DOES THE MONGE THEOREM APPLY TO SOME NON-EUCLIDEAN GEOMETRIES ?

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ABSTRACT. In geometry, Monge's theorem states that for any three non-overlapping circles of distinct radii in the two dimensional analytical plane equipped with the Euclidean metric, none of which is completely inside one of the others, the intersection points of each of the three pairs of external tangent lines are collinear. So, it is clearly observed that Monge's theorem is an application of Desargues' theorem. Our main motivation in this study is to show whether Monge theorem is still valid even if the plane is equipped with the metrics alpha and L_p.

1. INTRODUCTION AND MOTIVATIONS

Euclidean distance is the most common use of distance. In most cases when people said about distance, they will refer to Euclidean distance. The Euclidean distance between two points is defined as the length of the segment between two points. Although it is the most popular distance function, it is not practical when we measure the distance which we actually move in the real world (we live on a spherical Earth rather than on a Euclidean 3−space !). We must think of the distance as though a car would drive in the urban geography where physical obstacles have to be avoided. So, one had to travel through horizontal and vertical streets to get from one location to another. To compensate disadvantage of the Euclidean distance, the taxicab geometry was first introduced by K. Menger and has developed by E. F. Krause using the taxicab metric d_T of which paths composed of the line segments parallel to coordinate axes (see Figure 1) ([8], [13]). Its three-dimensional version has been introduced in [1]. Later, researchers have wondered whether there are alternative distance functions of which paths are different from path of Euclidean metric. For example, G. Chen [2] developed Chinese checker distance d_{CC} in $\mathbb{R}^2$ of which paths are similar to the movement made by Chinese checker (see Figure 1). Another example, S. Tian gave a family of metrics in [18], alpha distance d_α for $\alpha \in [0, \pi/4]$, which includes the taxicab and Chinese checker metrics as special cases. Then, Kaya et al. have given the most general form of d_α on $n−$dimensional analytical space for $\alpha \in [0, \pi/2)$ (see [4],[5]) . When we examine the common features of the metrics d_M, d_T, d_{CC} and d_α, we see that these metrics whose paths are parallel to at least one of the coordinate axes. So, it is a logical question " Are there metric or metrics of which

paths are not parallel to the coordinate axes" (Yes, there are !). The references [3], [6] and [14] can be reviewed for the answer to this question. Meanwhile, the alpha metric is defined as follows;

Definition 1.1. *Let* $A = (x_a, y_a)$ *and* $B = (x_b, y_b)$ *be two any points in* $\mathbb{R}^2$ *such that* $\Delta_{AB} = \max\{|x_a - x_b|, |y_a - y_b|\}$ *and* $\delta_{AB} = \min\{|x_a - x_b|, |y_a - y_b|\}$. *Then, the Euclidean metric can be given by*

$$d_E(A, B) = \left(\Delta_{AB}^2 + \delta_{AB}^2\right)^{1/2}.$$

Also, the metrics d_M, d_T, d_{CC} *and* d_α *are defined as following Figure 1.*

The path of distance d_M
$$d_M(A, B) = \Delta_{AB}$$

The path of distance d_T
$$d_T(A, B) = \Delta_{AB} + \delta_{AB}$$

The path of distance d_{CC}
$$d_{CC}(A, B) = \Delta_{AB} + \left(\sqrt{2} - 1\right)\delta_{AB}$$

The path of distance d_α
$$d_\alpha(A, B) = \Delta_{AB} + (\sec\alpha - \tan\alpha)\delta_{AB}$$
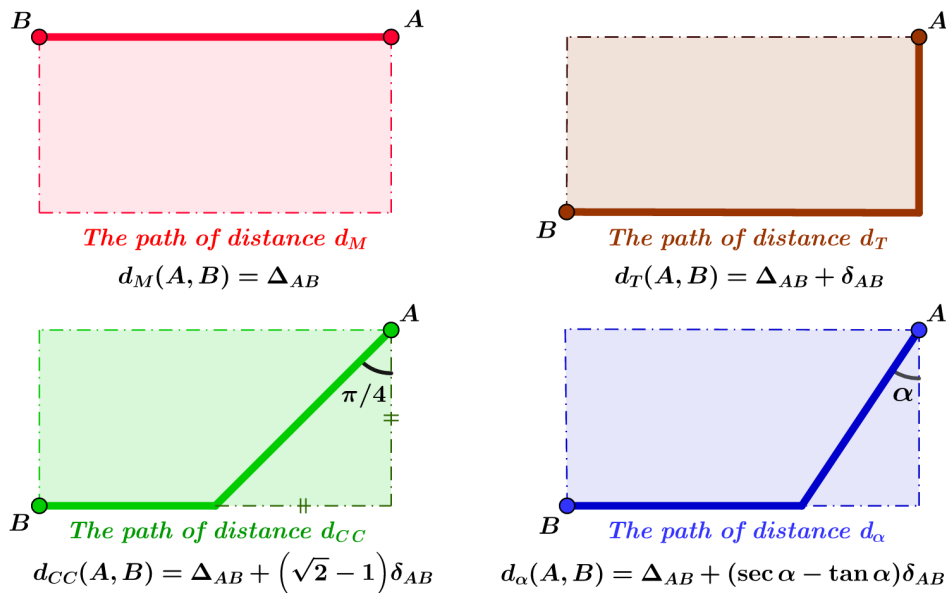
Figure 1. The paths of distances d_M, d_T, d_{CC} and d_α.

$\alpha-$(alpha) plane geometry, which includes the taxicab and Chinese checker geometry, is a Minkowski geometry. Minkowski geometry is a non Euclidean geometry in a finite number of dimensions that is different from elliptic and hyperbolic geometry (from Minkowskian geometry of space-time). Here the linear structure is same as the Euclidean one but distance is not uniform in all directions (see for details [7], [9], [11], [12], [15], [17]). That is, $\alpha-$plane is almost the same as Euclidean plane since the points are the same, the lines are the same, and the angles are measured in the same way. Instead of the usual circle in Euclidean plane geometry, unit ball is a certain symmetric closed (see Figure 2). Since the $\alpha-$plane geometry has a different distance function, it seems interesting to study the $\alpha-$analog of the topics that include the concepts of distance in the Euclidean geometry. One of the famous theorems that includes the concept of distance is the Monge Theorem [10], [19]. Monge's theorem says that for any three nonintersecting circles in a plane, none of which is equal radius, the intersection points of each of the three pairs of external tangent lines are collinear. For any two circles in a plane, an external tangent is a line that is tangent to both circles but does not pass between them. There

are two such external tangent lines for any two circles. Each such pair has a unique intersection point in the plane. In the case of two of the circles being of equal size, the two external tangent lines are parallel. If the two external tangents are considered to intersect at the point at infinity, then the other two intersection points must be on a line passing through the same point at infinity, so the line between them takes the same angle as the external tangent.

We will show that Monge theorem is still valid even if the plane is equipped with the metrics α and L_p in the next section. First, let us recall the definition of the well-known L_p-metric.

Definition 1.2. *Let $X = (x_1, \ldots, x_n)$ and $Y = (y_1, \ldots, y_n)$ be two vectors in the $n-$dimensional real vector space $\mathbb{R}^n$. The l_p-metric d_{l_p}, $1 \leq p \leq \infty$, is a norm metric on $\mathbb{R}^n$ (or on $\mathbb{C}^n$), is defined by*

$$\|X - Y\|_p,$$

where the l_p-norm $\|.\|_p$ is defined by

$$\|X\|_p = \left(\sum_{i=1}^{n} |x_i|^p \right)^{\frac{1}{p}}.$$

If $p \to \infty$, we obtain $\|X\|_\infty = \lim\limits_{p \to \infty} \left(\sum\limits_{i=1}^{n} |x_i|^p \right)^{\frac{1}{p}} = \max\limits_{1 \leq i \leq n} |x_i|$. Also, the the pair $\left(\mathbb{R}^n, d_{l_p} \right)$ is called l_p-(metric) space.

2. The Unit Circles in $\mathbb{R}^2$ and Monge' s Theorem

As is known to all, the circle is simply the set of points that are an equal distance from a certain central point. That is, let M be given point in the plane, and r be a positive real number. The set of points $\left\{ X \in \mathbb{R}^2 : d(M, X) = r \right\}$ is called a circle, the point M is called center of the circle, and r is called the length of the radius or simply radius of the circle. But there's a big assumption in definition of the circle: " distance ". While we usually
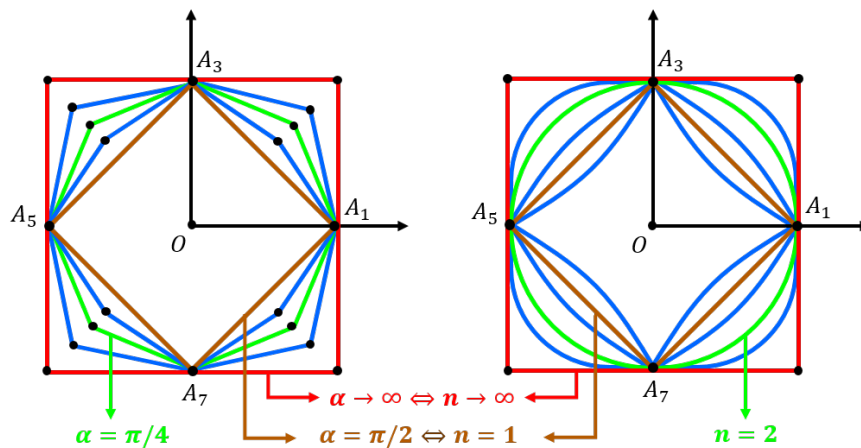

Figure 2. The unit circles in the α and l_p-plane.

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use the Euclidean distance d_E, there are other valid notions of distance we could have used instead, which we have mentioned above. It is easy to see that if $\alpha \in (0, \pi/2)$, then unit α–circle is an octagon with corner points $A_1 = (1, 0)$, $A_2 = (1/k, 1/k)$, $A_3 = (0, 1)$, $A_4 = (-1/k, 1/k)$, $A_5 = (-1, 0)$, $A_6 = (-1/k, -1/k)$, $A_7 = (0, -1)$ and $A_8 = (1/k, -1/k)$, where $k = 1 + \sec \alpha - \tan \alpha$. Note that A_2 and A_6 are on the line $y = x$; A_4 and A_8 are on the line $y = -x$. When $\alpha \to \pi/2$ and $\alpha = 0$, the α–circle is the circle with respect to the metrics d_M and d_T, respectively (see Figure 2). Also, α–circle is the circle with respect to the metrics d_{CC} such that $\alpha = \pi/4$. Similarly, when $p \to \infty$, $p = 1$ and $p = 2$, the circle in l_p–plane is the circle with respect to the metrics d_M, d_T and d_E, respectively (see Figure 2). Now, when we take the circles α and l_p instead of Euclidean circles, we will investigate whether the Monge theorem is still valid. Consider Figure 3. Let C_i be
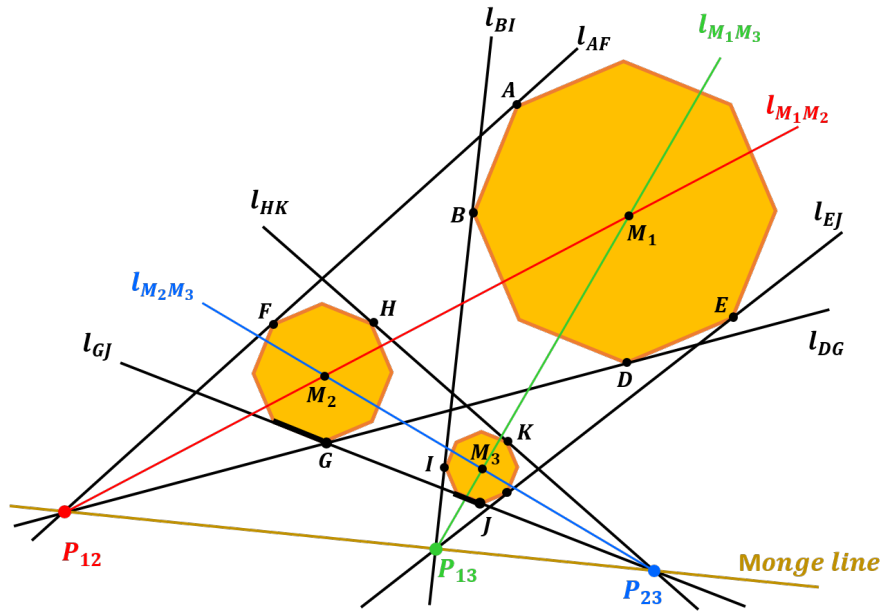

Figure 3

the α–circles (or l_p–circles) with center $M_i = (x_i, y_i)$ and radius r_i for $i = 1, 2, 3$. Then, the coordinates of points on the α–circles (or l_p–circles) C_1, C_2 and C_3 can be easily given with respect to the coordinates of circles's centers and radii. Because we know the coordinates of points A, F, D and G on the circles, we can find the equations of the tangent lines l_{AF} and l_{DG}. Consequently, coordinates of the point P_{12} of intersection of the tangent lines l_{AF} and l_{DG} be able to calculated as $P_{12} = \left(\dfrac{r_1 x_2 - r_2 x_1}{r_1 - r_2}, \dfrac{r_1 y_2 - r_2 y_1}{r_1 - r_2} \right)$.

Also, the line $l_{M_1 M_2}$ through the center points M_1 and M_2 passes through the point P_{12}. Similarly, after calculating the equations of tangent lines l_{BI} and l_{EJ} for the α–circles (or

l_p–circles) C_1 and C_3, their intersection point of lines l_{BI} and l_{EJ} is found as $P_{13} = \left(\dfrac{r_1 x_3 - r_3 x_1}{r_1 - r_3}, \dfrac{r_1 y_3 - r_3 y_1}{r_1 - r_3} \right)$. Finally, coordinates of the point P_{23} be able to calcu-

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lated as $P_{23} = \left(\dfrac{r_2 x_3 - r_3 x_2}{r_2 - r_3}, \dfrac{r_2 y_3 - r_3 y_2}{r_2 - r_3} \right)$. By simplifying long and boring calculations, we get the equation of the line passing through points P_{12} and P_{13} as

$$y = \frac{y_1 (r_2 - r_3) - y_2 (r_1 - r_3) + y_3 (r_1 - r_2)}{x_1 (r_2 - r_3) - x_2 (r_1 - r_3) + x_3 (r_1 - r_2)} x$$
$$+ \frac{r_1 (x_2 y_3 - x_3 y_2) - r_2 (x_1 y_3 - x_3 y_1) + r_3 (x_1 y_2 - x_2 y_1)}{x_1 (r_2 - r_3) - x_2 (r_1 - r_3) + x_3 (r_1 - r_2)}.$$

Similarly, we get the equation of the line passing through points P_{13} and P_{23} in the same way as the line above. Consequently, the points P_{12}, P_{13} and P_{23} are collinear since the these points determine one line called the axis of similitude (or Monge Line). Thus, using the notations in [16], we can give the following main theorem.

Theorem 2.1. *Let C_i and C_j be two $\alpha-$circles (or l_p-circles) with centers $M_i = (x_i, y_i)$ and $M_j = (x_i, y_i)$ of radii r_i and r_j for $i, j \in \{1, 2, 3\}$, respectively. Then, the common tangents of the circles C_i and C_j intersect at the point*

$$P_{ij} = \left(\frac{r_i x_j - r_j x_i}{r_i - r_j}, \frac{r_i y_j - r_j y_i}{r_i - r_j} \right).$$

Also, the points P_{12}, P_{13} and P_{23} of intersection of the three pairs of tangent lines lie on a line. This line is called Monge line whose equation

$$\begin{vmatrix} x_1 & x_2 & x_3 \\ r_1 & r_2 & r_3 \\ 1 & 1 & 1 \end{vmatrix} y = \begin{vmatrix} y_1 & y_2 & y_3 \\ r_1 & r_2 & r_3 \\ 1 & 1 & 1 \end{vmatrix} x - \begin{vmatrix} x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \\ r_1 & r_2 & r_3 \end{vmatrix}.$$

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