Global Journal of Advanced Research
on Classical and Modern Geometries
ISSN: 2284-5569, Vol.10, (2021), Issue 2, pp.137-152

CHEN'S INEQUALITY FOR C-TOTALLY REAL SUBMANIFOLDS IN A GENERALIZED (κ, μ)-SPACE FORM

MORTEZA FAGHFOURI* AND NARGES GHAFFARZADEH

ABSTRACT. In this paper, we obtain a basic Chen's inequality for a C-totally real submanifold in a generalized (κ, μ)-contact space form involving intrinsic invariants, namely a scalar curvature, sectional curvatures of the submanifold on the left-hand side and the main extrinsic invariant, namely a squared mean curvature on the right-hand side. Inequalities between the squared mean curvature, k-Ricci curvature and Ricci curvature, are also obtained here.

1. INTRODUCTION

In the theory of submanifolds, the study of immersibility of a Riemannian manifold in a Euclidean space is one of the fundamental problems. According to the well-known theorem of J. Nash in 1956 [14], every Riemannian manifold can be isometrically embedded in some Euclidean spaces with sufficiently high codimension.
In [5], B.-Y. Chen defined a Riemannian invariant $\delta_M = \tau - \inf K$ for any Riemannian manifold M, where τ is the scalar curvature of M and $(\inf K)(p) = \inf\{K(\pi)|$ plane sections $\pi \subset T_pM\}$. In [5], Chen also obtained a necessary condition for the existence of minimal isometric immersion from a given Riemannian manifold into Euclidean space and established a sharp inequality for a submanifold in a real space form using the scalar curvature, the sectional curvature and squared mean curvature. In [6], he gave a sharp relationship between the squared mean curvature and the Ricci curvature for the submanifolds in a real space form. These inequalities are also sharp, and many nice classes of submanifolds realize equality in inequalities. In [7], B. -Y. Chen introduced new types of curvature invariants, by defining two strings of scalar-valued Riemannian curvature functions, namely $\delta(n_1, \ldots, n_k)$ and $\tilde{\delta}(n_1, \ldots, n_k)$. The first string of δ-invariants, $\delta(n_1, \ldots, n_k)$, extends naturally the Riemannian invariant introduced in [5].
Many papers studied Chen's invariants and inequalities, such as complex space forms, cosymplectic space forms, warped product spaces, and Sasakian space forms [2,8–13,15].

2010 *Mathematics Subject Classification.* 53C40; 53C42; 53D10.
Key words and phrases. Generalized Sasakian-space-form, C-totally real submanifold,δ-invariant, k-Ricci curvature, (κ, μ)-space forms, Chen's inequality.
* Corresponding author.

In [4], A. Carriazo, V. Martín Molina, and M.M. Tripathi introduce generalized (κ,μ)-space forms as an almost contact metric manifold $(\tilde{M},\phi,\xi,\eta,\langle,\rangle)$ whose curvature tensor can be written as

$$R = f_1R_1 + f_2R_2 + f_3R_3 + f_4R_4 + f_5R_5 + f_6R_6,$$

where f_1, f_2, f_3, f_4, f_5, and f_6 are differentiable functions on $\tilde{M}$. Generalized (κ,μ)-space forms with divided R_5 whenever the curvature tensor can be written as

$$R = f_1R_1 + f_2R_2 + f_3R_3 + f_4R_4 + f_{5,1}R_{5,1} + f_{5,2}R_{5,2} + f_6R_6,$$

where $f_1, f_2, f_3, f_4, f_{5,1}, f_{5,2}$, and f_6 are differentiable functions on $\tilde{M}$. Obviously, any generalized Sasakian (κ,μ)-space form is a generalized Sasakian (κ,μ)-space form with divided R_5.

M.M. Tripathi and J.S. Kim [16, Theorem 5.2] studied the relationship between the scalar curvature, the sectional curvature and the squared mean curvature for C-totally real submanifolds in a (κ,μ)-contact space forms.

In this paper, we improved Theorem 5.2 in [16] for a C-totally real submanifold of generalized (κ,μ)-space form with divided R_5. In Section 2, we recall some necessary detailed background on Riemannian invariant in Riemannian manifolds, contact metric manifold, C-totally real submanifolds, and contact metric manifolds. In section 3, we establish a basic Chen's inequality for C-totally real submanifolds in a generalized (κ,μ)-space form with divided R_5. Sections 4 and 5 contain an inequality between the squared mean curvature, k-Ricci curvature and Ricci curvature. In Section 6, we finally apply these results to get corresponding results for C-totally real submanifolds in a generalized (κ,μ)-contact space form $\tilde{M}(f_1,\ldots,f_6)$ with $f_3 = f_1 - 1$.

2. Preliminaries

Riemannian invariants are the intrinsic characteristics of a Riemannian manifold. The scalar curvature is the most studied scalar valued Riemannian invariant on Riemannian manifolds.

Let M be an n-dimensional Riemannian manifold. We denote by $K(\pi)$ the sectional curvature of M for a plane section π in T_pM. For any orthonormal basis $\{e_1,\ldots,e_n\}$ for T_pM, the scalar curvature $\tau(p)$ of M at p is defined by $\tau(p) = \sum_{1\le i<j\le n} K(e_i \wedge e_j)$, where $K(e_i \wedge e_j)$ is the sectional curvature of the plane section spanned by e_i and e_j at $p \in M$.

Let Π_k be a k-plane section of T_pM and $\{e_1,\ldots,e_k\}$ any orthonormal basis of Π_k. The scalar curvature $\tau(\Pi_k)$ of Π_k is given by

$$\tau(\Pi_k) = \sum_{1\le i<j\le k} K(e_i \wedge e_j). \tag{2.1}$$

The scalar curvature $\tau(p)$ of M at p is identical with the scalar curvature of the tangent space T_pM of M at p, that is, $\tau(p) = \tau(T_pM)$. Also, we denote by $(\inf K)(p) = \inf\{K(\pi)|\pi \subset T_pM, \dim\pi = 2\}$, and introduce the first Chen invariant $\delta_M(p) = \tau(p) - (\inf K)(p)$, which is certainly an intrinsic character of M.

Suppose Π_k is a k-plane section of T_pM and U a unit vector in Π_k. We choose an orthonormal basis $\{e_1,\ldots,e_k\}$ of Π_k such that $e_1 = U$. The Ricci curvature Ric_{Π_k} of Π_k at U is given by

$$\underset{\Pi_k}{\mathrm{Ric}}(U) = K_{12} + \cdots + K_{1k}, \tag{2.2}$$

where K_{ij} is the sectional curvature of the plane section spanned by e_i and e_j. The $\mathrm{Ric}_{\Pi_k}(U)$ is called a k-Ricci curvature. For each integer k, $2 \leq k \leq n$, the Riemannian invariant θ_k on n-dimensional Riemannian Manifold M is defined by

$$\theta_k(p) = \left(\frac{1}{k-1}\right) \inf_{\Pi_k, X} \mathrm{Ric}_{\Pi_k}(X), p \in M, \tag{2.3}$$

where Π_k is k-plane sections in T_pM and X is unit vector in Π_k [6].
For an integer $k \geq 0$ denote by $\mathcal{S}(n,k)$ the finite set consisting of unordered k-tuples $(n_1, \ldots, n_k)$ of integers ≥ 2 satisfying $n_1 < n$ and $n_1 + \cdots + n_k \leq n$. Denote by $\mathcal{S}(n)$ the set of unordered k-tuples with $k \geq 0$ for a fixed n.
For each k-tuple $(n_1, \ldots, n_k) \in \mathcal{S}(n), \delta(n_1, \ldots, n_k)(p)$ is defined by

$$\delta(n_1, \ldots, n_k)(p) = \tau - \inf\{\tau(L_1) + \cdots + \tau(L_k)\}, \tag{2.4}$$

where $L_1, \ldots, L_k$ run over all k mutually orthogonal subspaces of T_pM such that $\dim L_j = n_j, j = 1, \ldots, k$. Similarly, $\tilde{\delta}(n_1, \ldots, n_k)(p)$ is defined by

$$\tilde{\delta}(n_1, \ldots, n_k)(p) = \tau - \sup\{\tau(L_1) + \cdots + \tau(L_k)\}. \tag{2.5}$$

Obviously, $\delta(\emptyset) = \tilde{\delta}(\emptyset) = \tau$ for $k = 0$. It is also clear that

$$\delta(n_1, \ldots, n_k) \geq \tilde{\delta}(n_1, \ldots, n_k)$$

for any k-tuple $(n_1, \ldots, n_k) \in \mathcal{S}(n)$.

Definition 2.1 ([7]). *A Riemannian n-manifold M is called an $\mathcal{S}(n_1, \ldots, n_k)$-space for a given k-tuple $(n_1, \ldots, n_k) \in \mathcal{S}(n)$ if it satisfies $\delta(n_1, \ldots, n_k) = \tilde{\delta}(n_1, \ldots, n_k)$, identically.*

For each $(n_1, \ldots, n_k) \in \mathcal{S}(n)$, let $c(n_1, \ldots, n_k)$ and $b(n_1, \ldots, n_k)$ denote

$$\begin{aligned} c(n_1, \ldots, n_k) =& \frac{n^2(n + k - 1 - \sum_{j=1}^{k} n_j)}{2(n + k - \sum_{j=1}^{k} n_j)}, \\ b(n_1, \ldots, n_k) =& \frac{1}{2}n(n-1) - \frac{1}{2}\sum_{j=1}^{k} n_j(n_j - 1). \end{aligned} \tag{2.6}$$

Let M be an n-dimensional submanifold in a manifold $\tilde{M}$ equipped with a Riemannian metric $\langle .,. \rangle$. The Gauss and Weingarten formulae are given respectively by $\tilde{\nabla}_X Y = \nabla_X Y + \sigma(X,Y)$ and $\tilde{\nabla}_X N = -A_N X + \nabla^{\perp}_X N$ for all $X, Y \in \mathfrak{X}(M)$ and $N \in T^{\perp}M$, where $\tilde{\nabla}$, ∇ and $\nabla^{\perp}$ are Riemannian, induced Riemannian and induced normal connections in $\tilde{M}$, M and the normal bundle $T^{\perp}M$ of M respectively, and σ is the second fundamental form related to the shape operator A_N in the direction of N by $\langle \sigma(X,Y), N \rangle = \langle A_N X, Y \rangle$. Then, the Gauss equation is given by

$$\begin{aligned} \tilde{R}(X,Y,Z,W) = R(X,Y,Z,W) &- \langle \sigma(X,W), \sigma(Y,Z) \rangle \\ &+ \langle \sigma(X,Z), \sigma(Y,W) \rangle \end{aligned} \tag{2.7}$$

for all $X, Y, Z, W \in \mathfrak{X}(M)$, where $\tilde{R}$ and R are the curvature tensors of $\tilde{M}$ and M respectively. The mean curvature vector H is expressed by $nH = \mathrm{trace}(\sigma)$. The submanifold M is totally geodesic in $\tilde{M}$ if $\sigma = 0$, and minimal if $H = 0$. If $\sigma(X,Y) = \langle X, Y \rangle H$, for all $X, Y \in \mathfrak{X}(M)$, then M is totally umbilical. The null space of M at a point $p \in M$ is $\mathcal{N}_p = \{X \in T_pM \,|\, \sigma(X,Y) = 0, \forall Y \in T_p(M)\}$. A manifold $\tilde{M}$ is called an almost contact metric manifold if there is an almost contact metric structure $(\phi, \xi, \eta, \langle , \rangle)$ consisting of a $(1,1)$ tensor field ϕ, a vector field ξ, a 1-form η and a compatible Riemannian metric $\langle ., . \rangle$ satisfying

$$\phi^2 = -I + \eta \otimes \xi, \eta(\xi) = 1, \eta \circ \phi = 0, \phi\xi = 0, \tag{2.8}$$

$$\langle X, Y \rangle = \langle \phi X, \phi Y \rangle + \eta(X)\eta(Y), \tag{2.9}$$

$$\langle X, \xi \rangle = \eta(X), \langle X, \phi Y \rangle = -\langle \phi X, Y \rangle, \tag{2.10}$$

for all $X, Y \in \mathfrak{X}(\tilde{M})$. An almost contact metric structure becomes a contact metric structure if $d\eta = \Phi$, where $\Phi(X, Y) = \langle X, \phi Y \rangle$ is the fundamental 2-form of $\tilde{M}$.
An almost contact metric structure of $\tilde{M}$ is said to be normal if the Nijenhuis torsion $[\phi, \phi] = -2d\eta \otimes \xi$. A normal contact metric manifold is called a Sasakian manifold. It can be proved that an almost contact metric manifold is Sasakian if and only if $(\nabla_X \phi)Y = \langle X, Y \rangle \xi - \eta(Y)X$, for any $X, Y \in \mathfrak{X}(\tilde{M})$ or equivalently, a contact metric structure is a Sasakian structure if and only if $\tilde{R}$ satisfies $\tilde{R}(X, Y)\xi = \eta(Y)X - \eta(X)Y$, for $X, Y \in \mathfrak{X}(\tilde{M})$. In a contact metric manifold $\tilde{M}$, the $(1,1)$-tensor field h is defined by $2h = \mathcal{L}_\xi \phi$, which is the Lie derivative of ϕ in the characteristic direction ϕ. It is symmetric and satisfies

$$h\xi = 0, \quad h\phi + \phi h = 0, \quad \text{trace}(h) = \text{trace}(\phi h) = 0, \tilde{\nabla}\xi = -\phi - \phi h,$$

where $\tilde{\nabla}$ is Levi-Civita connection.
Let $(M, \phi, \xi, \eta, \langle , \rangle)$ be an almost contact metric manifold. A ϕ-section of M at $p \in M$ is a section $\Pi \subset T_p\tilde{M}$ spanned by a unit vector X_p orthogonal to ξ_p, and ϕX_p. The ϕ-sectional curvature of Π is defined by $\tilde{K}(X, \phi X) = \tilde{R}(X, \phi X, \phi X, X)$. A Sasakian manifold with constant ϕ-sectional curvature c is called a Sasakian space form and is denoted by $\tilde{M}(c)$.
A contact metric manifold $(\tilde{M}, \phi, \xi, \eta, \langle , \rangle)$ is said to be a (κ, μ)-contact manifold if its curvature tensor satisfies the condition $\tilde{R}(X, Y)\xi = \kappa(\eta(Y)X - \eta(X)Y) + \mu(\eta(Y)hX - \eta(X)hY)$, where κ and μ are real constant numbers. If the (κ, μ)-contact metric manifold $\tilde{M}$ has constant ϕ-sectional curvature c, then it is said to be a (κ, μ)-contact space form.

Definition 2.2. *([4]) We say that an almost contact metric manifold $(\tilde{M}, \phi, \xi, \eta, \langle , \rangle)$ is a generalized (κ, μ)-space form if there exist functions $f_1, f_2, f_3, f_4, f_5, f_6$ defined on $\tilde{M}$ such that*

$$R = f_1 R_1 + f_2 R_2 + f_3 R_3 + f_4 R_4 + f_5 R_5 + f_6 R_6, \tag{2.11}$$

where $R_1, R_2, R_3, R_4, R_5, R_6$ are the following tensors

$$\begin{aligned}
R_1(X, Y)Z &= \langle Y, Z \rangle X - \langle X, Z \rangle Y, \\
R_2(X, Y)Z &= \langle X, \phi Z \rangle \phi Y - \langle Y, \phi Z \rangle \phi X + 2\langle X, \phi Y \rangle \phi Z, \\
R_3(X, Y)Z &= \eta(X)\eta(Z)Y - \eta(Y)\eta(Z)X + \langle X, Z \rangle \eta(Y)\xi - \langle Y, Z \rangle \eta(X)\xi, \\
R_4(X, Y)Z &= \langle Y, Z \rangle hX - \langle X, Z \rangle hY + \langle hY, Z \rangle X - \langle hX, Z \rangle Y, \\
R_5(X, Y)Z &= \langle hY, Z \rangle hX - \langle hX, Z \rangle hY + \langle \phi hX, Z \rangle \phi hY - \langle \phi hY, Z \rangle \phi hX, \\
R_6(X, Y)Z &= \eta(X)\eta(Z)hY - \eta(Y)\eta(Z)hX + \langle hX, Z \rangle \eta(Y)\xi - \langle hY, Z \rangle \eta(X)\xi.
\end{aligned}$$

for all vector fields X, Y, Z on $\tilde{M}$, where $2h = \mathcal{L}_\xi \phi$ and $\mathcal{L}$ is the Lie derivative. We will denote such a manifold by $\tilde{M}(f_1, \ldots, f_6)$.

For example, (κ, μ)-contact space forms are generalized (κ, μ)-space forms, with constant functions

$$f_1 = \frac{c+3}{4}, f_2 = \frac{c-1}{4}, f_3 = \frac{c+3}{4} - \kappa, f_4 = 1, f_5 = \frac{1}{2}, f_6 = 1 - \mu.$$

And also, generalized Sasakian space forms $\tilde{M}(f_1, f_2, f_3)$ are generalized (κ, μ)-space forms, with $f_4 = f_5 = f_6 = 0$ [1].

Definition 2.3 ([3]). *An almost contact metric manifold $(\tilde{M}, \phi, \xi, \eta, \langle, \rangle)$ is a generalized (κ, μ)-space form with divided R_5 if there exist function $f_1, f_2, f_3, f_4, f_{5,1}, f_{5,2}, f_6$ defined on $\tilde{M}$ such that*

$$R = f_1 R_1 + f_2 R_2 + f_3 R_3 + f_4 R_4 + f_{5,1} R_{5,1} + f_{5,2} R_{5,2} + f_6 R_6, \tag{2.12}$$

where $R_{5,1}, R_{5,2}$ are the following tensors

$$R_{5,1}(X,Y)Z = \langle hY, Z\rangle hX - \langle hX, Z\rangle hY,$$
$$R_{5,2}(X,Y)Z = \langle \phi hY, Z\rangle \phi hX - \langle \phi hX, Z\rangle \phi hY,$$

for all vector fields X, Y, Z on $\tilde{M}$.

We will denote such a manifold by $\tilde{M}(f_1, f_2, f_3, f_4, f_{5,1}, f_{5,2}, f_6)$. It follows that $R_5 = R_{5,1} - R_{5,2}$. It is obvious that, if $\tilde{M}(f_1, \ldots, f_6)$ is a generalized (κ, μ)-space form then $\tilde{M}$ is a generalized (κ, μ)-space form with divided R_5 with $f_{5,1} = f_5$ and $f_{5,2} = -f_5$. A non-Sasakian (κ, μ)-space form is the generalized (κ, μ)-space form with divided R_5 with

$$f_1 = \frac{2-\mu}{2}, f_2 = -\frac{\mu}{2}, f_3 = \frac{2-\mu-2\kappa}{2}, f_4 = 1, f_{5,1} = \frac{2-\mu}{2(1-\kappa)}, f_{5,2} = \frac{2\kappa-\mu}{2(1-\kappa)},$$

and $f_6 = 1 - \mu$ but not the generalized (κ, μ)-space form.

Theorem 2.1 ([4]). *Let $M(f_1, \ldots, f_6)$ be a generalized (κ, μ)-space form. If M is a Sasakian manifold, then $f_2 = f_3 = f_1 - 1$.*

Theorem 2.2 ([4]). *Let $M(f_1, \ldots, f_6)$ be a generalized (κ, μ)-space form. If M is a contact metric manifold with $f_3 = f_1 - 1$, then it is a Sasakian manifold.*

Let M be a submanifold in a contact manifold $\tilde{M}$. M is called a C-totally real submanifold if every tangent vector of M belongs to the contact distribution [17]. Thus, M is a C-totally real submanifold if ξ is normal to M. A submanifold M in an almost contact metric manifold $\tilde{M}$ is called anti-invariant if $\phi(TM) \subset T^{\perp}(M)$ [18]. If a submanifold M in a contact metric manifold is normal to the structure vector field ξ, then it is anti-invariant. Thus C-totally real submanifolds in a contact metric manifold are anti-invariant, as they are normal to ξ. For a C-totally real submanifold in a contact metric manifold we have $\langle A_\xi X, Y\rangle = -\langle \tilde{\nabla}_X \xi, Y\rangle = \langle \phi X + \phi h X, Y\rangle$, which implies that

$$A_\xi = (\phi h)^T, \tag{2.13}$$

where $(\phi h)^T$ is the tangential part of $\phi h X$ for all $X \in \mathfrak{X}(M)$.
We state the following Lemmas of Chen for later uses.

Lemma 2.1 (B. Y. Chen [5]). *If $a_1, \ldots, a_n, a_{n+1}$ are $n+1 (n > 1)$ real numbers such that*

$$\frac{1}{n-1}\left(\sum_{i=1}^{n} a_i\right)^2 = \sum_{i=1}^{n} a_i^2 + a_{n+1},$$

then $2a_1 a_2 \geq a_{n+1}$, with equality holding if and only if $a_1 + a_2 = a_3 = \ldots = a_n$.

3. Certain basic inequalities

In this section, We establish an inequality for submanifolds in a generalized (κ,μ)-contact space form with divided R_5, $\tilde{M}(f_1,\ldots,f_6)$ involving intrinsic invariants, namely the sectional curvature and the scalar curvature and the extrinsic invariant, namely the squared mean curvature.

Lemma 3.1. *In an n-dimensional C-totally real submanifold M in a $(2m+1)$-dimensional generalized (κ,μ)-contact space form with divided R_5, $\tilde{M}(f_1,\ldots,f_6)$ such that $\xi\in\mathfrak{X}(M)^{\perp}$, the scalar curvature and the squared mean curvature satisfy*

$$\begin{aligned}2\tau &= n(n-1)f_1 + 2(n-1)f_4\,\mathrm{trace}(h^T)\\ &\quad + f_{5,1}\left\{(\mathrm{trace}(h^T))^2 - \|h^T\|^2\right\} - f_{5,2}\left\{\|(\phi h)^T\|^2 - (\mathrm{trace}(\phi h)^T)^2\right\}\\ &\quad + n^2\|H\|^2 - \|\sigma\|^2,\end{aligned} \tag{3.1}$$

where

$$\|Q\|^2 = \sum_{i,j=1}^{n}\langle e_i, Qe_j\rangle^2,\quad Q\in\{(\phi h)^T, h^T\},$$
$$\|\sigma\|^2 = \sum_{i,j=1}^{n}\langle\sigma(e_i,e_j),\sigma(e_i,e_j)\rangle$$

and $(\phi h)^T X$and $h^T X$ are the tangential parts of ϕhX and hX respectively for $X\in\mathfrak{X}(M)$.

Proof. We choose a local orthonormal frame $\{e_1,\ldots,e_n\}$ such that $e_1,\ldots,e_n$ are tangent to M, e_{n+1} is parallel to the mean curvature vector H. Then from equation of Gauss (2.7), we have

$$K(e_i\wedge e_j) = \tilde{K}(e_i\wedge e_j) + \sum_{r=n+1}^{2m+1}(\sigma_{ii}^r\sigma_{jj}^r - (\sigma_{ij}^r)^2). \tag{3.2}$$

From equation (3.2), we get

$$2\tau = 2\tilde{\tau}(T_pM) + n^2\|H\|^2 - \|\sigma\|^2, \tag{3.3}$$

since

$$2\tilde{\tau}(T_pM) = \sum_{1\leqslant i\neq j\leqslant n}\tilde{K}(e_i\wedge e_j) = \sum_{1\leqslant i\neq j\leqslant n}\tilde{R}(e_i,e_j,e_j,e_i),$$

for to get 2τ to be enough that we get $\sum_{1\leqslant i\neq j\leqslant n}\tilde{R}(e_i,e_j,e_j,e_i)$. From (2.12), we have

$$\begin{aligned}\sum_{1\leqslant i\neq j\leqslant n}\tilde{R}(e_i,e_j,e_j,e_i) = f_1\{&\sum_{1\leqslant i\neq j\leqslant n}\langle e_j,e_j\rangle\langle e_i,e_i\rangle - \sum_{1\leqslant i\neq j\leqslant n}\langle e_i,e_j\rangle^2\}\\ &+ f_4\{\sum_{1\leqslant i\neq j\leqslant n}\langle e_j,e_j\rangle\langle h^T e_i,e_i\rangle - \sum_{1\leqslant i\neq j\leqslant n}\langle e_i,e_j\rangle\langle h^T e_j,e_i\rangle\\ &+ \sum_{1\leqslant i\neq j\leqslant n}\langle h^T e_j,e_j\rangle\langle e_i,e_i\rangle - \sum_{1\leqslant i\neq j\leqslant n}\langle h^T e_i,e_j\rangle\langle e_j,e_i\rangle\}\\ &+ f_{5,1}\{\sum_{1\leqslant i\neq j\leqslant n}\langle h^T e_j,e_j\rangle\langle h^T e_i,e_i\rangle - \sum_{1\leqslant i\neq j\leqslant n}\langle h^T e_i,e_j\rangle\langle h^T e_j,e_i\rangle\}\\ &- f_{5,2}\{\sum_{1\leqslant i\neq j\leqslant n}\langle(\phi h)^T e_i,e_j\rangle\langle(\phi h)^T e_j,e_i\rangle\\ &\quad - \sum_{1\leqslant i\neq j\leqslant n}\langle(\phi h)^T e_j,e_j\rangle\langle(\phi h)^T e_i,e_i\rangle\},\end{aligned}$$

or

$$\begin{aligned}2\tilde{\tau}(T_pM) = &\sum_{1\leqslant i\neq j\leqslant n}\tilde{R}(e_i,e_j,e_j,e_i)\\ =&n(n-1)f_1 + 2(n-1)f_4\,\mathrm{trace}(h^T)\\ &+ f_{5,1}\{(\mathrm{trace}(h^T))^2 - \sum_{1\leqslant i\neq j\leqslant n}\langle e_i,h^T e_j\rangle^2\}\\ &- f_{5,2}\{\sum_{1\leqslant i\neq j\leqslant n}\langle e_i,(\phi h)^T e_j\rangle^2 - (\mathrm{trace}(\phi h)^T)^2\}.\end{aligned}$$

We obtain

$$\begin{aligned}2\tilde{\tau}(T_pM) =&n(n-1)f_1 + 2(n-1)f_4\,\mathrm{trace}(h^T)\\ &+ f_{5,1}\left\{(\mathrm{trace}(h^T))^2 - \|h^T\|^2\right\}\\ &- f_{5,2}\left\{\|(\phi h)^T\|^2 - (\mathrm{trace}(\phi h)^T)^2\right\}.\end{aligned} \tag{3.4}$$

Now if we put (3.4) in (3.3), obtain (3.1). □

Theorem 3.1. *Let M be an n-dimensional $(n\geq 3)$ C-totally real submanifold in a $(2m+1)$-dimensional generalized (κ,μ)-contact space form with divided R_5, $\tilde{M}(f_1,\ldots,f_6)$. Then for each point $p\in M$ and each plane section $\pi\subset T_pM$, we have*

$$\begin{aligned}\tau - K(\pi) \leqslant &\frac{n^2(n-2)}{2(n-1)}\|H\|^2 + \frac{(n+1)(n-2)}{2}f_1\\ &+ f_4\{(n-1)\,\mathrm{trace}(h^T) - \mathrm{trace}(h|_\pi)\}\\ &+ \frac{1}{2}f_{5,1}\{(\mathrm{trace}(h^T))^2 - \|h^T\|^2 - 2\det(h|_\pi)\}\\ &- \frac{1}{2}f_{5,2}\{\|(\phi h)^T\|^2 - (\mathrm{trace}(\phi h)^T)^2 + 2\det((\phi h)|_\pi)\}.\end{aligned} \tag{3.5}$$

The equality in (3.5) holds at $p\in M$ if and only if there exist an orthonormal basis $\{e_1,\ldots,e_n\}$ of T_pM and an orthonormal basis $\{e_{n+1},\ldots,e_{2m+1}=\xi\}$ of $T_p^\perp M$ such that (a) $\pi = \mathrm{Span}\{e_1,e_2\}$

and (b) *the forms of shape operators* $A_r \equiv A_{e_r}$, $r = n+1, \ldots, 2m+1$, *become*

$$A_{n+1} = \begin{bmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & (a+b)I_{n-2} \end{bmatrix}, \tag{3.6}$$

$$A_r = \begin{bmatrix} c_r & d_r & 0 \\ d_r & -c_r & 0 \\ 0 & 0 & 0_{n-2} \end{bmatrix}, \; r = n+2, \ldots, 2m+1 \tag{3.7}$$

Proof. We set

$$\begin{aligned} \rho = 2\tau - n(n-1)f_1 - 2(n-1)f_4 \operatorname{trace}(h^T) \\ - f_{5,1}\left\{(\operatorname{trace}(h^T))^2 - \|h^T\|^2\right\} \\ + f_{5,2}\left\{\|(\phi h)^T\|^2 - (\operatorname{trace}(\phi h)^T)^2\right\} \\ - \frac{n^2(n-2)}{n-1}\|H\|^2. \end{aligned} \tag{3.8}$$

From (3.1) and (3.8), we get

$$n^2\|H\|^2 = (n-1)(\|\sigma\|^2 + \rho). \tag{3.9}$$

Let $\pi \subset T_pM$ be a plane section. We choose an orthonormal basis $\{e_1, \ldots, e_n\}$ for T_pM and an orthonormal basis $\{e_{n+1}, \ldots, e_{2m}, e_{2m+1}\}$ for the normal space $(T_p^\perp M)$ at p such that $\pi = \operatorname{Span}\{e_1, e_2\}$, the mean curvature vector $H(p)$ is parallel to e_{n+1} and $e_{2m+1} = \xi$ then the equation (3.9) can be written as

$$\left(\sum_{i=1}^{n} \sigma_{ii}^{n+1}\right)^2 = (n-1)\Bigg(\sum_{i=1}^{n}(\sigma_{ii}^{n+1})^2 + \sum_{i\neq j}(\sigma_{ij}^{n+1})^2 + \sum_{r=n+2}^{2m+1}\sum_{i,j=1}^{n}(\sigma_{ii}^r)^2 + \rho\Bigg). \tag{3.10}$$

Applying Lemma 2.1, from (3.10), we obtain

$$2\sigma_{11}^{n+1}\sigma_{22}^{n+1} \geq \sum_{i\neq j}(\sigma_{ij}^{n+1})^2 + \sum_{r=n+2}^{2m+1}\sum_{i,j=1}^{n}(\sigma_{ii}^r)^2 + \rho. \tag{3.11}$$

From equation (2.7), we also have

$$\begin{aligned} K(\pi) = f_1 + f_4 \operatorname{trace}(h|_\pi) + f_{5,1}\det(h|_\pi) + f_{5,2}\det((\phi h)|_\pi) \\ + \sigma_{11}^{n+1}\sigma_{22}^{n+1} - (\sigma_{12}^{n+1})^2 + \sum_{r=n+2}^{2m+1}(\sigma_{11}^r\sigma_{22}^r - (\sigma_{12}^r)^2), \end{aligned} \tag{3.12}$$

which in view of (3.11) gives

$$\begin{aligned} K(\pi) \geq f_1 + f_4 \operatorname{trace}(h|_\pi) + f_{5,1} \det(h|_\pi) + f_{5,2} \det((\phi h)|_\pi) + \frac{1}{2}\rho \\ + \sum_{r=n+1}^{2m+1} \sum_{j>2} ((\sigma_{1j}^r)^2 + (\sigma_{2j}^r)^2) + \frac{1}{2} \sum_{i \neq j > 2} (\sigma_{ij}^{n+1})^2 \\ + \frac{1}{2} \sum_{r=n+2}^{2m+1} \sum_{i,j>2} (\sigma_{ij}^r)^2 + \sum_{r=n+2}^{2m+1} (\sigma_{11}^r + \sigma_{22}^r)^2, \end{aligned} \tag{3.13}$$

or

$$K(\pi) \geq f_1 + f_4 \operatorname{trace}(h|_\pi) + f_{5,1} \det(h|_\pi) + f_{5,2} \det((\phi h)|_\pi) + \frac{1}{2}\rho. \tag{3.14}$$

In view of (3.8) and (3.14), we obtain (3.5). If the equality in (3.5) holds, then the inequalities given by (3.11) and (3.13) become equalities. In this case, we have

$$\begin{aligned} \sigma_{1j}^{n+1} = 0, \sigma_{2j}^{n+1} = 0, \sigma_{ij}^{n+1} = 0, i \neq j > 2; \\ \sigma_{1j}^r = \sigma_{2j}^r = \sigma_{ij}^r = 0, r = n+2, \ldots, 2m+1; i, j = 3, \ldots, n; \\ \sigma_{11}^{n+2} + \sigma_{22}^{n+2} = \cdots = \sigma_{11}^{2m+1} + \sigma_{22}^{2m+1} = 0. \end{aligned} \tag{3.15}$$

Furthermore, we may choose e_1 and e_2 so that $\sigma_{12}^{n+1} = 0$. Moreover, by applying Lemma 2.1, we also have

$$\sigma_{11}^{n+1} + \sigma_{22}^{n+1} = \sigma_{33}^{n+1} = \cdots = \sigma_{nn}^{n+1}. \tag{3.16}$$

Thus, after choosing a suitable orthonormal basis, the shape operator of M becomes of the form given by (3.6) and (3.7). The converse is straightforward. □

4. Squared mean curvature and Ricci curvature

Chen established a sharp relationship between the Ricci curvature and the squared mean curvature for submanifolds in real space forms [6]. We prove similar inequalities for certain submanifolds of a generalized (κ, μ)-contact space form with divided R_5, $\tilde{M}(f_1, \ldots, f_6)$.

Theorem 4.1. *Let M be an n-dimensional $(n \geq 3)$ C-totally real submanifold in a $(2m+1)$-dimensional generalized (κ, μ)-contact space form with divided R_5, $\tilde{M}(f_1, \ldots, f_6)$. Then for each point $p \in M$*

(1) For all unit vector $U \in T_pM$, we have

$$\begin{aligned} \operatorname{Ric}(U) \leq & \frac{1}{4} n^2 \|H\|^2 + (n-1) f_1 + f_4(\operatorname{trace}(h^T) + (n-2) g(h^T U, U)) \\ & + f_{5,1} \left(\operatorname{trace}(h^T) g(h^T U, U) - \|h^T U\|^2 \right) \\ & + f_{5,2} \left(\operatorname{trace}((\phi h)^T) g((\phi h)^T U, U) - \|(\phi h)^T U\|^2 \right). \end{aligned} \tag{4.1}$$

(2) For $H(p) = 0$, a unit tangent vector $U \in T_pM$ satisfies the equality case of (4.1) if and only if U belongs to the relative null space $\mathcal{N}_p$.

(3) the equality in (4.1) holds identically for all unit tangent vectors at p if and only if either p is a totally geodesic point or $n = 2$ and p is a totally umbilical point.

Proof. Let $U \in T_pM$ be a unit tangent vector. We choose an orthonormal basis $e_1, \ldots, e_n$, $e_{n+1}, \ldots, e_{2m+1}$ such that $e_1, \ldots, e_n$ are tangential to M at p with $e_1 = U$. Then, the squared second fundamental form and the squared mean curvature satisfy the following relation

$$\begin{aligned}\|\sigma\|^2 =&\frac{1}{2}n^2\|H\|^2 + \frac{1}{2}\sum_{r=n+1}^{2m+1}(\sigma_{11}^r - \sigma_{22}^r \cdots - \sigma_{nn}^r)^2 \\ &+ 2\sum_{r=n+1}^{2m+1}\sum_{j=2}^{n}(\sigma_{1j}^r)^2 - 2\sum_{r=n+1}^{2m+1}\sum_{2\le i<j\le n}(\sigma_{ii}^r\sigma_{jj}^r - (\sigma_{ij}^r)^2).\end{aligned} \tag{4.2}$$

From (3.1) and (4.2), we get

$$\begin{aligned}&\tau - \frac{n(n-1)}{2}f_1 - (n-1)f_4\,\mathrm{trace}(h^T) \\ &- \frac{1}{2}f_{5,1}\left\{(\mathrm{trace}(h^T))^2 - \|h^T\|^2\right\} + \frac{1}{2}f_{5,2}\left\{\|(\phi h)^T\|^2 - (\mathrm{trace}(\phi h)^T)^2\right\} \\ &+ \frac{1}{4}\sum_{r=n+1}^{2m+1}(\sigma_{11}^r - \sigma_{22}^r \cdots - \sigma_{nn}^r)^2 \\ &+ \sum_{r=n+1}^{2m+1}\sum_{j=2}^{n}(\sigma_{1j}^r)^2 - \sum_{r=n+1}^{2m+1}\sum_{2\le i<j\le n}(\sigma_{ii}^r\sigma_{jj}^r - (\sigma_{ij}^r)^2) \\ &= \frac{1}{4}n^2\|H\|^2.\end{aligned} \tag{4.3}$$

From (2.7) and (2.12), we also have

$$\begin{aligned}\sum_{2\le i<j\le n}K_{ij} =& \sum_{r=n+1}^{2m+1}\sum_{2\le i<j\le n}(\sigma_{ii}^r\sigma_{jj}^r - (\sigma_{ij}^r)^2) + \frac{(n-1)(n-2)}{2}f_1 \\ &+ \frac{1}{2}f_{5,1}\Big\{(\mathrm{trace}(h^T))^2 - 2\,\mathrm{trace}(h^T)\langle h^Te_1, e_1\rangle \\ &\qquad - \|h^T\|^2 + 2\|h^Te_1\|^2\Big\} \\ &- \frac{1}{2}f_{5,2}\Big\{\|(\phi h)^T\|^2 - (\mathrm{trace}(\phi h)^T)^2 - 2\|(\phi h)^Te_1\|^2 \\ &\qquad + 2\,\mathrm{trace}((\phi h)^T)\langle(\phi h)^Te_1, e_1\rangle\Big\} \\ &+ f_4(n-2)(\mathrm{trace}(h^T) - \langle h^Te_1, e_1\rangle).\end{aligned} \tag{4.4}$$

From (4.3) and (4.4), we get

$$\begin{aligned}\mathrm{Ric}(U) =& \frac{1}{4}n^2\|H\|^2 + (n-1)f_1 + f_4\,\mathrm{trace}(h^T) + (n-2)f_4\langle h^T U, U\rangle \\ &+ f_{5,1}\big(\mathrm{trace}(h^T)\langle h^T U, U\rangle - \|h^T U\|^2\big) \\ &+ f_{5,2}\big(\mathrm{trace}((\phi h)^T)\langle(\phi h)^T U, U\rangle - \|(\phi h)^T U\|^2\big) \\ &- \frac{1}{4}\sum_{r=n+1}^{2m+1}(\sigma^r_{11} - \sigma^r_{22}\cdots - \sigma^r_{nn})^2 \\ &- \sum_{r=n+1}^{2m+1}\sum_{j=2}^{n}(\sigma^r_{1j})^2,\end{aligned} \tag{4.5}$$

which implies (4.1).
Assuming $U = e_1$, from (4.5), the equality case of (4.1) is valid if and only if

$$\begin{aligned}&\sigma^r_{11} = \sigma^r_{22} + \cdots + \sigma^r_{nn}, \\ &\sigma^r_{12} = \cdots = \sigma^r_{1n} = 0, r = n+1, \ldots, 2m+1.\end{aligned} \tag{4.6}$$

If $H(p) = 0$, (4.6) implies that $U = e_1$ lies in the relative null space $\mathcal{N}_p$. Conversely, if $U = e_1$ lies in the relative null space $\mathcal{N}_p$, then (4.6) holds, since $H(p) = 0$ is assumed. Thus (2) is proved.
Now we prove (3). The equality case of (4.1) for all unit tangent vectors to M at p happens if and only if

$$\begin{aligned}2\sigma^r_{ii} &= \sigma^r_{11} + \cdots + \sigma^r_{nn}, i = 1, \ldots, n, r = n+1, \ldots, 2m+1, \\ \sigma^r_{ij} &= 0, i \neq j, r = n+1, \ldots, 2m+1.\end{aligned} \tag{4.7}$$

Thus, we have two cases, namely either $n = 2$ or $n \neq 2$. In the first case p is a totally umbilical point, while in the second case p is a totally geodesic point. The proof of converse part is straightforward. □

5. Squared mean curvature and k-Ricci curvature

Theorem 5.1. *Let M be an n-dimensional $(n \geq 3)$ C-totally real submanifold in a $(2m+1)$-dimensional generalized (κ, μ)-contact space form with divided R_5, $\tilde{M}(f_1, \ldots, f_6)$. Then we have*

$$\begin{aligned}n(n-1)\|H\|^2 \geq& 2\tau - n(n-1)f_1 - 2(n-1)f_4\,\mathrm{trace}(h^T) \\ &- f_{5,1}\left\{(\mathrm{trace}(h^T))^2 - \|h^T\|^2\right\} \\ &+ f_{5,2}\left\{\|(\phi h)^T\|^2 - (\mathrm{trace}(\phi h)^T)^2\right\}\end{aligned} \tag{5.1}$$

Proof. Let $X \in T_pM$ be a unit tangent vector X at p. We choose an orthonormal basis $\{e_1, \ldots, e_n, e_{n+1}, \ldots, e_{2m+1}\}$ such that $e_1, \ldots, e_n$ are tangential to M at p with $e_1 = X$. We recall the equation (3.1) as

$$\begin{aligned}n^2\|H\|^2 =& 2\tau + \|\sigma\|^2 - n(n-1)f_1 - 2(n-1)f_4\,\mathrm{trace}(h^T) \\ &- f_{5,1}\left\{(\mathrm{trace}(h^T))^2 - \|h^T\|^2\right\} + f_{5,2}\left\{\|(\phi h)^T\|^2 - (\mathrm{trace}(\phi h)^T)^2\right\}.\end{aligned} \tag{5.2}$$

Let the orthonormal basis $\{e_1, \ldots, e_n, e_{n+1}, \ldots, e_{2m+1}\}$ be such that e_{n+1} is parallel to the mean curvature vector $H(p)$ and $e_1, \ldots, e_n$ diagonalize the shape operator A_{n+1}. Then the shape operators take the forms

$$A_{n+1} = \begin{bmatrix} a_1 & 0 & \cdots & 0 \\ 0 & a_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & a_n \end{bmatrix}, \tag{5.3}$$

$$A_r = (\sigma_{ij}^r), i, j = 1, \ldots, n, r = n+2, \ldots, 2m+1, \mathrm{trace}(A_r) = \sum_{i=1}^{n} \sigma_{ii}^r = 0. \tag{5.4}$$

From (5.2), we get

$$\begin{aligned} n^2\|H\|^2 = 2\tau + \sum_{i=1}^{n} a_i^2 + \sum_{r=n+2}^{2m+1} \sum_{i,j=1}^{n} (\sigma_{ij}^r)^2 - n(n-1)f_1 - 2(n-1)f_4 \,\mathrm{trace}(h^T) \\ - f_{5,1}\left\{(\mathrm{trace}(h^T))^2 - \|h^T\|^2\right\} + f_{5,2}\left\{\|(\phi h)^T\|^2 - (\mathrm{trace}(\phi h)^T)^2\right\}. \end{aligned} \tag{5.5}$$

Since

$$0 \le \sum_{i<j} (a_i - a_j)^2 = (n-1)\sum_i a_i^2 - 2\sum_{i<j} a_i a_j. \tag{5.6}$$

Therefor, we get

$$n^2\|H\|^2 = \left(\sum_{i=1}^{n} a_i\right)^2 = \sum_{i=1}^{n} a_i^2 + 2\sum_{i<j} a_i a_j \le n\sum_{i=1}^{n} a_i^2, \tag{5.7}$$

which implies

$$\sum_{i=1}^{n} a_i^2 \ge n\|H\|^2.$$

In view of (5.5), we obtain

$$\begin{aligned} n^2\|H\|^2 \ge 2\tau + n\|H\|^2 - n(n-1)f_1 - 2(n-1)f_4 \,\mathrm{trace}(h^T) \\ - f_{5,1}\left\{(\mathrm{trace}(h^T))^2 - \|h^T\|^2\right\} + f_{5,2}\left\{\|(\phi h)^T\|^2 - (\mathrm{trace}(\phi h)^T)^2\right\}, \end{aligned} \tag{5.8}$$

which gives (5.1). □

Theorem 5.2. *Let M be an n-dimensional $(n \ge 3)$ C-totally real submanifold in a $(2m+1)$-dimensional generalized (κ, μ)-contact space form with divided R_5, $\tilde{M}(f_1, \ldots, f_6)$. Then we have*

$$\begin{aligned} n(n-1)\|H\|^2 \ge n(n-1)\theta_k(p) - n(n-1)f_1 - 2(n-1)f_4 \,\mathrm{trace}(h^T) \\ - f_{5,1}\left\{(\mathrm{trace}(h^T))^2 - \|h^T\|^2\right\} + f_{5,2}\left\{\|(\phi h)^T\|^2 - (\mathrm{trace}(\phi h)^T)^2\right\} \end{aligned} \tag{5.9}$$

Proof. Let $\{e_1, \ldots, e_n\}$ be an orthonormal basis of T_pM. We denote by $L_{i_1 \ldots i_k}$ the k-plane section spanned by $e_{i_1}, \ldots, e_{i_k}$. From (2.1) and (2.2), it follows that

$$\tau(L_{i_1 \ldots i_k}) = \frac{1}{2} \sum_{i \in \{i_1, \ldots, i_k\}} \mathrm{Ric}_{L_{i_1 \ldots i_k}}(e_i), \tag{5.10}$$

and

$$\tau(p)=\frac{1}{C_{k-2}^{n-2}}\sum_{1\le i_1<\cdots<i_k\le n}\tau(L_{i_1\ldots i_k}). \tag{5.11}$$

Combining (2.3), (5.10) and (5.11), we obtain

$$\tau(p)\ge\frac{n(n-1)}{2}\theta_k(p), \tag{5.12}$$

which in view of (5.1) implies (5.9). □

6. δ-INVARIANT AND INEQUALITIES FOR SUBMANIFOLDS IN A SASAKIAN GENERALIZED (κ,μ)-SPACE FORMS

In this section,we apply these results to get corresponding results for C-totally real submanifolds in a generalized (κ,μ)-contact space forms $\tilde{M}(f_1,\ldots,f_6)$ with $f_3=f_1-1$. If $\kappa=1=f_1-f_3$, the generalized (κ,μ)-contact space forms $\tilde{M}(f_1,\ldots,f_6)$ reduces to Sasakian space form; thus $h=0$ and (2.12) becomes

$$\tilde{R}(X,Y)Z=f_1R_1(X,Y)Z+f_2R_2(X,Y)Z+f_3R_3(X,Y)Z. \tag{6.1}$$

Moreover, for a C-totally real submanifolds in Sasakian space forms, from (2.13), we also get

$$A_\xi=0. \tag{6.2}$$

Thus, in view of Theorem 3.1, we can state the following.

Theorem 6.1. *Let M be an n-dimensional $(n\ge 3)$ C-totally real submanifold in a $(2m+1)$-dimensional generalized (κ,μ)-contact space forms $\tilde{M}(f_1,\ldots,f_6)$ satisfying $f_3=f_1-1$.We have*

$$\delta_M\leqslant\frac{n^2(n-2)}{2(n-1)}\|H\|^2+\frac{(n+1)(n-2)}{2}f_1. \tag{6.3}$$

The equality in (6.3) holds at $p\in M$ if and only if there exist an orthonormal basis $\{e_1,\ldots,e_n\}$ of T_pM and an orthonormal basis $\{e_{n+1},\ldots,e_{2m+1}=\xi\}$ of $T_p^\perp M$ such that (a) $\pi=\mathrm{Span}\{e_1,e_2\}$ *and* (b) *the forms of shape operators $A_r\equiv A_{e_r}$, $r=n+1,\ldots,2m+1$, become*

$$A_{n+1}=\begin{bmatrix} a & 0 & 0\\ 0 & b & 0\\ 0 & 0 & (a+b)I_{n-2}\end{bmatrix}, \tag{6.4}$$

$$A_r=\begin{bmatrix} c_r & d_r & 0\\ d_r & -c_r & 0\\ 0 & 0 & 0_{n-2}\end{bmatrix},\ r=n+2,\ldots,2m, \tag{6.5}$$

and $A_\xi=0$.

Theorem 6.2. *Let M be an n-dimensional $(n\ge 3)$ C-totally real submanifold in a $(2m+1)$-dimensional generalized (κ,μ)-contact space forms $\tilde{M}(f_1,\ldots,f_6)$ satisfying $f_3=f_1-1$. Then for each point $p\in M$*

(1) For all unit vector $U\in T_pM$, we have

$$\mathrm{Ric}(U)\le\frac{1}{4}n^2\|H\|^2+(n-1)f_1. \tag{6.6}$$

(2) For $H(p) = 0$, a unit tangent vector $U \in T_pM$ satisfies

$$\mathrm{Ric}(U) = (n-1)f_1 \tag{6.7}$$

if and only if U belongs to the relative null space $\mathcal{N}_p$.

(3) For each $p \in M$

$$4S \leq (n^2\|H\|^2 + 4(n-1)f_1)g. \tag{6.8}$$

where S is the Ricci tensor of the submanifold and g is the Riemannian metric. The equality in (6.8) holds if and only if either p is a totally geodesic point or $n = 2$ and p is a totally umbilical point.

Theorem 6.3. *Let M be an n-dimensional $(n \geq 3)$ C-totally real submanifold in a $(2m+1)$-dimensional generalized (κ, μ)-contact space form with divided R_5, $\tilde{M}(f_1, \ldots, f_6)$ satisfying $f_3 = f_1 - 1$. Then we have*

$$\delta(n_1, \ldots, n_k) \leq c(n_1, \ldots, n_k)\|H\|^2 + b(n_1, \ldots, n_k)f_1 \tag{6.9}$$

for any k-tuple $(n_1, \ldots, n_k) \in \mathcal{S}(n)$. The equality case of inequality (6.9) holds at a point $p \in M$ if and only if there exists an orthonormal basis $\{e_1, \ldots, e_{2m+1}\}$ at p such that the shape operators of M in $\tilde{M}$ at p take the following form:

$$A_r = \begin{pmatrix} A_1^r & \cdots & 0 & \\ \vdots & \ddots & \vdots & 0 \\ 0 & \cdots & A_k^r & \\ & 0 & & a_r I \end{pmatrix}, r = n+1, \ldots, 2m+1, \tag{6.10}$$

where I is an identity matrix and A_i^r are symmetric $n_i \times n_i$ submatrices such that

$$\mathrm{trace}(A_1^r) = \ldots = \mathrm{trace}(A_k^r) = a_r. \tag{6.11}$$

Proof. Let $\mathbf{N} \in \mathcal{S}(n)$. We set

$$\epsilon = 2\tau - 2c\mathbf{N}\|H\|^2 - n(n-1)f_1. \tag{6.12}$$

Substituting (3.1) in (6.12), we have

$$n^2\|H\|^2 = \gamma(\epsilon + \|\sigma\|^2), \qquad \gamma = n + k - \sum_{j=1}^{k} n_j. \tag{6.13}$$

Let $L_1, \ldots, L_k$ be mutually orthogonal subspaces of T_pM with $\dim L_j = n_j, j = 1, \ldots, k$. By choosing an orthonormal basis $\{e_1, \ldots, e_n, \ldots, e_{2m+1}\}$ at $p \in M$ such that $e_1, \ldots, e_n$ are tangent to M at p, $e_{n+1} = \frac{H}{\|H\|}, e_{2m+1} = \xi$, and

$$\begin{aligned} L_1 &= \mathrm{Span}\{e_1, \ldots, e_{n_1}\}, \\ L_2 &= \mathrm{Span}\{e_{n_1+1}, \ldots, e_{n_1+n_2}\}, \\ &\vdots \\ L_k &= \mathrm{Span}\{e_{n_1+\cdots+n_{k-1}+1}, \ldots, e_{n_1+\cdots+n_k}\}, \end{aligned}$$

then the equation (6.13) can be written as

$$\left(\sum_{i=1}^{n} \sigma_{ii}^{n+1}\right)^2 = \gamma\left(\epsilon + \sum_{i=1}^{n}(\sigma_{ii}^{n+1})^2 + \sum_{i\neq j}(\sigma_{ij}^{n+1})^2 + \sum_{r=n+2}^{2m+1}\sum_{i,j=1}^{n}(\sigma_{ii}^{r})^2\right). \tag{6.14}$$

We set

$$\Delta_1 = \{1, \ldots, n_1\}, \ldots, \Delta_k = \{n_1 + \cdots + n_{k-1} + 1, \ldots, n_1 + \cdots + n_k\}.$$

The equation (6.14) can be rewritten in the form

$$\begin{aligned}\left(\sum_{i=1}^{\gamma+1} a_i\right)^2 = \gamma\Bigg(\epsilon + \sum_{i=1}^{\gamma+1}(a_i)^2 + \sum_{i\neq j}(\sigma_{ij}^{n+1})^2 + \sum_{r=n+2}^{2m+1}\sum_{i,j=1}^{n}(\sigma_{ii}^r)^2 \\ - \sum_{2\le\alpha_1\neq\beta_1\le n_1}\sigma_{\alpha_1\alpha_1}^{n+1}\sigma_{\beta_1\beta_1}^{n+1} - \sum_{\alpha_2\neq\beta_2}\sigma_{\alpha_2\alpha_2}^{n+1}\sigma_{\beta_2\beta_2}^{n+1} - \cdots - \sum_{\alpha_k\neq\beta_k}\sigma_{\alpha_k\alpha_k}^{n+1}\sigma_{\beta_k\beta_k}^{n+1}\Bigg),\end{aligned} \tag{6.15}$$

where $\alpha_i, \beta_i \in \Delta_i, i = 2, \ldots, k$ and

$$\begin{aligned} a_1 = \sigma_{11}^{n+1}, a_2 = \sum_{i=2}^{n_1}\sigma_{ii}^{n+1}, a_3 = \sum_{i=n_1+1}^{n_1+n_2}\sigma_{ii}^{n+1}, \ldots, a_{k+1} = \sum_{i=n_1+\cdots+n_{k-1}+1}^{n_1+\cdots+n_k}\sigma_{ii}^{n+1}, \\ a_{k+2} = \sigma_{(n_1+\cdots+n_k+1)(n_1+\cdots+n_k+1)}^{n+1}, \ldots, a_{\gamma+1} = \sigma_{nn}^{n+1}.\end{aligned}$$

By applying Lemma 2.1 to (6.15), we can obtain the following inequality

$$\begin{aligned}\sum_{\alpha_1<\beta_1}\sigma_{\alpha_1\alpha_1}^{n+1}\sigma_{\beta_1\beta_1}^{n+1} + \sum_{\alpha_2<\beta_2}\sigma_{\alpha_2\alpha_2}^{n+1}\sigma_{\beta_2\beta_2}^{n+1} + \cdots + \sum_{\alpha_k<\beta_k}\sigma_{\alpha_k\alpha_k}^{n+1}\sigma_{\beta_k\beta_k}^{n+1} \\ \ge \frac{\epsilon}{2} + \sum_{i<j}(\sigma_{ij}^{n+1})^2 + \frac{1}{2}\sum_{r=n+2}^{2m+1}\sum_{i,j=1}^{n}(\sigma_{ij}^r)^2,\end{aligned} \tag{6.16}$$

where $\alpha_i, \beta_i \in \Delta_i, i = 1, \ldots, k$. From (2.1) and Gauss'equation, we see that

$$\tau(L_j) = \frac{n_j(n_j-1)}{2}f_1 + \sum_{r=n+1}^{2m+1}\sum_{\alpha_i<\beta_i}(\sigma_{\alpha_i\alpha_i}^r\sigma_{\beta_i\beta_i}^r - (\sigma_{\alpha_i\beta_i}^r)^2), \tag{6.17}$$

where $\alpha_i, \beta_i \in \Delta_i, i = 1, \ldots, k$. In view of (6.16) and (6.17), we obtain

$$\begin{aligned}\sum_{i=1}^{k}\tau(L_i) \ge& \frac{\epsilon}{2} + \sum_{i=1}^{k}\frac{n_i(n_i-1)}{2}f_1 \\ &+ \frac{1}{2}\sum_{r=n+1}^{2m+1}\sum_{(\alpha,\beta)\notin\Delta^2}(\sigma_{\alpha\beta}^r)^2 + \frac{1}{2}\sum_{r=n+2}^{2m+1}\sum_{i=1}^{k}\left(\sum_{\alpha_i\in\Delta_i}\sigma_{\alpha_i\alpha_i}^r\right)^2 \\ \ge& \frac{\epsilon}{2} + \sum_{i=1}^{k}\frac{n_i(n_i-1)}{2}f_1,\end{aligned} \tag{6.18}$$

where $\Delta^2 = (\Delta_1 \times \Delta_1) \cup \ldots \cup (\Delta_k \times \Delta_k)$. From (2.4), (6.12) and (6.18), we can obtain (6.9). If the equality in (6.9) holds at a point p, then the inequalities in (6.16) and (6.18) are actually equalities at p. In this case, by applying Lemma 2.1, (6.15), (6.16), (6.17) and (6.18), we also obtain (6.10) and (6.11). □

References

[1] Alegre, P., Blair, D.E., Carriazo, A.: Generalized Sasakian-space-forms. Israel J. Math. **141**, 157–183 (2004). DOI 10.1007/BF02772217. URL http://dx.doi.org/10.1007/BF02772217

[2] Alegre, P., Carriazo, A., Kim, Y.H., Yoon, D.W.: B. Y. Chen's inequality for submanifolds of generalized space forms. Indian J. Pure Appl. Math. **38**(3), 185–201 (2007)

[3] Carriazo, A., Martín-Molina, V.: Generalized (κ, μ)-space forms and D_a-homothetic deformations. Balkan J. Geom. Appl. **16**(1), 37–47 (2011)
[4] Carriazo, A., Martín Molina, V., Tripathi, M.M.: Generalized (κ, μ)-space forms. Mediterr. J. Math. **10**(1), 475–496 (2013). DOI 10.1007/s00009-012-0196-2. URL http://dx.doi.org/10.1007/s00009-012-0196-2
[5] Chen, B.Y.: Some pinching and classification theorems for minimal submanifolds. Arch. Math. (Basel) **60**(6), 568–578 (1993). DOI 10.1007/BF01236084. URL http://dx.doi.org/10.1007/BF01236084
[6] Chen, B.Y.: Relations between Ricci curvature and shape operator for submanifolds with arbitrary codimensions. Glasg. Math. J. **41**(1), 33–41 (1999). DOI 10.1017/S0017089599970271. URL http://dx.doi.org/10.1017/S0017089599970271
[7] Chen, B.Y.: Some new obstructions to minimal and Lagrangian isometric immersions. Japan. J. Math. (N.S.) **26**(1), 105–127 (2000)
[8] Faghfouri, M., Ghaffarzadeh, N.: Chen's inequality for invariant submanifolds in a generalized (κ, μ)-space forms. Glob. J. Adv. Res. Class. Mod. Geom. **4**(2), 86–101 (2015). URL http://gjarcmg.geometry-math-journal.ro/submission/
[9] Faghfouri, M., Ghaffarzadeh, N.: On doubly warped product submanifolds of generalized (κ, μ)-space forms. Afrika Matematika **26**(7-8), 1443–1455 (2015). DOI 10.1007/s13370-014-0299-y. URL http://dx.doi.org/10.1007/s13370-014-0299-y
[10] Faghfouri, M., Majidi, A.: On doubly warped product immersions. Journal of Geometry **106**(2), 243–254 (2015). DOI 10.1007/s00022-014-0245-z. URL http://dx.doi.org/10.1007/s00022-014-0245-z
[11] Hong, S., Tripathi, M.M.: Ricci curvature of submanifolds of sasakian space form. Iran. J. Math. Sci. Inform. **1**(2), 31–51 (2006)
[12] Kim, J.S., Choi, J.: A basic inequality for submanifolds in a cosymplectic space form. Int. J. Math. Math. Sci. (9), 539–547 (2003). DOI 10.1155/S0161171203202027. URL http://dx.doi.org/10.1155/S0161171203202027
[13] Mihai, I.: Ricci curvature of submanifolds in Sasakian space forms. J. Aust. Math. Soc. **72**(2), 247–256 (2002). DOI 10.1017/S1446788700003888. URL http://dx.doi.org/10.1017/S1446788700003888
[14] Nash, J.: The imbedding problem for Riemannian manifolds. Ann. of Math. (2) **63**, 20–63 (1956)
[15] Tripathi, M.M.: C-totally real warped product submanifolds. An. Ştiinţ. Univ. Al. I. Cuza Iaşi. Mat. (N.S.) **58**(2), 417–436 (2012)
[16] Tripathi, M.M., Kim, J.S.: C-totally real submanifolds in (κ, μ)-contact space forms. Bull. Austral. Math. Soc. **67**(1), 51–65 (2003). DOI 10.1017/S0004972700033517. URL http://dx.doi.org/10.1017/S0004972700033517
[17] Yamaguchi, S., Kon, M., Ikawa, T.: C-totally real submanifolds. J. Differential Geometry **11**(1), 59–64 (1976)
[18] Yano, K., Kon, M.: Anti-invariant submanifolds. Marcel Dekker Inc., New York (1976). Lecture Notes in Pure and Applied Mathematics, No. 21

Department of Pure Mathematics, Faculty of Mathematical Sciences, University of Tabriz, Tabriz, Iran.
E-mail address: faghfouri@tabrizu.ac.ir

Department of Pure Mathematics, Faculty of Mathematical Sciences, University of Tabriz, Tabriz, Iran.
E-mail address: n.ghaffarzadeh@tabrizu.ac.ir