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ON PENCILS OF HYPERBOLAS

PARIS PAMFILOS

ABSTRACT. In this article we study the structure of pencils of conics passing through four points of the plane in general position and consisting entirely of hyperbolas. We show that these pencils represent a generalization of the *"Poncelet pencil"*, whose members are all rectangular hyperbolas circumscribing a triangle. Our study uses properties deriving from a generalization of the notion of orthogonality, based on the conjugation of directions relative to a central circumconic. These include the generalization of the orthocenter, the Steiner lines and the generalization of the properties of rectangular hyperbolas and their relations to the orthocenter and the circumcircle of the triangle of reference.

1. INTRODUCTION

A central conic $\kappa(O)$ introduces a natural generalization of orthogonality of lines of the plane. Every line of the plane defines a parallel to it diameter α and we may use the *"conjugate diam-*

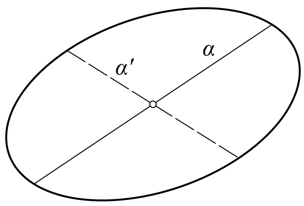

FIGURE 1. The generalized orthogonal direction α' to α

eter" α' of this to define a kind of generalized orthogonal direction to α (see Figure 1). In this article we fix a triangle ABC and a central circumconic κ of it. With a focus on the generation of circumconics of a triangle and especially hyperbolas, we study the generalizations of well known constructs and properties of the triangle related to standard orthogonality by replacing it with the aforementioned *"orthogonality w.r.t. the circumconic"* κ. In the second section we review the related generalization of *"orthocenter"* and its consequences. The results of this section should be well known, but, besides the allusion to some of them in [1], in the context of a generalization of the Wallace-Simson lines (WS-lines), I was not able to find a reference for their systematic exposition. In the third section we review the generalized notions of WS-lines

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and of Steiner lines, related to the orthogonality generalization of the previous section. Here we concentrate mainly on properties used in the following sections.

In the fourth, fifth and sixth sections, which represent the main parts of the article, we study the generation of the pencil $\mathcal{P}(ABC, H)$ of circumconics of $\triangle ABC$ passing through the *"generalized orthocenter of the triangle"* H, our prototype being the *"Poncelet pencil"*, of which this article proposes a generalization and an alternative way to generate the hyperbolas. The term, coined by Alperin ([2]), describes the pencil of all rectangular hyperbolas ([3, p.150]) circumscribing the $\triangle ABC$ and passing through its orthocenter. Our generalization method, bearing similarities to the one we used in the case of the *"Jerabek hyperbola"* in [4] and more recently in [5], characterizes the pencils $\mathcal{P}(ABC, H)$ whose all members are hyperbolas. These are precisely the pencils for which point H is inside the regions (I) to (IV) shown in figure 8, the standard Poncelet pencil obtained as the special case for which the circumscribing $\triangle ABC$ conic κ is its circumcircle.

2. The Orthocenter

Fixing the triangle ABC and an arbitrary central circumconic $\kappa(O)$ of it, the conjugate directions to the directions of its sides are determined by the lines $\{OA', OB', OC'\}$ joining the center of the conic with the middles of corresponding sides (see Figure 2). Using barycentric

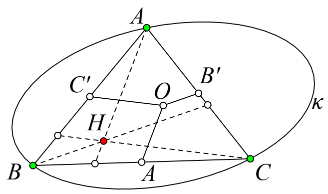

FIGURE 2. The generalized orthocenter H w.r.t. conic κ

coordinates or *"barycentrics"* w.r.t. $\triangle ABC$ ([6], [7]), we prove the following generalization of the notion of orthocenter.

Theorem 1. *The generalized orthogonals to the sides of the triangle from the opposite vertices concur at a point* H.

Proof. In fact, the conic κ is represented in barycentrics by an equation of the form:

$$(u, v, w)\begin{pmatrix} 0 & r & q \\ r & 0 & p \\ p & q & 0 \end{pmatrix}\begin{pmatrix} u \\ v \\ w \end{pmatrix} = 0 \quad \Leftrightarrow \quad pvw + qwu + ruv = 0. \tag{2.1}$$

The conjugate diameter to the direction of a line $\alpha : ku + lv + mw = 0$ is given by the polar of the point at infinity of this line ([8, II, p.65]). The point at infinity of the given line is

$$(l - m\ ,\ m - k\ ,\ k - l)$$

and its polar, representing the generalized orthogonal to α, is the line:

$$\alpha' : (r(m-k) + q(k-l)u \ + \ (p(k-l) + r(l-m))v \ + \ (q(l-m) + p(m-k))w \ = \ 0.$$

Applying this procedure to the lines $\{BC, CA, AB\}$ we obtain the coefficients of the equations of the lines $\{\alpha', \beta', \gamma'\}$ representing the conjugate diameters (generalized orthogonals) to the directions of the sides of the triangle:

$$\begin{aligned} BC &: (k=1, l=0, m=0), \quad & \alpha' &: (\ q-r\ \ p\ ,\ -p\ ), \\ CA &: (k=0, l=1, m=0), \quad & \beta' &: (\ -q\ ,\ r-p\ ,\ q\ ), \\ AB &: (k=0, l=0, m=1), \quad & \gamma' &: (\ r\ ,\ -r\ ,\ p-q\ ). \end{aligned}$$

The points at infinity A'', B'', C'' of these lines, representing the directions of the *"generalized orthogonals"* to the sides of the triangle, are found by taking the respective vector products with $(1,1,1)$, which represents the coefficients of the line at infinity:

$$A'' = (\ 2p\ ,\ -s_r\ ,\ -s_q\ )\ ,\ \ B'' = (\ -s_r\ ,\ 2q\ ,\ -s_p\ )\ ,\ \ C'' = (\ -s_q\ ,\ -s_p\ ,\ 2r\ )\ , \tag{2.2}$$

$$\text{where we have set:}\quad s_p = q+r-p\ \ ,\ \ s_q = r+p-q\ \ ,\ \ s_r = p+q-r\ . \tag{2.3}$$

The generalized perpendiculars from the vertices of the triangle are found again by the respective vector products of these vectors with $\{A(1,0,0), B(0,1,0), C(0,0,1)\}$ leading to the lines with respective coefficients:

$$A''A = (0, -s_q, s_r),\ \ B''B = (s_p, 0, -s_r),\ \ C''C = (-s_p, s_q, 0)\ .$$

It is then readily verified that these lines pass through the point

$$H(\ s_q s_r\ ,\ s_r s_p\ ,\ s_p s_q\ )\ , \tag{2.4}$$

which we call *"generalized orthocenter"* w.r.t. the circumconic κ, since it reduces to the traditional orthocenter in the case κ coincides with the circumcircle ([6, p.33]). ☐

The following lemmata show that the generalized orthocenter has properties analogous to those of the standard orthocenter, the analogy reaching to the existence of a generalized *"Euler line"* a generalization of the *"Euler circle"* and the existence of a generalized *"orthocentric quadrilateral"* ([9, p.109]).

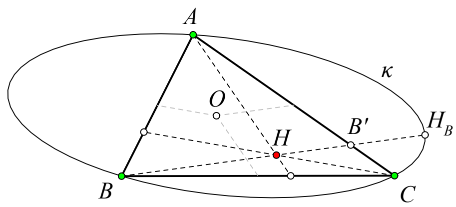

FIGURE 3. The symmetrics $\{H_A, H_B, H_C\}$ of H w.r.t. the traces of H are on κ

Lemma 2. *If $\{A', B', C'\}$ are the traces of H on the sides of the triangle, then the symmetrics $\{H_A, H_B, H_C\}$ of H correspondingly w.r.t. $\{A', B', C'\}$ lie on the conic κ (see Figure 3).*

Proof. In fact, the traces of H are found to be

$$A'(0, s_r, s_q)\ ,\ B'(s_r, 0, s_p)\ ,\ C'(s_q, s_p, 0)\ . \tag{2.5}$$

Passing to absolute barycentrics and applying the formula for the point symmetry, which in absolute barycentrics with center of symmetry S is ([10]) $\{Y = -X + 2S\}$, we find the barycentrics of $\{H_A, H_B, H_C\}$ and verify that they satisfy equation (2.1). This involves standard calculations, which, typically for H_A, lead to a non zero multiple of:

$$H_A = (\, p s_q s_r \,,\; s_r((r-q)^2 - p(r+q)) \,,\; s_q((r-p)^2 - p(r+q)) \,)\,,$$

seen easily to verify equation (2.1). □

Lemma 3. *There is a conic κ' homothetic to κ w.r.t. H with ratio $1/2$, which passes through the middles of the sides $\{M_A, M_B, M_C\}$, through the traces $\{A', B', C'\}$ of H and through the middles $\{A_H, B_H, C_H\}$ of the segments joininig H to the vertices (see Figure 4).*

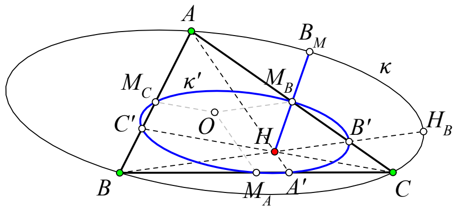

FIGURE 4. The *"nine point conic"* κ of the quadrangle $ABCH$

Proof. Since the homothety $H(1/2)$ maps $\{A, \ldots\}$ to $\{A_H, \ldots\}$ and $\{H_A, \ldots\}$ to $\{A', \ldots\}$, it suffices to show that the symmetrics $\{A_M, B_M, C_M\}$ of H w.r.t. $\{M_A, M_B, M_C\}$ are on the conic κ. This leads to a calculation similar to that of the preceding lemma. Typically, it is easily seen that the barycentrics of B_M are non zero multiples of:

$$B_M = (\, 2ps_p,\; -s_p s_r,\; 2rs_r \,)\,,$$

which satisfies equation (2.1) of the conic κ. We call κ' the *"generalized Euler conic"* of $\triangle ABC$ w.r.t. to the circumconic κ. □

The conic κ' coincides with the well known *"nine point conic"* of the quadrangle defined by the four points $\{A, B, C, H\}$ ([11, p.171], [12, p.387]), its center coinciding with the centroid of the quadrangle ([13, p.294]). It is also well known ([13, p.292]) and we'll prove it also below (proposition 16) that κ' carries the centers of the conics which pass through these four points.

Next theorem shows that an arbitrary point H of the plane, not contained in any sideline of the triangle ABC, can play the role of the generalized orthocenter w.r.t. an appropriate conic κ circumscribing $\triangle ABC$. We notice here, that the determination of the conic κ from H has been handled in [1] and [14] in connection with a generalization of the concept of WS-lines.

Theorem 4. *Conversely to theorem 1, an arbitrary point H of the plane, not contained in a sideline of the triangle ABC and not contained in the line at infinity of the plane, is the generalized orthocenter of the triangle w.r.t. an appropriate central conic κ uniquely determined by H.*

Proof. If (u, v, w) are the absolute barycentrics of H, then the points at infinity of the lines $\{HA, HB, HC\}$ are $\{A-H, B-H, C-H\}$. The coefficients of the lines parallel to HA, HB, HC through the middles $\{A', B', C'\}$ of the sides are computed by the corresponding vector products of coordinate vectors in absolute barycentrics:

$$\begin{aligned}(B+C)\times(A-H) &= (v-w\,,\ v+w\,,\ -v-w)\,,\\ (C+A)\times(B-H) &= (-u-w\,,\ w-u\,,\ u+w)\,,\\ (A+B)\times(C-H) &= (u+v\,,\ -u-v\,,\ u-v)\,.\end{aligned}$$

It is readily seen that these three lines intersect at the point

$$O(v+w,\ w+u,\ u+v) \tag{2.6}$$

which is not contained in the line at infinity and defines uniquely the desired circumconic having this point as its center. □

The generalized *"Euler line"* of ABC w.r.t. κ is defined to be the line OH.

Lemma 5. *The generalized Euler line of $\triangle ABC$ w.r.t. κ passes through the centroid G of $\triangle ABC$ and defines the oriented euclidean segments with ratio $HG/GO=2$.*

Proof. In order to prove the collinearity claim we evaluate the determinant of the barycentrics of points $\{G,H,O\}$ using equations (2.4) and (2.6) and find that it vanishes. For the ratio we notice that in absolute barycentrics $G=(2/3)O+(1/3)H$ from which follows immediately ([10]) the claim about the ratio. □

Lemma 6. *The three triangles $\{HBC,HCA,HAB\}$ defined by the generalized orthocenter H of triangle ABC w.r.t. the conic κ, are respectively inscribed in three conics $\{\kappa_A,\kappa_B,\kappa_C\}$ which have corresponding generalized orthocenters the vertices $\{A,B,C\}$ of the triangle. The three conics are translates of κ by the euclidean vectors $\{2OM_A,2OM_B,2OM_C\}$. In addition the corresponding generalized Euler conics of these triangles w.r.t. these circumconics coincide with κ'. The triangles $\{ABC,\ O_AO_BO_C\}$ are symmetric w.r.t. the middle N of the segment HO, which is also the center of conic κ' (see Figure 5).*

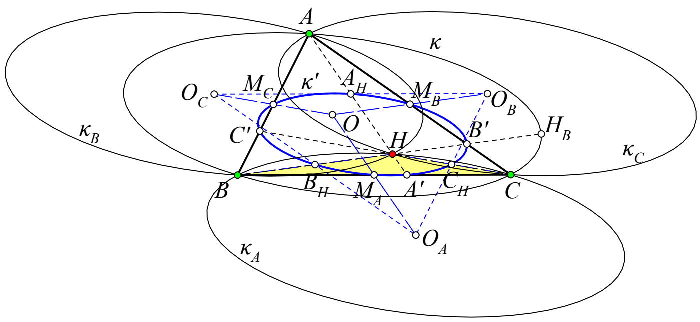

FIGURE 5. The generalized orthocentric quadrilateral $ABCH$ w.r.t. κ

Proof. The properties result easily from the preceding lemmata. For example, the conic κ_A circumscribing $\triangle ABC$ with center the translated O_A of O by $2OM_A$ defines for $\triangle HBC$ the corresponding generalized orthocenter A, since at this point intersect the parallels to lines $\{O_AM_A,O_AB_H,O_AC_H\}$ from the corresponding vertices $\{H,C,B\}$. The proofs of the other claims are equally easy and are left as exercises for the reader. The point N could be called the *"generalized Euler center"* or *"generalized nine point center"* of $\triangle ABC$ w.r.t. κ. □

The quadrilateral $ABCH$ is called the *"generalized orthocentric quadrilateral"* w.r.t. κ. The preceding lemma shows that it has also properties completely analogous to those of the standard orthocentric quadrilateral ([9, p.109]).

The description of the conic κ in barycentrics results from the expression $O(x,y,z)$ of its center in barycentrics and the fact that the center is the pole of the line at infinity, leading to the equation for the coefficients $\{p,q,r\}$ of the matrix of the conic:

$$\begin{pmatrix} 0 & r & q \\ r & 0 & p \\ q & p & 0 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \quad \Rightarrow \quad \begin{pmatrix} p \\ q \\ r \end{pmatrix} = \begin{pmatrix} x(y+z-x) \\ y(z+x-y) \\ z(x+y-z) \end{pmatrix} = \begin{pmatrix} u(v+w) \\ v(w+u) \\ w(u+v) \end{pmatrix},$$

where (u,v,w) are the coordinates of the orthocenter H. Using the notions of *"perspector"* ([6])) and *"barycentric product"* ([7], [15]), we have proved the following theorem.

Theorem 7. *The center O and the perspector $P(p,q,r)$ of the triangle conic κ with given generalized orthocenter $H(u,v,w)$ are respectively $O(v+w,w+u,u+v)$ and $P = H \cdot O$.*

Remark. From equations (2.4) and (2.6) we see that the coordinates of the center O of the conic κ, expressed in terms of the coordinates of the perspector $P(p,q,r)$ of the conic, are

$$O(ps_p, qs_q, rs_r). \tag{2.7}$$

The condition for the conic κ to be *"central"* , assumed from the beginning of this study, is equivalent to the condition for point O to *"not lie on the line at infinity"*. Since the line at infinity in barycentrics is given by $u+v+w=0$, later condition is equivalent to

$$ps_p + qs_q + rs_r = 0 \quad \Leftrightarrow \quad g(p,q,r) = p2 + q2 + r2 - 2(pq+qr+rp) = 0. \tag{2.8}$$

The points $P(p,q,r)$ satisfying the last equation are known to belong to the *"inner Steiner ellipse"* κ_S of the triangle of reference ABC. This is ([6, p.128]) the ellipse, seen in Figure 6, with center at the centroid G of the triangle touching the sides at their middles. The figure

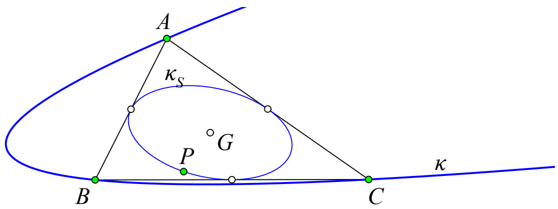

FIGURE 6. The inner Steiner ellipse κ_S of $\triangle ABC$

shows also a typical case of a parabola κ with corresponding perspector $P \in \kappa_S$. It is also known ([6, p.128]), that for perspectors P lying in the inner region of κ_S, where $g(p,q,r) < 0$, the corresponding conic κ is an ellipse, whereas for perspectors P lying outside κ_S, where $g(p,q,r) > 0$, the conic is a hyperbola.

The relation $HG = 2GO$ between the triangle centers $\{O,H\}$ is often expressed by saying that H is the *"anticomplement"* of O, latter being the *"complement"* of H ([16])). The discussion so far shows that the conic κ determines the points $\{H,O,P\}$ and also either of these points determines the other two and the conic κ. Figure 7 shows the geometric locus ν_ε of the perspectors $\{P\}$ of the conics $\{\kappa\}$ for $\{H,O\}$ varying on a line ε through the centroid G of $\triangle ABC$.

Next lemmata lead to the determination of the positions of H for which the corresponding central conic $\kappa(O)$ is an ellipse.

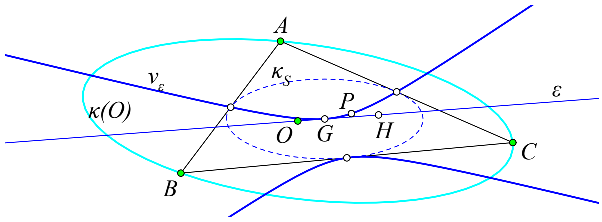

FIGURE 7. The conic ν_ε locus of perspectors $P = H \cdot O$ for $\{O, H \in \varepsilon\}$

Lemma 8. *For points $\{H, O\}$ satisfying $HG = 2GO$ and varying on a line ε through the centroid G of the triangle ABC, the corresponding locus of points $P = H \cdot O$ is a hyperbola ν_ε tangent to ε at G and passing through the middles of the sides of the triangle of reference ABC.*

Proof. Since the line $\varepsilon : au + bv + cw = 0$ passes through G, its coefficients satisfy the equation $a + b + c = 0$. Assuming $E(k, l, m)$ is the point at infinity of this line, point H can be expressed as combination $H = E + t \cdot G$. Also applying the relation between absolute barycentrics ([10]) of collinear points

$$\frac{XA}{XB} = r \quad \Leftrightarrow \quad X = \frac{1}{1-r}(A - rB),$$

in the case $OH/OG = 3$, we find, the hat denoting the corresponding coordinate vectors in absolute barycentrics and "$\cong$" denoting equality up to non-zero multiplicative constant,

$$\hat{O} = \frac{1}{1-3}(\hat{H} - 3\hat{G}) = \frac{1}{6t}(2t - k, 2t - l, 2t - m) \quad \Rightarrow$$
$$P = O \cdot H \cong ((2t - k)(t + k), (2t - l)(t + l), (2t - m)(t + m)). \tag{2.9}$$

This is the parametric representation of a conic ν_ε easily seen to satisfy the conditions of the lemma. The equation satisfied by this conic is also easily seen to be

$$\nu_\varepsilon : a(u^2 + vw) + b(v^2 + wu) + c(w^2 + uv) = 0.$$

□

Lemma 9. *Let $\hat{H}(h_1, h_2, h_3)$ be the generalized orthocenter expressed in absolute barycentrics. Then, the positions of H on the plane for which the corresponding conic κ is an ellipse are characterized by the inequality $h_1h_2h_3 > 0$.*

Proof. Figure 8 shows the four regions for which the condition $h_1h_2h_3 > 0$ is satisfied. The proof amounts to show that for H in these regions, and only there, the corresponding perspector $P(p, q, r)$ of κ is inside the inner Steiner ellipse κ_S. By equation (2.8) the inner points of κ_S are characterized by the inequality $g(p, q, r) < 0$. Hence, we have to prove the relation

$$h_1h_2h_3 > 0 \quad \Leftrightarrow \quad g(p, q, r) < 0.$$

Using the representation of H introduced in the preceding lemma $H = E + tG$ with $E(k, l, m)$ the point at infinity of the line ε through G, we have:

$$h_1h_2h_3 = \frac{1}{(3t)^3}(k + t)(l + t)(m + t).$$

On the other side, evaluating $g(p,q,r)$ for the arguments $\{p(t),q(t),r(t)\}$ given by equation (2.9), we find a polynomial $\gamma(t) = g(p(t),q(t),r(t))$ of degree four. The crucial fact, due to the condition $k+l+m=0$, satisfied by the coordinates of the point at infinity E, is that this

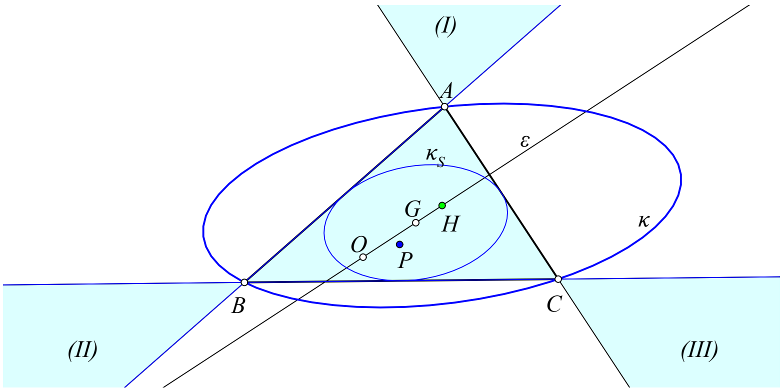

FIGURE 8. Positions of H for which the corresponding conic κ is an ellipse

polynomial is a multiple of $\eta(t) = (k+t)(l+t)(m+t)$. In fact, doing the division of the two polynomials $\gamma(t)/\eta(t)$ and some simplifications, which are a bit tedious and I omit, we find the simple relation:

$$\gamma(t) \;=\; -12\cdot\eta(t) \;=\; -12\cdot t\cdot(3t)^3(h_1h_2h_3)\,.$$

This proves the lemma for points along the line ε but also for all points of the plane, since latter is totally swept by the different positions of the line ε through G. □

3. STEINER LINES

It is well known ([17, p.8], [9, p.140], [18, p.137]) that the projections $\{P_a,P_b,P_c\}$ of a point $P(u,v,w)$ of the circumcircle κ of the triangle ABC on its side-lines (see Figure 9) lie on a line σ_P traditionally called the *"Wallace-Simson line of P"* (WS-line). It is possible ([1], [14]) to

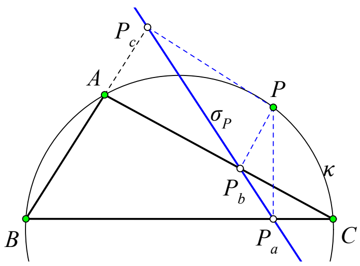

FIGURE 9. WS-line σ_P of $\triangle ABC$ w.r.t. P

generalize the definition of this line, by replacing the circumcircle with an arbitrary central circumconic $\kappa(O)$ of the triangle of reference ABC and the projection directions with the *"generalized orthogonals"* introduced in section two (see Figure 10). The projections $\{P_A,P_B,P_C\}$ on the sides are found as intersections:

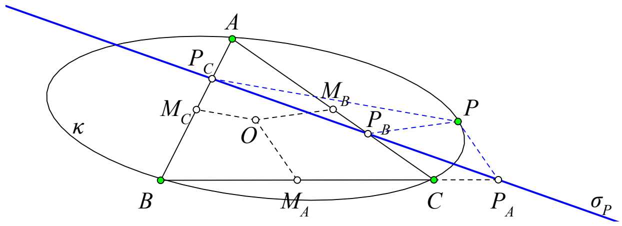

FIGURE 10. Wallace line σ_P of $\triangle ABC$ w.r.t. P and the conic κ

$$\begin{aligned} P_A &= PA'' \cap BC = uA'' - 2pP\,, \\ P_B &= PB'' \cap CA = vB'' - 2qP\,, \\ P_C &= PC'' \cap AB = wC'' - 2rP\,, \end{aligned}$$

where $\{A'', B'', C''\}$ are the points at infinity respectively of the lines $\{OM_A, OM_B, OM_C\}$ calculated in section two. The collinearity of $\{P_A, P_B, P_C\}$ can be expressed by the vanishing of the determinant of their barycentrics, latter found to be:

$$det(P_A, P_B, P_C) = 2(p^2 + q^2 + r^2 - 2(pq + qr + rp))(u + v + w)(pvw + qwu + ruv),$$

in which the last factor vanishes, since this is the equation expressing $P \in \kappa$. The line σ_P is called the *"generalized WS-line"* of P w.r.t. κ.

Of interest for our study is the parallel line τ_P to σ_P passing through the reflected point of P in σ_P. This line, which we call *"generalized Steiner line"* of P w.r.t. κ, has the following property analogous to that of the traditional Steiner line ([19, p.127]) of the triangle (see Figure 11).

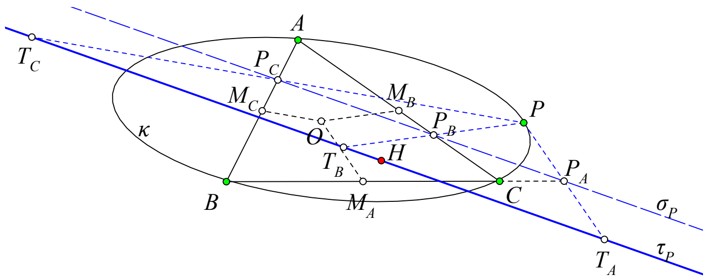

FIGURE 11. The generalized Steiner line τ_P of P w.r.t. κ

Theorem 10. *The generalized Steiner lines $\{\tau_P : P \in \kappa\}$ pass all through the generalized orthocenter H of $\triangle ABC$ w.r.t. κ.*

Proof. This involves a calculation of the symmetric points $\{T_A, T_B, T_C\}$ of $P(u, v, w)$ correspondingly w.r.t. $\{P_A, P_B, P_C\}$ leading to:

$$T_A = pP - uA''\,, \quad T_B = qP - vB''\,, \quad T_C = rP - wC''\,.$$

Using the barycentrics vectors of these points and of $H(s_q s_r, s_r s_p, s_p s_q)$ we compute the determinant of the triple:

$$det(T_A, T_B, H) = s_r[2(pq + qr + rp) - (p^2 + q^2 + r^2)](pvw + quw + ruv) = 0\,,$$

the last factor vanishing since $P(u,v,w) \in \kappa$. This shows that $\{T_A, T_B, H\}$ are collinear and completes the proof. $\square$

The coefficients of the Steiner line τ_P in barycentrics and in symmetric form for $P(u,v,w) \in \kappa$ can be found by computing the sum of the barycentrics vectors:

$$\tau_P \ : \ T_A \times T_B + T_B \times T_C + T_C \times T_A \ = \\ (s_p((r-p)v + (q-p)w)\ ,\ s_q((p-q)w + (r-q)u)\ ,\ s_r((q_r)u + (p_r)v)). \tag{3.1}$$

4. The pencil $\mathcal{P}(ABC, H)$

Given the central circumconic $\kappa(O)$ of the triangle ABC, we denote by $\mathcal{P}(ABC, H)$ the pencil of all conics passing through the vertices of the triangle and the generalized orthocenter H of $\triangle ABC$ w.r.t. κ. A member conic μ_L of the pencil intersects κ at a fourth point L, which can be used to parameterize the pencil $\mathcal{P}(ABC, H)$, since the five points $\{A, B, C, H, L\}$ uniquely define a conic. A result at the foundations of this article is formulated through the following theorem (see Figure 12).

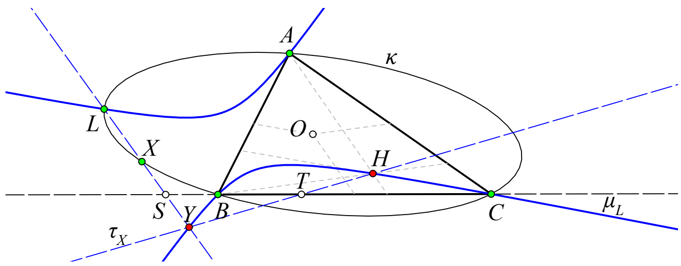

FIGURE 12. Conic μ_L of the pencil $\mathcal{P}(ABC, H)$

Theorem 11. *The member-conic $\mu_L \in \mathcal{P}(ABC, H)$ passing through $L \in \kappa$ is generated by the intersections $Y = LX \cap \tau_X$, where τ_X is the generalized Steiner line of $X \in \kappa$ w.r.t. $\triangle ABC$ and the reference conic κ.*

Proof. The proof can be reduced to the *"Chasles-Steiner principle"* of generation of conics ([20, p.5], [3, p.72], [21, p.259]), applied to the pencils $\{L^*, H^*\}$ of lines passing respectively through $\{L, H\}$. According to this principle, a homographic transformation $f : L^* \to H^*$ defines a conic through the intersections of homologous rays $\{Y = \ell \cap f(\ell), \ell \in L^*\}$. In addition, the conic passes through the centers $\{L, H\}$ of the pencils. In our case we define f to be the map which associates to the line LX through L for $X \in \kappa$, the corresponding Steiner line τ_X through H. To show that this map is a homography, we parameterize the pencils through the intersection of their member-lines with the line BC. Thus, we consider the lines through L intersecting BC at points $S = B + sC$ say, and the lines through H intersecting BC at $T = B + tC$. It suffices to show that the map f is represented through these parameters in the form

$$t \ = \ \frac{as+b}{cs+d} \quad \text{with} \quad ad - bc \neq 0\,.$$

Fixing $L(l,m,n) \in \kappa$ and representing the second intersection point X of LS with κ in the form $X = L + \sigma S = (l, m+\sigma, n+\sigma s)$ and requiring from X to satisfy equation (2.1), we find

$$\sigma = -\frac{(lq+mp)s+lr+np}{ps}.$$

Denoting by $\{f_1(u,v,w), f_2(u,v,w), f_3(u,v,w)\}$ the coefficients of the Steiner line of equation (3.1), we find its intersection with line $BC(u=0) : T = (0, f_3, -f_2)$. The two functions, evaluated for $X = L + \sigma S = (l, m+\sigma, n+\sigma s)$, lead to the expressions:

$$f_2 = 2s_q \frac{(q-p)(lq+mp)s+lq(r-p)}{p}, \quad f_3 = 2s_r \frac{lr(q-p)s+(r-p)(lr+np)}{ps} \quad \Rightarrow$$

$$t = -\frac{f_2}{f_3} = -\frac{ss_q}{s_r} \cdot \frac{(q-p)(lq+mp)s+lq(r-p)}{lr(q-p)s+(r-p)(lr+np)}.$$

Dividing however the polynomials in s appearing in the last quotient we find:

$$\frac{(q-p)(lq+mp)s+lq(r-p)}{lr(q-p)s+(r-p)(lr+np)} = \frac{lq+mp}{lr} - \frac{p(r-p)(pmn+qnl+rlm)}{lr(lr(q-p)s+(r-p)(lr+np))},$$

the factor $(pmn+qnl+rlm)$ in last quotient vanishing, since $L(l,m,n)$ satisfies equation (2.1). Thus, the relation between $\{t,s\}$ is indeed homographic of the simplest form:

$$t = -\frac{s_q}{s_r} \cdot \frac{lq+mp}{lr} s = \frac{s_q m}{s_r n} s. \tag{4.1}$$

The last expression resulting by multiplying the preceding one with $1 = n/n$ and taking into account equation (2.1). This completes the proof of the theorem, since it is trivially seen that the conic generated by $\{Y = LX \cap \tau_X : X \in \kappa\}$ passes through $\{A,B,C\}$ and by the Chasles-Steiner principle passes also through $\{H,L\}$. □

Theorem 12. *The map $f_L : \kappa \to \mu_L$, with $f(X) = Y$, is a projectivity between conics, mapping κ onto μ_L (see Figure 12).*

Proof. The proof results from general properties of projectivities, according to which ([22, I, p.213]), given three distinct points $\{A,B,C\}$ on the conic κ and three other distinct points $\{A',B',C'\}$ on the conic κ', there is a unique projectivity $f : \kappa \to \kappa'$ between the conics with the property $\{f(A) = A', f(B) = B', f(C) = C'\}$. Considering the vertices of the triangle of reference ABC, as points of κ and also as points of μ_L, we deduce the existence of a projectivity f mapping κ to μ_L and also fixing these three points. We show that this projectivity coincides with f_L. This follows directly from the preservation of the cross ratios by projectivities. In fact, referring to figure 12, the cross ratios $(AB;CX)$ on κ and $(AB;CY) = (f(A)f(B); f(C)f(X))$ on μ_L are per definition both equal to the cross ratio of the pencil of lines through $\{L : LA, LB, LC, LX\}$, which is the same with the cross ratio $(A'B;CS)$ of their intersections with line BC. Thus, $f(X) = fL(X) = Y$ for every $X \in \kappa$, which proves the claim. □

Remark. It is easy to see that every projectivity f between two conics $f : \kappa \to \mu_L$ of the plane extends to a projectivity of the whole plane onto itself ([23, II, p.179]). It suffices to consider four points $\{A,B,C,D \in \kappa\}$ and their images $\{A',B',C',D' \in \mu_L\}$ under f. By the general properties of projectivities there is a projectivity f' of the plane mapping the first quadruple to the second and coinciding with f on κ. In the following we'll work with this extension of f_L to a projectivity of the whole plane, denoting the extension by the same symbol f_L.

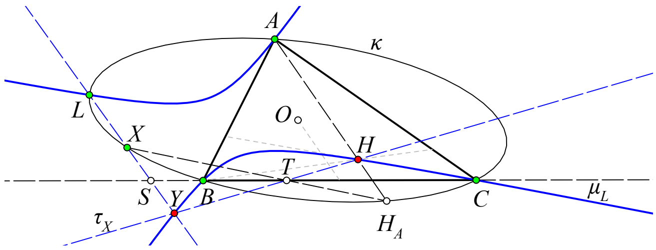

FIGURE 13. Lines $\{XT\}$ pass through $H_A \in \kappa$

Corollary 13. *The cross ratio of the four points* $(BC;ST) = \frac{SB}{SC} : \frac{TB}{TC}$ *is constant for* $X \in \kappa$ *and the lines* $\{XT\}$ *pass through a fixed point coinciding with the second intersection* H_A *of line* AH *with* κ *(see Figure 13).*

Proof. The first claim follows from equation (4.1) and the fact that the corresponding ratios $\{SB/SC = -s, TB/TC = -t\}$. The second claim follows from the constancy of the cross ratio of four points on a conic ([21, p.298]) measured by the cross ratio of the pencil centered at an arbitrary fifth point on the conic. Here the fifth point is X and the pencil is $X(BCST)$ $= X(BCLZ)$, where $Z \in \kappa$. The location of Z is seen to coincide with H_A by letting X take the position of the vertex A. □

Since by lemma 2 the points $\{H_A, H_B, H_C\}$ depend directly on $\triangle ABC$ and the location of H, we obtain, by means of the previous corollary, an alternative way to generate the conic κ.

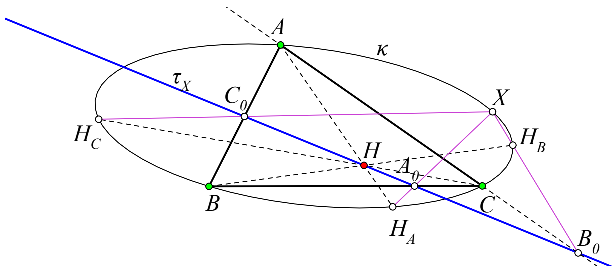

FIGURE 14. Generation of κ by revolving τ_X about H

Corollary 4.1. The lines $\{XH_A, XH_B, XH_C\}$ for $X \in \kappa$, intersect the respective sides of the triangle $\{BC, CA, AB\}$ at points $\{A_0, B_0, C_0\}$ lying on the generalized Steiner line τ_X. The conic κ is generated by the points $\{X\}$ defined as intersections of the lines $\{A_0H_A, B_0H_B, C_0H_C\}$, where $\{A_0, B_0, C_0\}$ are the intersections of the sides with a line τ_X revolving about the orthocenter H (see Figure 14).

5. PROPERTIES OF THE MEMBER CONICS

In this section, using mainly the theorems of the preceding section, we derive various properties of the member conics of the pencil $\mathcal{P}(ABC, H)$ formulated as propositions. We use also the terms *"WS-line"* and *"Steiner line"* in the generalized sense. Figure 15 shows three remarkable points $\{M, N, P\}$, next proposition listing some of their properties.

Proposition 14. *The second intersection N of line HL with κ maps via f_L to H and the Steiner line of N is the tangent t_H of μ_L at H. The second intersection M of κ with the tangent t_L to μ_L at L, maps via f_L to L and the Steiner line of M is line LH. The second intersection P of the tangent to κ at L is the image via f_L of L and the Steiner line of L is line PH.*

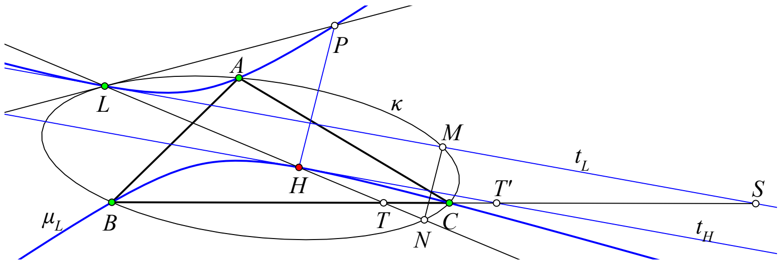

FIGURE 15. Three remarkable points $\{M, N, P\}$

Proof. All claims are trivial consequences of theorem 11. For example, as $X \in \kappa$ approaches M the corresponding line XL tends to the tangent t_L of μ_L at L and the corresponding Steiner line HY tends to coincide with line LH. The other claims are proved similarly. □

Proposition 15. *The tangent t_H to μ_L at H is parallel to its tangent t_L at $L(l, m, n) \in \kappa$.*

Proof. A way to see this is to show the equality of the ratios $TH/TL = TT'/TS$ (see Figure 15), where $\{T' = t_H \cap BC, S = t_L \cap BC\}$. The first ratio is easily computed by expressing T as a combination

$$T = l \cdot H - s_q s_r \cdot L = l\sigma_H \cdot \hat{H} - s_q s_r \sigma_L \cdot \hat{L}.$$

Here, for a point $X(u, v, w)$ expressed in barycentrics we use the notation $\sigma_X = u + v + w$ and $\hat{X} = X/\sigma_X$ for the corresponding absolute barycentrics vector. From a standard ratio computation in barycentrics ([10])) we deduce:

$$\frac{TH}{TL} = \frac{s_q s_r \sigma_L}{l\sigma_H} = \frac{s_q s_r (l + m + n)}{l(s_p s_q + s_q s_r + s_r s_p)}. \tag{5.1}$$

To compute the ratio TT'/TS we start with line LH, its intersection $T = LH \cap BC$, and also $S = B + sC$, the parameters $\{s, t\}$ being related by equation (4.1):

$$T = B + tC \quad \text{with} \quad t = -\frac{s_q(ns_r - ls_p)}{s_r(ls_p - ms_q)} \quad \Rightarrow \quad s = -\frac{n}{m} \cdot \frac{ns_r - ls_p}{ls_p - ms_q}. \tag{5.2}$$

By proposition 14, the relation of $T' = B + t'C$ to $T = B + tC$ is given by equation (4.1):

$$t' = -\frac{m}{n} \cdot \frac{s_q^2}{s_r^2} \cdot \frac{ns_r - ls_p}{ls_p - ms_q}. \tag{5.3}$$

Expressing T as a linear combination of $\{T', S\}$ we find, "$\cong$" meaning equality up to non zero multiplicative constant:

$$T \cong \mu \cdot T' + \nu \cdot S \quad \Leftrightarrow \quad B + tC \cong \mu(B + t'C) + \nu(B + sC) \quad \Leftrightarrow$$
$$(\mu, \nu) \cong \left(\frac{t - s}{t' - s}, \frac{t' - t}{t' - s}\right) \cong \left(\frac{ns_r}{m \cdot s_q + n \cdot s_r}, \frac{m \cdot s_q}{m \cdot s_q + n \cdot s_r}\right).$$

Passing to absolute barycentrics and writing $T = \mu\sigma_{T'}\hat{T}' + \nu\sigma_S\hat{S}$, we see that

$$\frac{TT'}{TS} = -\frac{\nu \cdot \sigma_S}{\mu \cdot \sigma_{T'}} = -\frac{\nu}{\mu} \cdot \frac{1+s}{1+t'} .$$

Finally computing the difference $\frac{TH}{TL} - \frac{TT'}{TS}$ and doing some simplifications, we find its expression as a product, one of the factors being $pmn + qnl + rlm = 0$, because of equation 2.1. This completes the proof of the proposition. □

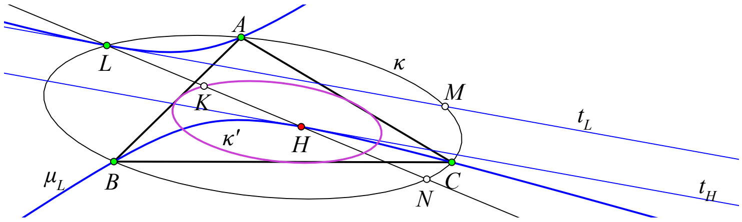

FIGURE 16. The center K of μ_L lying on the Euler conic κ'

Proposition 16. *Line LH is the conjugate diameter w.r.t. the conic μ_L of the common direction of the parallel tangents $\{t_H, t_L\}$ of μ_L. The center of μ_L is the middle K of the segment HL, and lies on the Euler conic κ' of $\triangle ABC$ w.r.t. κ (see Figure 16).*

Proof. The first claim is an immediate consequence of proposition 15. The second claim follows from the first and lemma 3. □

Proposition 17. *The line $\zeta = OQ$, where Q is the pole of line MN w.r.t. κ maps via f_L to the line at infinity (see Figure 17).*

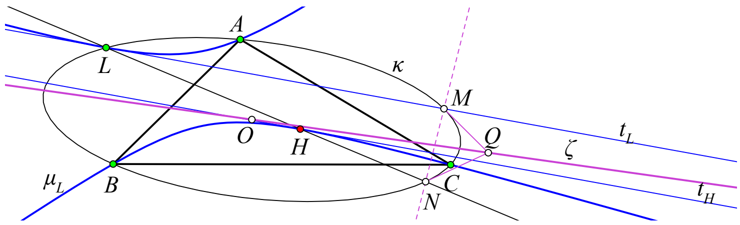

FIGURE 17. The line $\zeta = OQ$ mapping via f_L to the line at infinity

Proof. Since $\{f_L(N) = H, f_L(M) = L\}$ the corresponding tangents to κ at $\{N, M\}$ map via f_L to the corresponding tangents $\{t_H, t_L\}$ of μ_L at H and L. Since latter intersect at a point X at infinity, the intersection point Q of their pre-images, which is the pole Q of MN, maps to X at infinity. To complete the proof we show that also O maps via f_L to infinity. For this we compute the representation of f_L in barycentrics, considering it as the projectivity which fixes the vertices of $\triangle ABC$ and maps N to H. The barycentrics of N can be expressed as coordinates of the second intersection point of line HL with κ:

$$N \cong \begin{pmatrix} s_q s_r k_1 \\ s_r s_p k_2 \\ s_p s_q k_3 \end{pmatrix} \quad \text{with} \quad \begin{pmatrix} k_1 \\ k_2 \\ k_3 \end{pmatrix} = \begin{pmatrix} -ps_p & rs_r + ps_p & qs_q + ps_p \\ rs_r + qs_q & -qs_q & qs_q + ps_p \\ rs_r + qs_q & rs_r + ps_p & -rs_r \end{pmatrix} \begin{pmatrix} s_p l \\ s_q m \\ s_r n \end{pmatrix} . \quad (5.4)$$

Since f_L fixes the points $\{A, B, C\}$ its matrix M_f has a diagonal form and the diagonal elements are determined from the equality $f_L(N) = H$:

$$M_f \;\cong\; \begin{pmatrix} 1/k_1 & 0 & 0 \\ 0 & 1/k_2 & 0 \\ 0 & 0 & 1/k_3 \end{pmatrix}. \tag{5.5}$$

Using the coordinates of the center O expressed by equation (2.7) we can find the coordinates of $O' = f_L(O)$ by means of the matrix M_f. Taking then the resulting sum of coordinates $\sigma_{O'}$, we find that this is expressible through a product, one of its factors being $pmn + qnl + rlm = 0$, because of equation (2.1). This means that O' satisfies the equation $u + v + w = 0$ of the line at infinity and completes the proof of the proposition. □

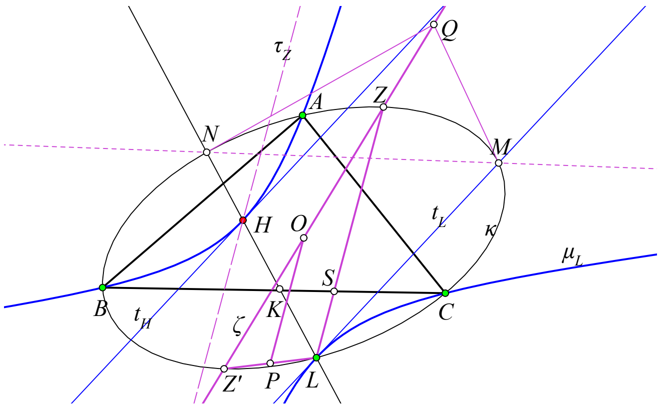

FIGURE 18. The asymptotes are the WS-lines of $\{Z, Z'\}$ and parallel to $\{LZ, LZ'\}$

Proposition 18. *If the line ζ intersects the conic κ at two points $\{Z, Z'\}$, the conic μ_L is a hyperbola and its asymptotes are parallel respectively to $\{LZ, LZ'\}$. The asymmptotes coincide with the generalized WS-lines of $\{Z, Z'\}$ w.r.t. κ, which are the parallels to $\{LZ, LZ'\}$ from K.*

Proof. This is an immediate consequence of proposition 17 (see Figure 18). For example, since the point Z maps via f_L to a point X at infinity, lines $\{LZ, \tau_Z\}$, intersecting at X, are parallel in the direction of LZ. Thus, X is a point at infinity of μ_L and its tangent there, which is an asymptote of the conic, passes through its center K coinciding with WS-line of Z which is parallel to the Steiner line τ_X. The same argument is valid, and proves, the analogous property for the point Z'. □

If P is the middle of LZ', then the segment OP is parallel to LZ and $\{LZ', OP\}$ define the directions of two conjugate diameters of κ, implying the property ([24, p.199]):

Proposition 19. *The asymptotes of μ_L, if any, are parallel to two conjugate diameters of κ. In this case, the diameter ZZ' of the conic κ uniquely determines the conic μ_L, which in barycentrics is described by the equation*

$$\mu_L \;:\; \left(\frac{p}{bc}\right) vw \;+\; \left(\frac{q}{ca}\right) wu \;+\; \left(\frac{r}{ab}\right) uv \;=\; 0\,,$$

where $au + bv + cw = 0$ is the equation of line ZZ'.

Proof. The first claim follows from the preceding remark. The second claim can be proved by solving the trivial diagonal linear equation $M_f \cdot E = Z\ (*)$, w.r.t. the coefficients of the diagonal matrix M_f of f_L, given by equation (16). In this $\{E(1,1,1), Z(a,b,c)\}$ and equation $(*)$ represents in barycentrics proposition 17 and leads to:

$$k_1 = 1/a\ ,\ k_1 = 1/b\ ,\ k_3 = 1/c\ .$$

Having f_L, the conic is obtained by theorem 12 as the image $\mu_L = f_L(\kappa)$. □

Proposition 20. *If the line ζ intersects the conic κ at points $\{Z, Z'\}$, the tangents $\{\alpha, \beta\}$ to κ at these points map via f_L to the asymptotes of μ_L and their point at infinity maps to the center K of μ_L. All parallels to $\{\alpha, \beta\}$ map to lines through K. In particular the line γ through Q parallel to MN maps to the middle-parallel γ' of the two tangents $\{t_H, t_L\}$ of μ_L (see Figure 19).*

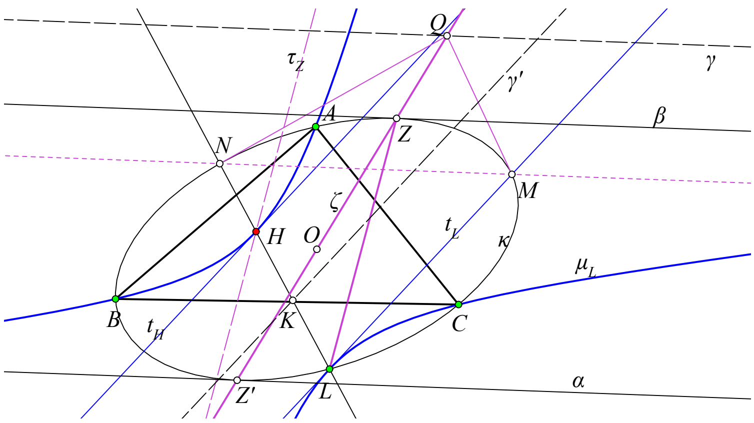

FIGURE 19. The tangents $\{\alpha, \beta\}$ to κ map via f_L to the asymptotes

Proof. The first claim is a trivial consequence of the fact that $\{f_L(Z), f_L(Z')\}$ are the points at infinity of μ_L, hence the tangents to κ at $\{Z, Z'\}$ map to the asymptotes, which are the tangents of the hyperbola at the points at infinity. If X is the point at infinity of line α, $f_L(X)$ is the intersection point of the asymptotes, which is K. The statement on the parallels to α follows at once. Line γ is the harmonic conjugate of ζ w.r.t. the lines $\{QN, QM\}$ mapping via f_L correspondingly to $\{t_H, t_L\}$. Since f_L preserves the cross ratio and $f_L(\zeta)$ is the line at infinity, its conjugate w.r.t. $\{t_H, t_L\}$ is the middle parallel of these two lines, as stated. □

The lines through a point $X \in \zeta$ map via f_L to parallels, since they all contain the same point at infinity $f_L(X)$. In particular the parallels to ζ, which pass through the point at infinity X_0 of ζ, map also to a special pencil of parallel lines through the point at infinity $X_1 = f_L(X)$. Next proposition reveals the direction determined by X_1.

Proposition 21. *Let M' be the second intersection of κ with the parallel to ζ from M. Then line MM' maps via f_L to LM' and all parallels to ζ map to parallels to LM'. In particular the line at infinity maps via f_L to the parallel η to LM' through the center K of μ_L (see Figure 20).*

Proof. The first claim is obvious, since $f_L(M) = L$ and the point $M'' = f_L(M')$ is on the line LM'. By the preceding remark all lines parallel to ζ map to parallels to LM'. In particular the line at infinity maps also via f_L to such a parallel η to LM'. Since the line at infinity contains the point at infinity of MN which maps via f_L to K, line η passes through K as claimed. Notice that the symmetry of κ about O implies that M' is the diametral of N. □

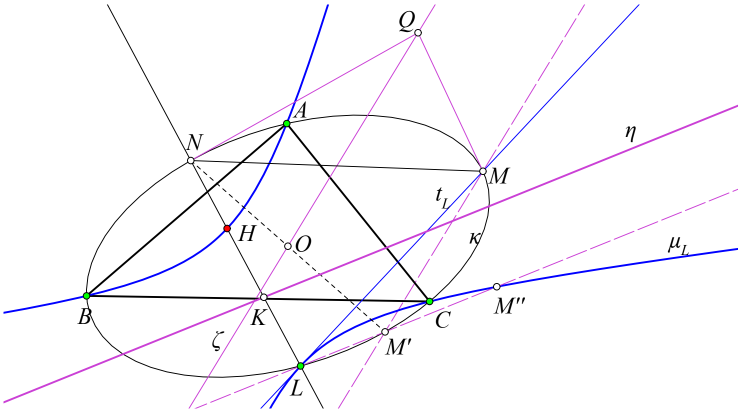

FIGURE 20. The image line η via f_L of the line at infinity

Proposition 22. *Let* $L' = \zeta \cap HL$, $P = f_L(L) \in \mu_L$ *and* L'' *be the second intersection of* ML' *with* κ *with* $P' = f_L(L'')$. *Then, the lines* $\{NM, PH, LL''\}$ *are parallel (see Figure 21).*

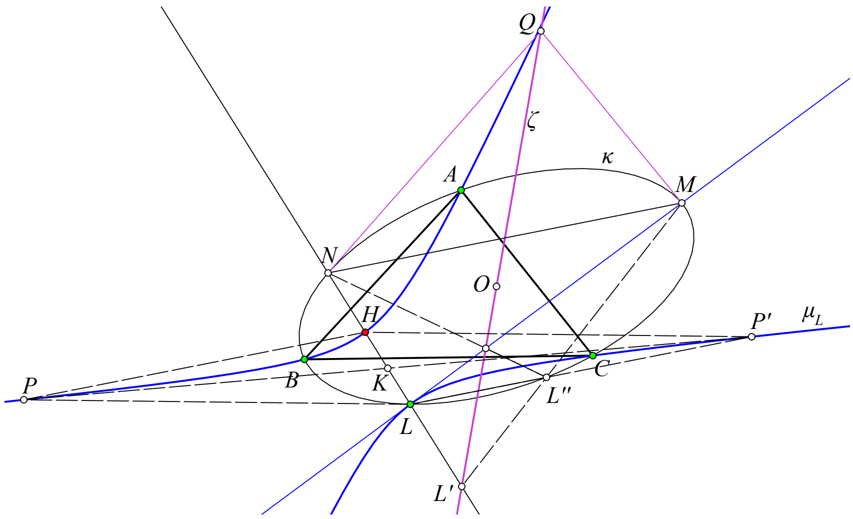

FIGURE 21. The three parallels $\{NM, PH, LP'\}$

Proof. As we noticed above, the lines through L' map via f_L to lines parallel to a fixed direction. Since line NL maps to PH the direction is that of PH. It follows that line ML' maps via f_L to a parallel LP' to PH, with $P' = f_L(L'')$. Since K is the center of μ_L, point P' is the symmetric of P w.r.t. K and $HPLP'$ is a parallelogram. Since the segments $\{\alpha = NL'', \beta = LM\}$ map via f_L to $\{\alpha' = HP', \beta' = LP\}$ the intersection of the corresponding lines $\alpha \cap \beta$ maps to the intersection of $\alpha' \cap \beta'$, which is a point at infinity. This means that the intersection point $\alpha \cap \beta$ is a point of ζ. Since ζ passes through the middle of NM, this implies that $\{LL'', NM\}$ are parallel and completes the proof. □

6. THE NATURE OF PENCILS $\mathcal{P}(ABC, H)$

By theorem 4, any quadruple of points in general position and none lying at infinity, can be considered as a set of (i) the vertices of a triangle ABC and (ii) the generalized orthocenter H w.r.t. to a conic κ circumscribing the triangle. Figure 22 shows a typical case of some members of such a pencil through the four points $\{A, B, C, H\}$. In general, as seen in the figure, the pencil comprises conics of all kinds, hyperbolas, ellipses and also two parabolas if H is outside of ABC ([25]). The fact that the displayed in the figure conic κ is a hyperbola, is characteristic for the existence of pencil members of different kinds.

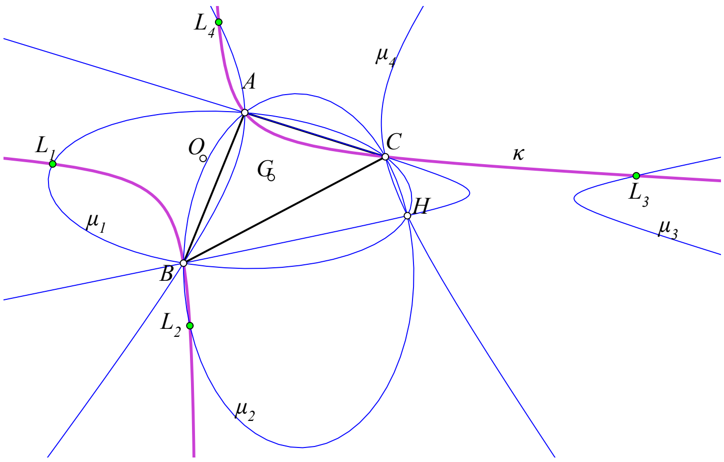

FIGURE 22. A generic pencil of conics through the points $\{A, B, C, H\}$

Theorem 23. *The member conics of the pencil $\mathcal{P}(ABC, H)$ are all hyperbolas, if and only if the associated to H central conic κ is an ellipse.*

In fact, if the conic κ defined by $\{ABC, H\}$ is an ellipse, then by propositions 17 and 18 the line ζ passing through the center O of the ellipse must intersect κ at two diametral points $\{Z, Z'\}$ whose images via f_L are two points at infinity of μ_L. This shows that for all $L \in \kappa$ the corresponding member conic μ_L is a hyperbola. Since for varying $L \in \kappa$ we obtain all member conics of the pencil, we see that $\mathcal{P}(ABC, H)$ consists entirely of hyperbolas. The positions of H for which this happens are in the regions (I) to (IV) of figure 8.

On the other side, if the conic κ is a hyperbola, then by lemma 9, point H is in the complement of the regions (I) to (IV) of figure 8 and there is a side of $\triangle ABC$, AC say, intersected by the segment BH at an inner point of both segments $\{BH, AC\}$ $(^{**})$. Figure 23 shows such a case and displays also the corresponding hyperbola, which, depending on the location of H, may have both branches carrying vertices of the triangle ABC or having all the vertices on one branch. To prove the theorem it suffices to show that in these cases there are always conics μ_L of the pencil

$\mathcal{P}(ABC, H)$, which depending on the location of $L \in \kappa$, are of different kinds. This is handled in the following lemma.

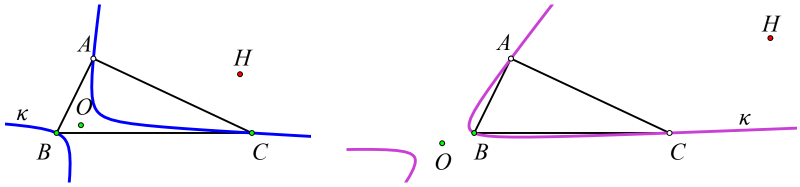

FIGURE 23. The two cases κ is a hyperbola

Lemma 24. *In the case the conic κ is a hyperbola, the points $L \in \kappa$ which are on the branch of κ carrying an odd number of vertices of the triangle of reference define member conics of the pencil $\mu_l \in \mathcal{P}(ABC, H)$ which are ellipses, whereas points L lying on the branch of κ carrying an even number of vertices of $\triangle ABC$ define member conics μ_L which are hyperbolas.*

Proof. The case concerning the members μ_L which are hyperbolas is easily seen, a typical case shown in figure 24. In this triangle ABC has the vertices $\{A, B\}$ on one branch of κ and the third vertex on the other branch. For L varying on the branch of κ carrying $\{A, B\}$ the

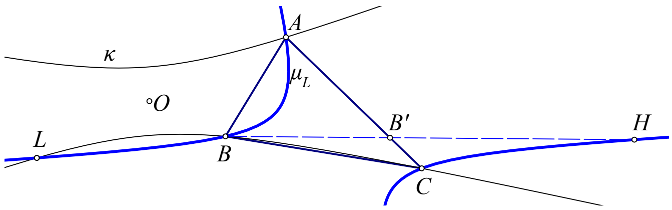

FIGURE 24. κ hyperbola, L on the branch carrying two vertices of ABC

four points $\{A, B, C, L\}$ can always be grouped so that three of them make a triangle and the fourth point is inside that triangle. Since this cannot happen for ellipses passing through such arrangements of four points, the conic μ_L must be a hyperbola. The same argument works also in the case in which all three vertices $\{A, B, C\}$ are on a branch of κ and L is on the other branch.

The case of members μ_L which are ellipses can be handled using proposition 15. According to this the tangents to μ_L at $\{H, L\}$ are parallel. Figure 25 shows again a typical case, in which $L \in \kappa$ is contained in the branch of the hyperbola which contains only one vertex of $\triangle ABC$. The conic μ_L is contained between the two parallel tangents at $\{H, L\}$. This is easily seen when L obtains the position of the vertex A, in which the conics $\{\kappa, \mu_A\}$ have a common tangent at A and AH becomes a diameter of μ_A. By the continuity, connectedness of the branch of κ containing L and the fact, that under our hypothesis $(^{**})$ made in the proof of theorem 23, $|LH|$ is positive for all L on that branch, the conic μ_L will be contained between the parallel tangents for all points L on that branch. But the central conics having points between two parallel tangents are ellipses, hence the proof of the claim in this case. The other case in which all three vertices $\{A, B, C\}$ are on a branch of κ and L is on the same branch can be handled similarly. □

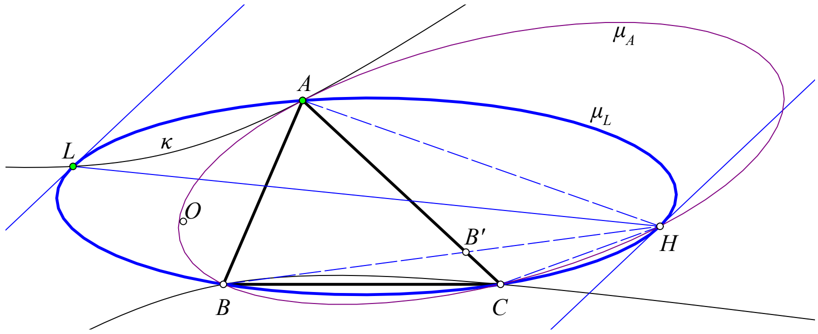

FIGURE 25. κ hyperbola, L on the branch carrying one vertex of ABC

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DEPARTMENT OF MATHEMATICS, UNIVERSITY OF CRETE, HERAKLION, 70013 GR
Email address: pamfilos@uoc.gr