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A NOTE ON THE BARYCENTRIC SQUARE

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ABSTRACT. In this note we make some remarks concerning the barycentric quadratic transformation $P \mapsto P^2$ relative to a triangle of reference ABC and its application on lines, creating the ellipses inscribed in the triangle. In addition, we discuss its relation to the Newton line of a complete quadrilateral, a property of inscribed ellipses passing also through the centroid, and certain hyperbolas related to lines passing through the vertices of the triangle of reference.

1. INTRODUCTION

Using barycentric coordinates (barycentrics [2]) relative to the triangle of reference ABC one can define the transformation of *"Barycentric square"* $f : P \mapsto P^2$ ([1, p.100]) of the projective plane into itself, which to the point $P(p : q : r)$ corresponds *"its barycentric square"* $P^2 = (p^2 : q^2 : r^2)$. The following properties result immediately from the definition (see Figure 1):

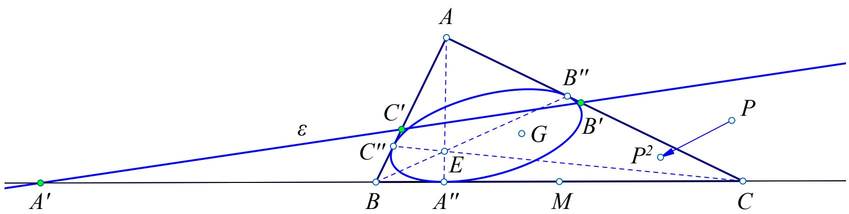

Figure 1. The map $f : P \mapsto P$ and the image $f(\varepsilon)$ of a generic line ε

(1) Since the points $f(P)$ have positive coordinates f maps the whole plane into the inner domain of the triangle of reference ABC and its sides.
(2) f is a $\{4 \textit{ to } 1\}$ transformation, mapping a point $P(u : v : w)$ and its *"harmonic associates"* ([1, p.102]) $\{(-u : v : w), (u : -v : w), (u : v : -w)\}$ to $P^2(u^2 : v^2 : w^2)$.
(3) In particular, the side-line carrying a side, maps onto that side of the triangle and fixes the vertices and the middle of the side.

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(4) Also a median AM maps via f onto the median segment AM fixing $\{A, M\}$ and the centroid G of the triangle.
(5) The seven points consisting of the vertices, the middles of the sides and the centroid of the triangle ABC are the only points which remain fixed under f.
(6) f commutes with the isotomic conjugation $\tau : (u : v : w) \mapsto (1/u : 1/v : 1/w)$.
(7) A generic line ε of the plane (i.e. line not passing through the vertices of $\triangle ABC$) maps to an inconic $f(\varepsilon)$, which is an ellipse tangent to the sides of the triangle at the points $A'' = f(A')$, $B'' = f(B')$, $C'' = f(B')$, where $\{A' \in BC$, $B' \in CA\}$ and $C' \in AB$ are the intersections of the line ε with the sides.

Last *nr* follows from *nr-1* and the fact that a quadratic map transforms lines into conics. From *nr-2* follows also that the intersection $A' = \varepsilon \cap BC$ maps to the unique point $A'' = f(A')$ common to the ellipse and the side BC. Thus, it is a contact point and the ellipse is tangent at A'' to BC.
In section 2 we calculate the barycentrics of the perspector of the ellipse $f(\varepsilon)$ and give some examples of lines defining well known inscribed ellipses of the triangle. In section 3 we discuss the relation of the quadratic transformation f to *"complete quadrilaterals"*. General information on this kind of quadrilaterals can be found in [3]. Some properties related to the Newton line of *cyclic* quadrilaterals are discussed in [4]. In section 4 we examine the conics created by lines passing through the middles of the sides of the triangle of reference. In the last section 5 we discuss the case of non-generic lines, i.e. lines that pass through some vertex of the triangle of reference.

2. The perspector

With the notation and conventions of the preceding section, assume that the generic line ε has the coefficients $(p : q : r)$. Then the intersection points with the sides of the triangle and their images via f are respectively

$$A' = (0 : r : -q)\ ,\ B' = (-r : 0 : p)\ ,\ C' = (q : -p : 0)\ , \tag{2.1}$$

$$A'' = (0 : r^2 : q^2)\ ,\ B'' = (r^2 : 0 : p^2)\ ,\ C'' = (q^2 : p^2 : 0)\ . \tag{2.2}$$

Later are the three traces of the point

$$E = (\ q^2r^2\ :\ r^2p^2\ :\ p^2q^2\ ) = \left(\frac{1}{p^2}\ :\ \frac{1}{q^2}\ :\ \frac{1}{r^2}\right)\ , \tag{2.3}$$

the perspector of the ellipse. It follows that the equation of the ellipse is

$$p^4x^2 + q^4y^2 + r^4z^2 - 2(p^2q^2xy + q^2r^2yz + r^2p^2zx)\ =\ 0\ . \tag{2.4}$$

We notice that the correspondence $\{$*generic lines* $\mapsto$ *inellipses*$\}$ is a $\{4$ *to* $1\}$, since the lines

$$\left.\begin{aligned} \varepsilon :\ px + qy + rz &= 0\ ,\\ -px + qy + rz &= 0\ ,\\ px - qy + rz &= 0\ ,\\ px + qy - rz &= 0\ , \end{aligned}\right\} \tag{2.5}$$

define the same perspector E, as in formula (2.3), consequently define also the same ellipse. We call these three lines *"companion lines"* of ε. Figure 2 shows the line ε and its companion lines defining the same perspector and the same ellipse. Besides the points $\{A', B', C'\}$ where ε intersects the sides of the triangle, the points $\{A_1, B_1, C_1\}$ at which

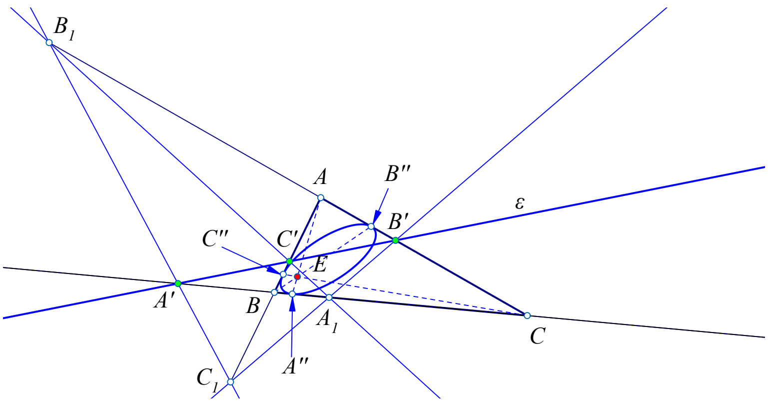

Figure 2. Four lines defining the same inscribed ellipse via f

the companion lines intersect the sides of the triangle are harmonic conjugate to the former:

$$A_1 = A'(BC)\ ,\ B_1 = B'(CA)\ ,\ C_1 = C'(AB)\ .$$

If $A' = \lambda B + \mu C = (0 : \lambda : \mu)$, then $A_1 = \lambda B - \mu C = (0 : \lambda : -\mu)$, both defining

$$A'^2 = A_1^2 = (0 : \lambda^2 : \mu^2),$$

and analogous relations holding for the pairs $\{(B', B_1), (C', C_1)\}$.
Since every inner point of the triangle has barycentrics which can be written in the form of squares, as in equation (2.3), it follows that every ellipse inscribed in the triangle ABC can be obtained as the image via f of a line of the plane, and conversely, every generic line of the plane produces via f an ellipse inscribed in the triangle.
Figure 3 shows the case of the incircle, whose perspector is the *"Gergonne center"* of the triangle

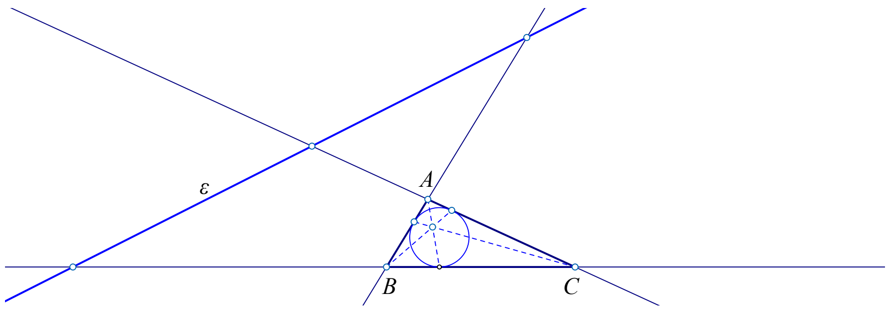

Figure 3. Incircle as image via f of the line ε

$$G_e = \left(\frac{1}{b+c-a} : \frac{1}{c+a-b} : \frac{1}{a+b-c}\right),$$

with $\{a = |BC|, b = |CA|, c = |AB|\}$ the side-lengths of the triangle. The line ε producing the incircle via f is

$$\varepsilon :\ x\sqrt{b+c-a} + y\sqrt{c+a-b} + z\sqrt{a+b-c} = 0\,.$$

Figure 4 shows the case of the *"Brocard ellipse"*, whose perspector is the *"Symmedian*

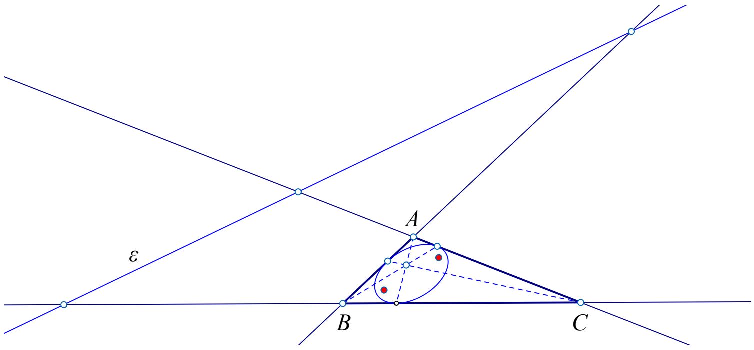

Figure 4. Brocard ellipse as image via f of the line ε

center" of the triangle

$$K = (a^2 : b^2 : c^2) ,$$

The line ε producing this ellipse via f is

$$\varepsilon : \frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 0 .$$

The focal points of this ellipse, which is the *"dual"* inconic of the circumcircle, sharing with it the same perspector, are the *"Brocard points"* 1st and 2nd:

$$\text{1st} : B_1 = \left(\frac{1}{b^2} : \frac{1}{c^2} : \frac{1}{a^2}\right) \quad \text{and 2nd} : \quad B_2 = \left(\frac{1}{c^2} : \frac{1}{a^2} : \frac{1}{b^2}\right) .$$

Figure 5 shows the case of the ellipse, whose perspector is the *"3rd Brocard center"* of the triangle $B_3 = (a^{-2} : b^{-2} : c^{-2})$, The line ε producing this ellipse via f is

$$\varepsilon : ax + by + cz = 0 .$$

The simplest case is the *"Steiner inellipse"*, which is the image via f of the line at infinity

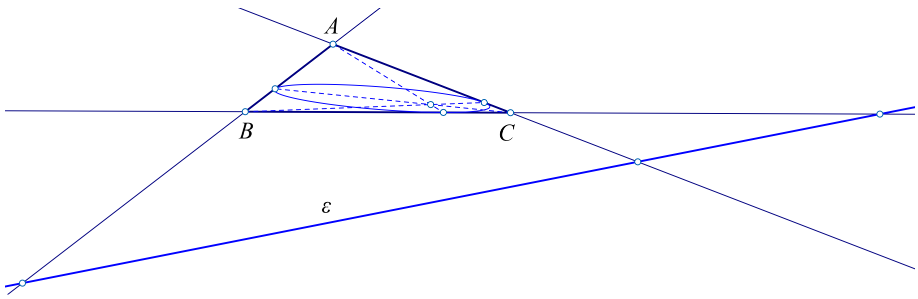

Figure 5. Ellipse, image via f of the line $ax + by + cz = 0$

$x + y + z = 0$ (see Figure 6). By the remark made at the beginning of the section, the Steiner inellipse is also the image $f(\varepsilon)$ of the three lines $\{\varepsilon\}$ which pass through the middles of the sides of $\triangle ABC$, and are the *companion lines* of the line at infinity.

We notice that the ellipses considered, appearing as images $f(\varepsilon)$ of generic lines of the plane, and which we could call *"squares of lines"*, are not the only ellipses inscribed in the triangle, i.e. tangent to its sides. There are many other, usually called *"escribed"*, which run totally outside the triangle (see Figure 7). These correspond to perspectors E

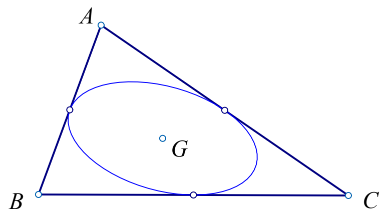

Figure 6. Steiner inellipse, image via f of the line at infinity $x+y+z=0$

which are inside the *"Steiner ellipse"* κ of the triangle, whose center is at the centroid G, but outside the inner domain of the triangle, as seen in the figure.

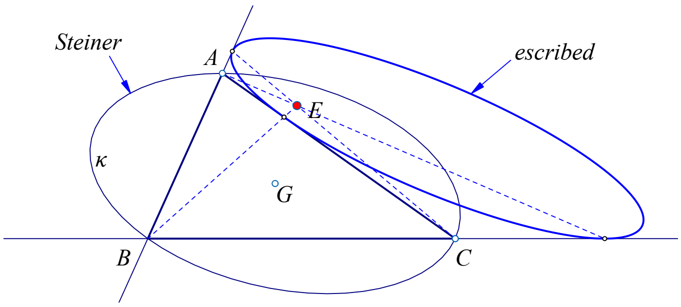

Figure 7. Escribed ellipses to the triangle

3. Complete quadrilateral and Newton line

By the remark at the beginning of the preceding section, the triangle ABC together with a generic line ε and its companion lines define a complete quadrilateral, whose *"diagonal triangle"* is the triangle of reference ABC (see Figure 8).

Theorem 3.1. *With the notation and conventions adopted so far, the Newton line of the complete quadrilateral defined by the triangle ABC and the line ε is the trilinear polar of the perspector E of the corresponding ellipse $f(\varepsilon)$.*

Proof. From line $\varepsilon: \; px+qy+rz=0$ and equations (2.1) and (2.2) we have the coordinates of the middle A_0 of $A'A_1$

$$A'(0:r:-q)\,,\; A_1(0:r:q) \quad \Rightarrow$$
$$A_0 = (r+q)A' + (r-q)A_1 \;=\; (0:2r^2:-2q^2) \;=\; A''(BC)\,,$$

and analogously the middles $\{B_0, C_0\}$ respectively of $\{B'B_1, C'C_1\}$ are the harmonic conjugates $\{B_0 = B''(CA), C_0 = C''(AB)\}$, showing that they lie on the trilinear polar of the perspector E. □

We should notice at this point a certain symmetry concerning the complete quadrilateral. Considering a triangle ABC and the corresponding to it barycentrics, every line $\varepsilon: \; px+qy+rz=0$, not passing through any of its vertices, defines, together with its companions (2), a complete quadrilateral having $\triangle ABC$ as its *diagonal triangle*. Every complete quadrilateral can be defined this way and its four lines can be described by

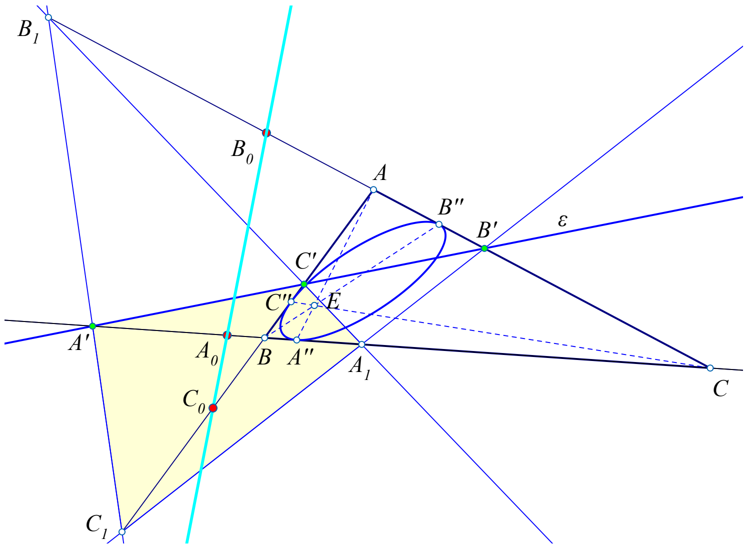

Figure 8. Complete quadrilateral and its Newton line defined from $\triangle ABC$ and ε

equations with coefficients differing only on their signs. By theorem 3.1, the Newton line is then represented by equation

$$p^2x + q^2y + r^2z = 0 . \tag{3.1}$$

This shows that, given the triangle ABC, any complete quadrilateral having that as its diagonal triangle is completely determined by its Newton line and its coefficients $(p^2 : q^2 : r^2)$. These correspond to a line running totally outside the domain of the triangle ABC, and any such line can be represented in the form $p^2x + q^2y + r^2z = 0$, the corresponding quadrilateral consisting of lines with coefficients $\{(\pm p : \pm q : \pm r)\}$.
As an application of this remark we give a proof of the following well known couple of results ([5, p.154], [3]).

Theorem 3.2. *The centers of the conics inscribed in a complete quadrilateral lie on the Newton line of the quadrilateral.*

Proof. We work with barycentrics w.r.t. the diagonal triangle ABC of the quadrilateral. Its sides are represented by a line $\varepsilon : px + qy + rz = 0$ and its companions (2). The theorem of Desargues ([6, p.25]) implies that the conics inscribed in the quadrilateral have $\triangle ABC$ as *"self-polar"*. This implies that they can be represented in the form $\{Ux^2 + Vy^2 + Wz^2 = 0\}$. The center of a conic is the pole of the line at infinity $x + y + z = 0$, thus having barycentrics $(1/U : 1/V : 1/W)$. On the other side, line ε is tangent to the conic and satisfies the dual conic equation ([1, p.125]):

$$\frac{p^2}{U} + \frac{q^2}{V} + \frac{r^2}{W} = 0 .$$

This, for variable $(U : V : W)$, shows that the conic centers lie on the Newton line. □

The center of a parabola is considered to be its point at infinity at which is tangent to the line at infinity. It represents also the direction of its axis. From this and the preceding theorem follows the corollary:

Corollary 3.1. *The Newton line of the complete quadrilateral is parallel to the axis of the parabola inscribed in the quadrangle.*

4. MIDDLES OF SIDES AND MEDIANS

Besides the points $\{A'', B'', C''\}$ in equation (2.2), inscribed ellipses which are obtained as images of generic lines via f have three additional points on the medians. These are easily determined from the intersections of the line ε with the medians:

$$\varepsilon: \; px + qy + rz = 0\,,$$

$$\text{A-median: } (0:-1:1), \text{ intersection: } (q+r:-p:-p), \text{ on } f(\varepsilon): \; ((q+r)^2 : p^2 : p^2)\,,$$

$$\text{B-median: } (1:0:-1), \text{ intersection: } (-q:r+p:-q), \text{ on } f(\varepsilon): \; (q^2 : (r+p)^2 : q^2)\,,$$

$$\text{C-median: } (-1:1:0), \text{ intersection: } (-r:-r:(p+q)), \text{ on } f(\varepsilon): \; (r^2 : r^2 : (p+q)^2)\,.$$

When line ε passes through the middle, M say of side BC, then the corresponding ellipse $f(\varepsilon)$ is tangent to BC at M, since later is a fixed point of f. Next theorem lists some additional properties of this ellipse.

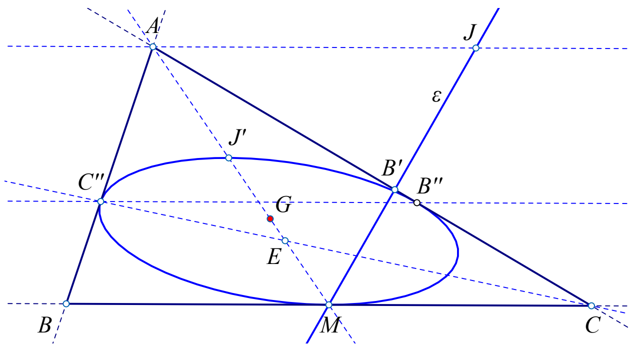

Figure 9. Line ε through the middle M and corresponding ellipse $f(\varepsilon)$

Theorem 4.1. *For a line ε through the middle M of the side BC of $\triangle ABC$ the corresponding ellipse $f(\varepsilon)$ has the following properties:*

1. *(1) It is tangent to the sides $\{AB, AC\}$ at points correspondingly $\{C'', B''\}$ whose line $B''C''$ is parallel to BC.*
2. *(2) The median AM is conjugate to the direction of BC and carries the center of the ellipse.*
3. *(3) The diametral point J' of the ellipse is the image $J' = f(J)$ of the intersection J of ε with the parallel to BC from A.*

Proof. Nr-1. Line $B''C''$ is the polar of A w.r.t. the ellipse $f(\varepsilon)$ and using the equation (2.4) of the $f(\varepsilon)$ we determine its coefficients

$$\begin{pmatrix} p^4 & -p^2q^2 & -p^2r^2 \\ -p^2q^2 & q^4 & -q^2r^2 \\ -p^2r^2 & -q^2r^2 & r^4 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = p^2 \begin{pmatrix} p^2 \\ -q^2 \\ -r^2 \end{pmatrix}\,.$$

The direction of this line, i.e. its intersection with the line at infinity is

$$(-q^2 + r^2 : -r^2 - p^2 : p^2 + q^2) \;=\; (0 : -q^2 - p^2 : p^2 + q^2)\,,$$

later because ε passing through $M(0:1:1)$ satisfies $q+r=0$. Thus, the direction of the line $B''C''$ is the same with that of $BC:(0:-1:1)$.
Nr-2 follows from *nr-1*.
Nr-3 follows also from a simple computation. The line parallel to BC has coefficients $\varepsilon_A=(0:1:1)$ and intersects ε at $J(r-q:p:-p)$, with $J'=f(J)=((r-q)^2:p^2:p^2)$ a point on the median AM.
As we noticed in § 2, when ε passes also through the middle of AC, then the corresponding ellipse $f(\varepsilon)$ is the Steiner inellipse of $\triangle ABC$. □

Ellipses $f(\varepsilon)$ produced as images of lines ε through the middle M of BC coincide with the ellipses produced from lines ε parallel to BC (see Figure 10). This is immediately seen by the location of the perspector, which in both cases is on the median AM. This follows also from the form of the coefficients $(p:q:-q)$ of a line through M and the corresponding form of *companion lines* considered in § 2. In this case the coefficients of line ε and its companions are

$$(p:q:-q)\,,\ (-p:q:-q)\,,\ (p:-q:-q)\,,\ (p:q:q)\,.$$

The second corresponds to the affine reflection of the first in the median, the third is parallel to BC and the fourth is also parallel to BC through the harmonic conjugate $B'(AC)$ of the point B'

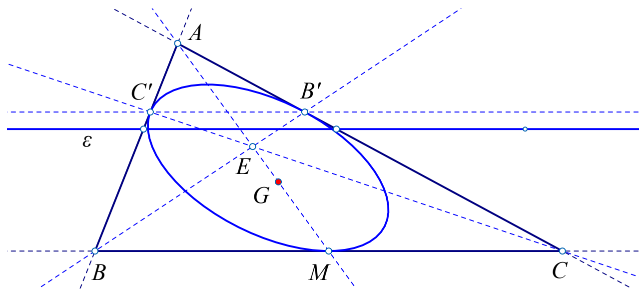

Figure 10. Ellipse $f(\varepsilon)$ for line ε parallel to BC

Theorem 4.2. *Generic lines ε through the centroid G define ellipses $f(\varepsilon)$ tangent to ε at G.*

Proof. Since the line ε passes through G it satisfies the condition $p+q+r=0$, which implies $p^2=q^2+r^2+2qr$ and similar relations for the cyclic permutations of $\{p,q,r\}$. Obviously also, since f fixes G, the ellipse, in this case, passes through G. Hence using equation (2.4), the tangent at G has coefficients

$$(\,p^2(p^2-q^2-r^2)\ :\ q^2(q^2-p^2-r^2)\ :\ r^2(r^2-p^2-q^2)\,)\ =\ 2pqr(p:q:r)\,,$$

thereby proving the claim (see Figure 11). □

Corollary 4.1. *All the inscribed in the triangle ABC ellipses, which pass through the centroid G, are images $f(\varepsilon)$ of lines ε through G.*

Proof. In fact, consider the tangent ε of an ellipse ε' passing through G. The ellipses ε' and $f(\varepsilon)$ are tangent to the same four lines $\{BC,CA,AB,\varepsilon\}$ and also tangent to ε at the same point G. Thus, they coincide and $\varepsilon'=f(\varepsilon)$ ([7, § 6.1]). □

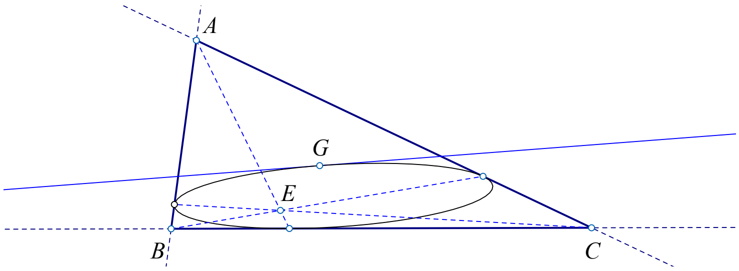

Figure 11. Ellipse $f(\varepsilon)$ for a line ε through the centroid G

Theorem 4.2 could be roughly expressed by saying, that *"the lines through G are tangent there to their squares"*. Next theorem shows that these are the only inscribed ellipses that are the squares of one of their tangents.

Theorem 4.3. *If an inscribed to the triangle ABC ellipse is the image $f(\varepsilon)$ of one tangent ε of it, then ε contacts the ellipse at the centroid G of the triangle.*

Proof. Let us assume that $Q(u : v : w)$ is the contact point of the ellipse $\varepsilon' = f(\varepsilon)$ with the line $\varepsilon : px + qy + rz = 0$ and $pqr \neq 0$. If the image $f(Q) = Q^2 = Q$, then the theorem holds, since then we have for a constant $k \neq 0$:

$$\{\, u^2 = ku\,,\ v^2 = kv\,,\ w^2 = kw \,\} \quad \Rightarrow \quad u(u-k) = v(v-k) = w(w-k) = 0,$$

implying that $Q = G$. We complete the proof by showing that $Q^2 \neq Q$ is impossible. In fact, if $Q^2 \neq Q$, then Q belonging to the ellipse is the square $Q = P^2$ of another point $P \in \varepsilon$. Then, both $\{P(x : y : z)\}$ and $P^2(x^2 : y^2 : z^2) = Q$ are distinct points of ε, and we can represent the coefficients of ε through the exterior product of the vectors of coordinates:

$$(p : q : r) = P \times P^2 = (yz^2 - y^2z : zx^2 - z^2x : xy^2 - x^2y) = \left(\frac{y-z}{x} : \frac{z-x}{y} : \frac{x-y}{z}\right).$$

Substitution of these values in equation (2.4) gives the equation of the ellipse. The coefficients of the tangent at $Q(x^2 : y^2 : z^2)$ is then found to be a constant non-zero multiple of

$$\left(\frac{y-z}{x^2} : \frac{z-x}{y^2} : \frac{x-y}{z^2}\right),$$

and this must be a constant non-zero multiple of the previously found values of $(p : q : r)$:

$$\left(\frac{y-z}{x^2} : \frac{z-x}{y^2} : \frac{x-y}{z^2}\right) = k \cdot \left(\frac{y-z}{x} : \frac{z-x}{y} : \frac{x-y}{z}\right) \quad \Rightarrow$$
$$x = y = z\,,$$

which contradicts to the hypothesis. □

5. Non generic lines

As we noticed in § 1, we consider as *"non generic"* the lines which pass through a vertex of the triangle of reference ABC. A line through a vertex, say $A(1 : 0 : 0)$ of the triangle maps to a line through the same vertex. In fact, such a line ε can be described by its intersection $P(0:q:r)$ with the line BC

$$\varepsilon : P_t = (1-t)A + tB = (1-t : tq : tr) \quad \Rightarrow \quad P_t^2 = ((1-t)^2 : t^2q : t^2r)\,,$$

which is a line passing through $P^2(0:p^2:q^2)$, as is seen through the vanishing of the determinant

$$\begin{vmatrix} 1 & 0 & 0 \\ 0 & q^2 & r^2 \\ (1-t)^2 & t^2q^2 & t^2r^2 \end{vmatrix} = 0\,.$$

Next theorem deals with basic properties of the curve enveloping the lines $\{\varepsilon_t = P_tP_t^2\}$ for $t \in \mathbb{R}$ (see Figure 12).

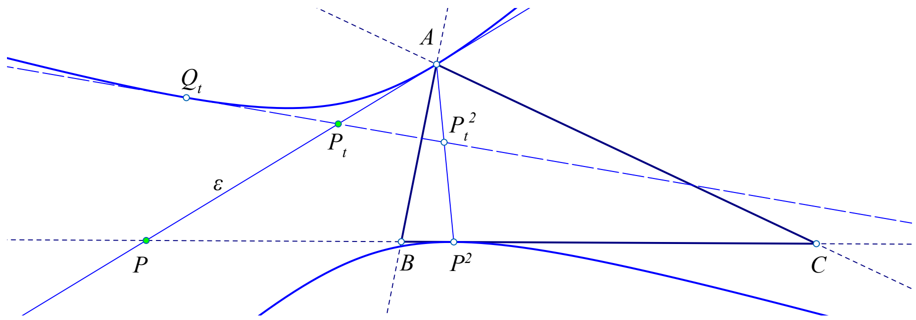

Figure 12. Hyperbola enveloping lines $\{P_tP_t^2, P_t \in \varepsilon, t \in \mathbb{R}\}$

Theorem 5.1. *With the notation and conventions of this section, for a line ε through $A(1:0:0)$ intersecting BC at the point $P(0:q:r)$ with $q \neq r$, the lines $\{\varepsilon_t = P_tP_t^2, P_t \in \varepsilon, t \in \mathbb{R}\}$ have the following properties.*

(1) They envelope a conic passing through $\{A, P^2\}$ and having there correspondingly tangents the lines $\{\varepsilon, BC\}$.
(2) The lines $\{BG, CG\}$, where G is the centroid of $\triangle ABC$, are tangent to the conic.
(3) The conic is a hyperbola and in case the line ε is parallel to BC through A, the hyperbola is tangent to the middle of BC. When ε coincides with the median AM, the lines $\{P_tP_t^2, P_t \in \varepsilon\}$ coincide all with line AM.

Proof. The line ε in parametric form has the coefficients

$$\varepsilon : \; P_t = (1-t)A + tP = (1-t : tq : tr)\,, \quad P_t^2 = ((1-t)^2 : t^2q^2 : t^2r^2) \quad \Rightarrow$$

$$\varepsilon_t = P_tP_t^2 = P_t \times P_t^2 = \begin{pmatrix} (1-t) & tq & tr \\ (1-t)^2 & t^2q^2 & t^2r^2 \end{pmatrix} \quad \Rightarrow$$

$$\varepsilon_t = (\; t^2qr(r-q) \; : \; (1-t)r(1-t(1+r)) \; : \; (1-t)q(t(q+1)-1)\;)\,. \tag{5.1}$$

Since the coefficients of the lines $\{\varepsilon_t\}$ are quadratic functions of the parameter t they envelope a conic ([8, p.248]). The equation of the conic is determined by writing the line explicitly

$$\begin{aligned} \varepsilon_t &= t^2qr(r-q)\cdot x + (1-t)r(1-t(1+r))\cdot y + (1-t)q(t(q+1)-1)\cdot z \\ &= t^2[qr(r-q)x + r(r+1)y - q(q+1)z] + t[q(q+2)z - r(r+2)y] + [ry - qz] \\ &= t^2\alpha(x:y:z) + t\beta(x:y:z) + \gamma(x:y:z)\,. \end{aligned}$$

The equation of the conic is then

$$4\alpha\gamma - \beta^2 \;=\; 4qr(r-q)x(qz-ry) + (q^2z - r^2y)^2 \;=\; 0\,. \qquad (5.2)$$

The matrix M, expressing the conic in the form $X^tMX = 0$, has a simple inverse, which up to a non-zero factor is

$$M' \;=\; \begin{pmatrix} 0 & q^2 & r^2 \\ q^2 & 2q^2 & 2qr \\ r^2 & 2qr & 2r^2 \end{pmatrix}\,. \qquad (5.3)$$

Applying this to the vector (5.1) of coefficients of the lines ε_t, we determine their contact

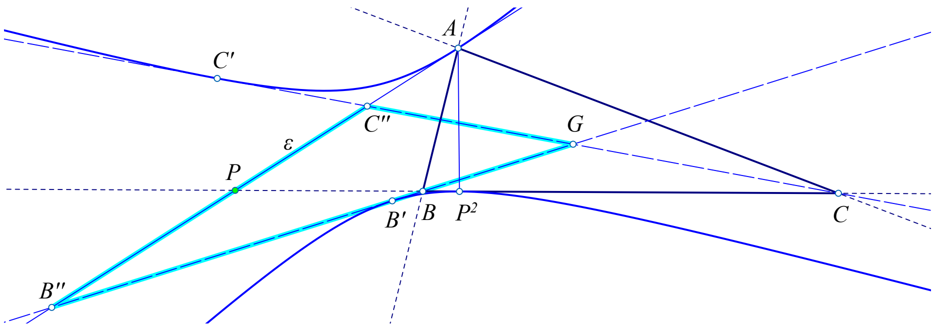

Figure 13. Lines $\{BG, CG\}$ are tangent to the conic

points with the conic, which are

$$Q_t \;=\; (\,-(t-1)^2 \;:\; qt((q+2)t-2) \;:\; rt((r+2)t-2)\,)\,. \qquad (5.4)$$

Using this and equation (5.2) we complete the proof of *nr-1*.
Nr-2. Using equation (5.1) we see that the line ε_t passes through B when $t = 1/(1+r)$. Replacing this value into (5.1) we see also that $G(1:1:1)$ satisfies the equation of the corresponding tangent line (see Figure 13). Analogously is proved the tangency of GC.
Nr-3 is proved by applying the kind-criterion $G^tM^\#G < 0$ through the adjoint matrix $M^\#$ ([1, p.127]). In fact, $M^\# = |M|M^{-1}$ and it is seen that

$$G^tM^\#G \;=\; -8q^2r^2(r-q)^2(r^2+qr+q^2)\,,$$

thereby proving the claim about the kind of the conic.
Concerning the case ε is parallel to BC through A, analogous arguments show that the equation of the conic is

$$(y-z)^2 + 8x(y+z) \;=\; 0\,.$$

This is again a hyperbola with tangents the lines $\{GB, GC\}$ and center at the middle of the median AB. (see Figure 14). The last statement is trivially verified.

□

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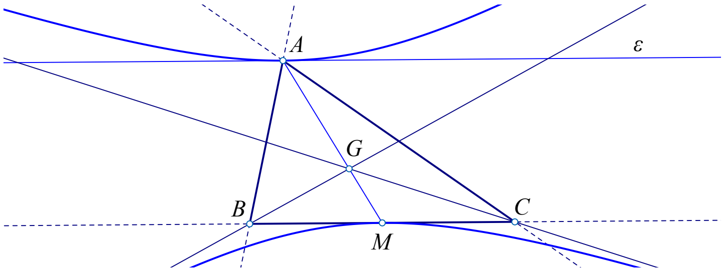

Figure 14. The envelope of lines $\varepsilon_t = P_t P_t^2$, $P_t \in \varepsilon$

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