^2\right)} \left(\frac{1}{\left|4l_-^2 - l_+^2\right|} - \frac{1}{4l_+^2 - l_-^2}\right)$$

and by a simple calculation we get the formula (6.1).
If $4l_-^2 - l_+^2 > 0$, then $N_- - K - N_+$ and, hence, we have $N_+N_- = KN_+ + KN_-$. Again we find the formula (6.1).
Second solution. By applying Stewart's theorem to the triangle N_-OF_+ and to the point N_+ (by Lema 1, $O - N_+ - F$) we have (Fig. 3 or 4):

$$N_+N_-{}^2 \cdot F_+O = N_-O^2 \cdot N_+F_+ + N_-F_+{}^2 \cdot N_+O - F_+O \cdot N_+F_+ \cdot N_+O.$$

Using the formulas (2.1), (2.2), (2.6), (2.8), (5.2) and performing routine calculations we will obtain (6.1). The proof is finished. □

Remark 6.1. *In addition to the polynomial φ defined by (4.5), we also introduce the following homogeneous symmetric polynomial:*

$$\psi(a,b,c) = 7a^4 + 7b^4 + 7c^4 - 11a^2b^2 - 11a^2c^2 - 11b^2c^2. \tag{6.2}$$

Then, the above formula for distance N_+N_- is written:

$$N_+N_-{}^2 = \frac{3\varphi}{\psi^2}, \tag{6.3}$$

where φ and ψ are given by (4.5) and (6.2). The reader can express other distances above only in a, b, c.

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