Global Journal of Advanced Research
on Classical and Modern Geometries
ISSN: 2284-5569, Vol.10, (2021), Issue 2, pp.153-165

# GEODESIC VECTORS OF RANDERS METRIC ON GENERALIZED SYMMETRIC SPACES

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ABSTRACT. In this paper we study homogeneous geodesics in five-dimensional generalized symmetric spaces equipped with a left invariant Randers metric and we obtain homogeneous geodesics of this space in some special cases.

## 1. INTRODUCTION AND MOTIVATIONS

Let $(M, F)$ be a connected homogeneous Finsler space. If $G$ is any connected transitive group of isometries of $M$ and $H$ is the isotropy subgroup at a point $o \in M$, then $M$ is naturally identified with the coset space $G/H$ with $G-$invariant Finsler metric $F$. The Lie algebra $\mathfrak{g}$ of $G$ has a reductive decomposition $\mathfrak{g} = \mathfrak{m} + \mathfrak{h}$, where $\mathfrak{m} \subset \mathfrak{g}$ is a subspace of $\mathfrak{g}$ isomorphic to the tangent space $T_oM$ and $\mathfrak{h}$ is the Lie algebra of $H$. A geodesic $\gamma(t)$ through the $o \in M$ is called a homogeneous geodesic if it is an orbit of a one-parameter subgroup of $G$, that is

$$\gamma(t) = \exp(tZ)(o), \quad t \in R,$$

where $Z$ is a nonzero vector of $\mathfrak{g}$. In differential geometry homogeneous geodesics have been studied by many authors. In 1965 R. Hermann showed that homogeneous geodesics which are orbits of a given 1-parameter group of isometries $a(t)$ correspond to the critical points of the norm of Killing vector field $X$ which generates $a(t)$. B. Kostant [8] and E. B. Vinberg [14] and O. Kowalski and L. Vanhecke [11] found a simple condition that the orbit $\gamma(t) = a(t)\circ$ through the point $\circ = eK$ of an 1-parameter subgroup $a(t) = exptX \subset G$ of the isometry group $G$ of a homogeneous Riemannian manifold $M = G/K$, is a geodesic. In [12], the author studied homogenous geodesics in homogeneous Finsler space. The existence of at least one homogeneous geodesic in a general homogeneous Riemannian manifold was proved by O. Kowalski and J. Szenthe in [10]. Generalization of this existence result to Finsler space was proved in the series of papers [15, 16]. Several authors homogeneous geodesics on some types of homogeneous spaces [3, 6]. In this paper we study homogeneous geodesics (their generators geodesic vector) of left invariant Randers metrics on five dimensional generalized symmetric spaces.

2010 *Mathematics Subject Classification.* 22E46, 53C60, 53C30.
*Key words and phrases.* Randers metric, homogeneous geodesics, geodesic vector.

## 2. Preliminaries

In this section, we recall briefly some known facts about Finsler spaces. For details, see [1]. Let $M$ be a $n$-dimensional $C^\infty$ manifold and $TM = \bigcup_{x\in M} T_xM$ the tangent bundle. A Finsler metric on a manifold $M$ is a non-negative function $F : TM \to \mathbb{R}$ with the following properties:
(1) $F$ is smooth on the slit tangent bundle $TM_0 := TM \setminus \{0\}$,
(2) $F(x, \lambda y) = \lambda F(x, y)$ for any $x \in M$, $y \in T_xM$ and $\lambda > 0$,
(3) the $n \times n$ Hessian matrix $[g_{ij}] = [\frac{1}{2}\frac{\partial^2 F^2}{\partial y^i \partial y^j}]$ is positive definite at every point $(x, y) \in TM_0$.
The following bilinear symmetric form $g_y : T_xM \times T_xM \to \mathbb{R}$ is positive definite:

$$g_y(u,v) = \frac{1}{2}\frac{\partial^2}{\partial s \partial t}[F^2(x, y + su + tv)]|_{s=t=0}.$$

By the homogeneity of $F$, we have

$$g_y(u,v) = g_{ij}(x,y)u^i v^j \quad , \quad F = \sqrt{g_{ij}(x,y)u^i v^j}.$$

In 1941, G. Randers [13] studied a very interesting class of Finsler metrics. Let $M$ be an $n$-dimensional manifold. A Randers metric is a Finsler structure $F$ on $TM$ that has the form

$$F(x,y) := \alpha(x,y) + \beta(x,y),$$

where

$$\alpha(x,y) = \sqrt{\tilde{a}_{ij} y^i y^j}, \qquad \beta(x,y) = \tilde{b}_i(x) y^i.$$

The $\tilde{a}_{ij}$ are the components of a Riemannian metric and the $\tilde{b}_i$ are those of a 1-form. Due to the presence of the $\beta$ term, Randers metrics do not satisfy $F(x, -y) = F(x, y)$ when $\tilde{b}_i \neq 0$. In fact, the Finsler function of a Randers space is absolutely homogeneous if and only if it is Riemannian. Also, in order for $F$ to be positive if and only if

$$\| \tilde{b} \| := \sqrt{\tilde{b}_i \tilde{b}^i} < 1, \quad \text{where} \quad \tilde{b}^i := \tilde{a}^{ij}\tilde{b}_j.$$

The Riemannian metric $\tilde{a}$ induce a linear isomorphism between $T_x^*M$ and $T_xM$. Then the $1-$form $\beta$ corresponds to a vector field $X$ on $M$ such that

$$\tilde{a}(X_x, y) = \beta(x, y).$$

Therefore we can write the Randers metric $F = \alpha + \beta$ as following:

$$F(x,y) = \sqrt{\tilde{a}(y,y)} + \tilde{a}(X,y), \qquad \forall y \in T_xM$$

Let $\pi^*TM$ be the pull-back of the tangent bundle $TM$ by $\pi : TM_0 \to M$. Unlike the Levi-Civita connection in Riemannian geometry, there is no unique natural connection in the Finsler case. Among these connections on $\pi^*TM$, we choose the Chern connection whose coefficients are denoted by $\Gamma^i_{jk}$ (see [1], p. 38). This connection is almost $g$-compatible and has no torsion. Since, in general, the Chern connection coefficients $\Gamma^i_{jk}$in natural coordinates have a directional dependence, we must define a fixed reference vector. Let $\sigma(t)$ be a smooth regular curve in $M$, with velocity field $T$. Let $W(t) := W^i(t)\frac{\partial}{\partial x^i}$ be a vector field along $\sigma$. The expression

$$[\frac{dW^i}{dt} + W^j T^k (\Gamma^i_{jk})_(\sigma, T)]\frac{\partial}{\partial x^i}|_{\sigma(t)}$$

would have defined the covariant derivative $D_T W$ with reference vector $T$. A curve $\sigma(t)$, with velocity $T = \dot{\sigma}(t)$ is a Finslerian geodesic if

$$D_T[\frac{T}{F(T)}] = 0, \quad \text{with reference vector T},$$

that the constant speed geodesics are precisely the solution of

$$D_T T = 0, \quad \text{with reference vector T}.$$

Since $T = \frac{d\sigma^i}{dt}\frac{\partial}{\partial x^i}$, this differential equations that describe constant speed geodesics are:

$$\frac{d^2\sigma^i}{dt^2} + \frac{d\sigma^j}{dt}\frac{d\sigma^k}{dt}(\Gamma^i_{jk})(\sigma, T) = 0.$$

## 3. Homogeneous geodesics and generalized symmetric spaces

Let $(M = G/H, F)$ be a homogeneous Finsler manifold with a fixed origin $o$, $\mathfrak{g}$ and $\mathfrak{h}$ the Lie algebra of $G$ and $H$, respectively, and $\mathfrak{g} = \mathfrak{m} + \mathfrak{h}$ a reductive decomposition. A homogeneous geodesic through the origin $o \in M = G/H$ is a geodesic $\gamma(t)$ which is an orbit of a one-parameter subgroup of $G$, that is

$$\gamma(t) = \exp(tZ)(o) \quad , \quad t \in \mathbb{R} \tag{3.1}$$

where $Z$ is a nonzero vector of $\mathfrak{g}$. A nonzero vector $Z \in \mathfrak{g}$ for which $\gamma(t) = \exp(tZ)(o), t \in \mathbb{R}$ is a geodesic is called a geodesic vector. Geodesic vectors are characterized by the geodesic lemma proved in Riemannian geometry by O. Kowalski and L. Vanhecke in [10]. They proved that a vector $X \in \mathfrak{g} - \{0\}$ is a geodesic vector if and only if

$$\langle [X, Y]_{\mathfrak{m}}, X_{\mathfrak{m}} \rangle = 0 \quad \forall Y \in \mathfrak{m} \tag{3.2}$$

(See [4, 5, 7], for the pseudo-Riemannian case). The Finslerian version of geodesic lemma was proved in [12].

**Lemma 3.1.** *Let $(G/H, F)$ be a homogeneous Finsler space with a reductive decomposition $\mathfrak{g} = \mathfrak{h} + \mathfrak{m}$, then a vector $Y \in \mathfrak{g} \setminus \{0\}$ is a geodesic vector if and only if*

$$g_{Y_{\mathfrak{m}}}(Y_{\mathfrak{m}}, [Y, Z]_{\mathfrak{m}}) = 0, \ \forall Z \in \mathfrak{m},$$

*where the subscript $\mathfrak{m}$ indicates the projection of a vector from $\mathfrak{g}$ to $\mathfrak{m}$.*

Let $(M, \tilde{a})$ be a connected Riemannian manifold. A symmetry at $x \in M$ is a isometry of $(M, \tilde{a})$ for which $x$ is an isolated fixed point. An $s-$structure on $(M, \tilde{a})$ is a family $\{s_x\}_{x\in M}$ such that $s_x$ is a symmetry at $x \in M$, for each $x \in M$. An $s-$structure is called regular if for any two point $x, y \in M$

$$s_x \circ s_y = s_z \circ s_x, \ \ z = s_x(y).$$

A generalized symmetric space is a connected Riemannian manifold $(M, \tilde{a})$ admitting a regular $s-$structure. Every generalized symmetric space is a homogeneous Riemannian space $(G/H, \tilde{a})$. An $s-$structure $\{s_x\}_{x\in M}$ is called of order $k$ if $(s_x)^k = id_M$ for all $x \in M$ and $k$ is the minimal number with this property.

## 4. Homogeneous geodesics of generalized symmetric spaces of type 2

A five-dimensional generalized symmetric space $M$ of type 2 is $\mathbb{R}^5(x,y,z,w,t)$ equipped with the Riemannian metric

$$\begin{aligned}\tilde{a} =& e^{-2\lambda_1 t}dx^2 + e^{2\lambda_1 t}dy^2 + e^{-2\lambda_2 t}dz^2 + e^{2\lambda_2 t}dw^2 + dt^2 + 2\alpha[e^{-(\lambda_1+\lambda_2)t}dxdz + e^{(\lambda_1+\lambda_2)t}dydw] \\ &+ 2\beta[e^{(\lambda_1-\lambda_2)t}dydz - e^{(\lambda_2-\lambda_1)t}dxdw],\end{aligned}$$

where either $(2a)$ $\lambda_1 > \lambda_2 > 0$, $\alpha^2+\beta^2 < 1$, $(2b)$ $\lambda_1 = \lambda_2 > 0$, $\alpha = 0$, and $0 \le \beta < 1$, or $(2c)$ $\lambda_1 < 0$, $\lambda_2 = 0$, $\alpha = 0$ and $0 < \beta < 1$. As homogeneous space, $M = \frac{G}{H}$, Where $G$ is the group of all matrices of the form

$$\begin{pmatrix} e^{\lambda_1 t} & 0 & 0 & 0 & x \\ 0 & e^{-\lambda_1 t} & 0 & 0 & y \\ 0 & 0 & e^{\lambda_2 t} & 0 & z \\ 0 & 0 & 0 & e^{-\lambda_2 t} & w \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

The linear subspace $\mathfrak{m}$ of $\mathfrak{g}$ admits a basis $\{X_1, X_2, Y_1, Y_2, W\}$ such that $[X_j, W] = -\lambda_j X_j$, $[Y_j, W] = \lambda_j Y_j$. It is considered that the other multiplication are zero.

**2a**) $\lambda_1 > \lambda_2 > 0$ and $\alpha^2+\beta^2 < 1$

In this case $\mathfrak{h} = 0$, [9]. The Lie bracket $[\ ,\ ]$ and the Riemannian metric $\tilde{a}$ are respectively determined by

| $[\ ,\ ]$ | $X_1$ | $X_2$ | $Y_1$ | $Y_2$ | $W$ |
|---|---|---|---|---|---|
| $W$ | $\lambda_1 X_1$ | $\lambda_2 X_2$ | $-\lambda_1 Y_1$ | $-\lambda_2 Y_2$ | 0 |

| $\tilde{a}(,)$ | $X_1$ | $X_2$ | $Y_1$ | $Y_2$ | $W$ |
|---|---|---|---|---|---|
| $X_1$ | 1 | $\alpha$ | 0 | $-\beta$ | 0 |
| $X_2$ | $\alpha$ | 1 | $\beta$ | 0 | 0 |
| $Y_1$ | 0 | $\beta$ | 1 | $\alpha$ | 0 |
| $Y_2$ | $-\beta$ | 0 | $\alpha$ | 1 | 0 |
| $W$ | 0 | 0 | 0 | 0 | 1 |

we construct an orthonormal frame field $\{e_1, e_2, e_3, e_4, e_5\}$, by setting

$$e_1 = X_1,\quad e_2 = \frac{X_2 - \alpha X_1}{\sqrt{1-\alpha^2}},\quad e_3 = \frac{(1-\alpha^2)Y_1 - \beta X_2 + \alpha\beta X_1}{\sqrt{1-\alpha^2}\sqrt{1-\alpha^2-\beta^2}},\quad e_4 = \frac{Y_2 + \beta X_1 - \alpha Y_1}{\sqrt{1-\alpha^2-\beta^2}},\quad e_5 = W \tag{4.1}$$

We use the above table of Lie bracket and (4.1) to calculate the Lie brackets $[e_i, e_j]$ for all $i, j$.

$$\begin{aligned}
&[e_1, e_5] = -\lambda_1 e_1, \qquad [e_2, e_5] = -\lambda_2 e_2 - \frac{\alpha\lambda_2 e_1}{\sqrt{1-\alpha^2}} + \frac{\alpha\lambda_1 e_1}{\sqrt{1-\alpha^2}}, \\
&[e_3, e_5] = \lambda_1 e_3 + \frac{\beta e_2}{\sqrt{1-\alpha^2-\beta^2}}(\lambda_1+\lambda_2) + \frac{\alpha\beta e_1}{\sqrt{1-\alpha^2}\sqrt{1-\alpha^2-\beta^2}}(\lambda_2-\lambda_1), \\
&[e_4, e_5] = \lambda_2 e_4 - \frac{\beta e_1}{\sqrt{1-\alpha^2-\beta^2}}(\lambda_1+\lambda_2) - \frac{\alpha e_3}{\sqrt{1-\alpha^2}}(\lambda_1-\lambda_2) - \frac{\alpha\beta e_2}{\sqrt{1-\alpha^2}\sqrt{1-\alpha^2-\beta^2}}(\lambda_1-\lambda_2).
\end{aligned}$$

In the following we consider homogeneous geodesics of left invariant Randers metrics defined by the Riemannian metric $\tilde{a}$ and vector field $X = \sum_{i=1}^{5} x_i e_i$ on five dimensional generalized symmetric spaces of type $(2a)$. By using the formula $g_y(u,v) = \frac{1}{2}\frac{\partial^2}{\partial s \partial t}[F^2(x, y+su+tv)]|_{s=t=0}$ and some computations, for the Randers metric $F$ we have:

$$\begin{aligned} g_y(u,v) =& \tilde{a}(u,v) + \tilde{a}(X,u)\tilde{a}(X,v) - \frac{\tilde{a}(X,y)\tilde{a}(y,u)\tilde{a}(y,v)}{\tilde{a}(y,y)^{\frac{3}{2}}} \\ &+ \frac{1}{\sqrt{\tilde{a}(y,y)}}\Big\{\tilde{a}(X,u)\tilde{a}(y,v) + \tilde{a}(X,y)\tilde{a}(u,v) + \tilde{a}(X,v)\tilde{a}(y,u)\Big\}. \end{aligned} \tag{4.2}$$

So for all $z \in \mathfrak{g}$ we have

$$g_y(y,[y,z]) = \tilde{a}(X + \frac{y}{\sqrt{\tilde{a}(y,y)}}\ ,\ [y,z])F(y). \tag{4.3}$$

By using Lemma (3.1) and equation (4.3), a vector $Y \in \mathfrak{g}$ is a geodesic vector of $(M,F)$ if and only if

$$\tilde{a}(\sum_{i=1}^{5} x_i e_i + \frac{\sum_{i=1}^{5} y_i e_i}{\sqrt{\tilde{a}(y,y)}}\ ,\ [\sum_{i=1}^{5} y_i e_i, e_i]) = 0. \tag{4.4}$$

for each $i = 1,2,3,4,5$. So we get

$$\begin{cases} y_5\lambda_1(x_1+\frac{y_1}{\sqrt{\tilde{a}(y,y)}})=0,\\ y_5\Big(\lambda_2((x_2+\frac{y_2}{\sqrt{\tilde{a}(y,y)}})+\frac{\alpha}{\sqrt{1-\alpha^2}}(x_1+\frac{y_1}{\sqrt{\tilde{a}(y,y)}}))-\frac{\alpha\lambda_1}{\sqrt{1-\alpha^2}}(x_1+\frac{y_1}{\sqrt{\tilde{a}(y,y)}})\Big)=0,\\ y_5\Big(\lambda_1(x_3+\frac{y_3}{\sqrt{\tilde{a}(y,y)}})+\frac{\beta(\lambda_1+\lambda_2)}{\sqrt{1-\alpha^2-\beta^2}}(x_2+\frac{y_2}{\sqrt{\tilde{a}(y,y)}})+\frac{\alpha\beta(\lambda_2-\lambda_1)}{\sqrt{1-\alpha^2}\sqrt{1-\alpha^2-\beta^2}}(x_1+\frac{y_1}{\sqrt{\tilde{a}(y,y)}})\Big)=0,\\ y_5\Big(-\lambda_2(x_4+\frac{y_4}{\sqrt{\tilde{a}(y,y)}})+\frac{\beta(\lambda_1+\lambda_2)}{\sqrt{1-\alpha^2-\beta^2}}(x_1+\frac{y_1}{\sqrt{\tilde{a}(y,y)}})\\ +\frac{\alpha(\lambda_1-\lambda_2)}{\sqrt{1-\alpha^2}}(x_3+\frac{y_3}{\sqrt{\tilde{a}(y,y)}})+\frac{\alpha\beta(\lambda_1-\lambda_2)}{\sqrt{1-\alpha^2}\sqrt{1-\alpha^2-\beta^2}}(x_2+\frac{y_2}{\sqrt{\tilde{a}(y,y)}})\Big)=0,\\ y_1(-\lambda_1(x_1+\frac{y_1}{\sqrt{\tilde{a}(y,y)}}))+y_2(-\lambda_2(x_2+\frac{y_2}{\sqrt{\tilde{a}(y,y)}})-\frac{\lambda_2\alpha}{\sqrt{1-\alpha^2}}(x_1+\frac{y_1}{\sqrt{\tilde{a}(y,y)}})+\frac{\alpha\lambda_1}{\sqrt{1-\alpha^2}}(x_1+\frac{y_1}{\sqrt{\tilde{a}(y,y)}})\\ +y_3\Big(\lambda_1(x_3+\frac{y_3}{\sqrt{\tilde{a}(y,y)}})+\frac{\beta(\lambda_1+\lambda_2)}{\sqrt{1-\alpha^2-\beta^2}}(x_2+\frac{y_2}{\sqrt{\tilde{a}(y,y)}})+\frac{\alpha\beta(\lambda_2-\lambda_1)}{\sqrt{1-\alpha^2}\sqrt{1-\alpha^2-\beta^2}}(x_1+\frac{y_1}{\sqrt{\tilde{a}(y,y)}})\Big)\\ +y_4\Big(\lambda_2(x_4+\frac{y_4}{\sqrt{\tilde{a}(y,y)}})-\frac{\beta(\lambda_1+\lambda_2)}{\sqrt{1-\alpha^2-\beta^2}}(x_1+\frac{y_1}{\sqrt{\tilde{a}(y,y)}})-\frac{\alpha(\lambda_1-\lambda_2)}{\sqrt{1-\alpha^2}}(x_3+\frac{y_3}{\sqrt{\tilde{a}(y,y)}})-\frac{\alpha\beta(\lambda_1-\lambda_2)}{\sqrt{1-\alpha^2}\sqrt{1-\alpha^2-\beta^2}}(x_2+\frac{y_2}{\sqrt{\tilde{a}(y,y)}})\Big)=0. \end{cases} \tag{4.5}$$

If $y_5 \neq 0$, and $\tilde{a}(y,y) = 1$ then (4.5) gives $y_1 = -x_1, y_2 = -x_2, y_3 = -x_3, y_4 = -x_4$. Next assume that $y_5 = 0$ , $X = x_5e_5$ and $\tilde{a}(y,y) = 1$ then (4.5) reduces to the last equation, which gives

$$y_1(-\lambda_1 y_1)+y_2\Big(-\lambda_2(y_2+\frac{\alpha}{\sqrt{1-\alpha^2}}y_1)+\frac{\alpha\lambda_1}{\sqrt{1-\alpha^2}}y_1\Big)+y_3\Big(\lambda_1 y_3+\frac{\beta(\lambda_1+\lambda_2)}{\sqrt{1-\alpha^2-\beta^2}}y_2+\frac{\alpha\beta(\lambda_2-\lambda_1)}{\sqrt{1-\alpha^2}\sqrt{1-\alpha^2-\beta^2}}y_1\Big)$$
$$+y_4\Big(\lambda_2 y_4-\frac{\beta(\lambda_1+\lambda_2)}{\sqrt{1-\alpha^2-\beta^2}}y_1-\frac{\alpha(\lambda_1-\lambda_2)}{\sqrt{1-\alpha^2}}y_3-\frac{\alpha\beta(\lambda_1-\lambda_2)}{\sqrt{1-\alpha^2}\sqrt{1-\alpha^2-\beta^2}}y_2\Big)=0. \tag{4.6}$$

the solutions of (4.6) are given by $(y_1, y_2, y_3, y_4, 0)$, satisfying in (4.6).
Hence, we proved that $Y$ is a geodesic vector of a generalized symmetric space of type $(2a)$ equipped with a left invariant Randers metric defined by the Riemannian metric $\tilde{a}$ and vector field $X$. if and only if

$$Y = -\sum_{i=1}^{4} x_i e_i + y_5 e_5, or$$

$$Y = \sum_{i=1}^{4} y_i e_i \quad and \quad (4.6) \quad hold.$$

If $\alpha = \beta = 0$ and $y_5 \neq 0$, (4.5), gives $y_1 = -x_1, y_2 = -x_2, y_3 = -x_3, y_4 = -x_4$. So $Y = \sum_{i=1}^{5} y_i e_i$ is geodesic vector. For $y_5 = 0$ and $X = x_5e_5$, we get $\lambda_1(y_1^2 - y_3^2) + \lambda_2(y_2^2 - y_4^2) = 0$. Therefore we have $y_1 = \pm y_3$ and $y_2 = \pm y_4$. So we get $Y = y_3(e_1 + e_3) + y_4(e_2 + e_4)$ and $Y = y_3(e_3 - e_1) + y_4(e_4 - e_2)$. If $\alpha = 0, \beta \neq 0$ and $\frac{\beta}{\sqrt{1-\beta^2}} = 1$ or $\alpha \neq 0, \beta = 0$ and $\frac{\alpha}{\sqrt{1-\alpha^2}} = 1$, Then for $y_5 \neq 0$ we get $Y = \sum_{i=1}^{5} y_i e_i$ and for $y_5 = 0$ and $X = x_5e_5$, we get $Y = y_3(e_1 + e_3) + y_4(e_2 + e_4)$ and $Y = y_3(e_3 - e_1) + y_4(e_4 - e_2)$.

**Theorem 4.1.** *Let $(M, F)$ be a five-dimensional generalized symmetric spaces of type $(2a)$ equipped with a left invariant Randers metric defined by the Riemannian metric $\tilde{a}$ and vector field $X = x_5e_5$. Then $Y \in \mathfrak{g}$ is a geodesic vector of $(M, F)$ if and only if $Y$ is a geodesic vector of $(M, \tilde{a})$.*

*Proof.* Let $Y = \sum_{i=1}^{5} y_i e_i \in \mathfrak{g}$ is a geodesic vector of $(M, \tilde{a})$. By using (3.2) we have $\tilde{a}(Y, [Y, e_i]) = 0, \quad for\ i = 1,2,3,4,5$. Therefore by using equation (4.4) $Y \in \mathfrak{g}$ is a geodesic vector of $(M, F)$.

Conversely, let $Y = \sum_{i=1}^{5} y_i e_i \in \mathfrak{g}$ is a geodesic vector of $(M, F)$, because $\tilde{a}(X, [Y, e_i]) = 0$, for $i = 1, 2, 3, 4, 5$ by using (4.4) we have $\tilde{a}(Y, [Y, e_i]) = 0,\ for\ i = 1, 2, 3, 4, 5.$ □

**2b** ) $\lambda_1 = \lambda_2 > 0$ , $\alpha = 0$ and $\beta > 0$
In this case, $\mathfrak{h} = \mathfrak{so}(2) = Span(A)$, where $A$ is determined by $AX_1 = X_2$, $AX_2 = X_1$, $AY_1 = Y_2$, $AY_2 = Y_1$, $AW = 0$ (see **[?]**). The Lie bracket [ , ] and the Riemannian metric $\tilde{a}$ are respectively determined by

| [ , ] | $X_1$ | $X_2$ | $Y_1$ | $Y_2$ | $W$ | $A$ |
|---|---|---|---|---|---|---|
| $W$ | $\lambda_1 X_1$ | $\lambda_2 X_2$ | $-\lambda_1 Y_1$ | $-\lambda_2 Y_2$ | 0 | 0 |
| $A$ | $X_2$ | $-X_1$ | $Y_2$ | $-Y_1$ | 0 | 0 |

small

$$\tilde{a}(X_1, X_1) = \tilde{a}(X_2, X_2) = \tilde{a}(Y_1, Y_1) = \tilde{a}(Y_2, Y_2) = \tilde{a}(W, W) = 1, \quad \tilde{a}(X_1, Y_2) = -\beta, \quad \tilde{a}(X_2, Y_1) = \beta.$$

we construct an orthonormal frame field $\{e_1, e_2, e_3, e_4, e_5\}$, by setting

$$e_1 = X_1, \quad e_2 = X_2, \quad e_3 = \frac{Y_1 - \beta X_2}{\sqrt{1-\beta^2}}, \quad e_4 = \frac{Y_2 + \beta X_1}{\sqrt{1-\beta^2}}, \quad e_5 = W. \tag{4.7}$$

We use the above table of Lie bracket and (4.7) to calculate the Lie brackets $[e_i, e_j]$ for all $i, j$.

$$[e_1, e_5] = -\lambda_1 e_1, \quad [e_2, e_5] = -\lambda_2 e_2, \quad [e_3, e_5] = \lambda_1 e_3 + \frac{\beta e_2}{\sqrt{1-\beta^2}}(\lambda_1 + \lambda_2), \quad [e_4, e_5] = \lambda_2 e_4 - \frac{\beta e_1}{\sqrt{1-\beta^2}}(\lambda_1 + \lambda_2),$$

$$[e_1, A] = -e_2, \ [e_2, A] = e_1, \ [e_3, A] = -e_4 \ [e_4, A] = e_3.$$

In the following we consider homogeneous geodesics of invariant Randers metrics defined by the Riemannian metric $\tilde{a}$ and vector field $X$ on five dimensional generalized symmetric spaces of type $(2b)$. By using Lemma (3.1) and equation (4.3), a vector $Y \in \mathfrak{g}$ is a geodesic vector of $(M, F)$ if and only if

$$\tilde{a}\Big(\sum_{i=1}^{5} x_i e_i + \frac{\sum_{i=1}^{5} y_i e_i}{\sqrt{\tilde{a}(y,y)}}\ ,\ [\sum_{i=1}^{5} y_i e_i + aA, e_i]\Big) = 0. \tag{4.8}$$

for each $i = 1, 2, 3, 4, 5$. So we get

$$\begin{cases} y_5\lambda_1(x_1 + \frac{y_1}{\sqrt{\tilde{a}(y,y)}}) + a(x_2 + \frac{y_2}{\sqrt{\tilde{a}(y,y)}}) = 0, \\ y_5\lambda_1(x_2 + \frac{y_2}{\sqrt{\tilde{a}(y,y)}}) - a(x_1 + \frac{y_1}{\sqrt{\tilde{a}(y,y)}}) = 0, \\ y_5\Big(-\lambda_1(x_3 + \frac{y_3}{\sqrt{\tilde{a}(y,y)}}) - \frac{2\beta\lambda_1}{\sqrt{1-\beta^2}}(x_2 + \frac{y_2}{\sqrt{\tilde{a}(y,y)}})\Big) + a(x_4 + \frac{y_4}{\sqrt{\tilde{a}(y,y)}}) = 0, \\ y_5\Big(-\lambda_1(x_4 + \frac{y_4}{\sqrt{\tilde{a}(y,y)}}) + \frac{2\beta\lambda_1}{\sqrt{1-\beta^2}}(x_1 + \frac{y_1}{\sqrt{\tilde{a}(y,y)}})\Big) - a(x_3 + \frac{y_3}{\sqrt{\tilde{a}(y,y)}}) = 0, \\ -y_1(x_1 + \frac{y_1}{\sqrt{\tilde{a}(y,y)}}) - y_2(x_2 + \frac{y_2}{\sqrt{\tilde{a}(y,y)}}) + y_3\Big((x_3 + \frac{y_3}{\sqrt{\tilde{a}(y,y)}}) + \frac{2\beta}{\sqrt{1-\beta^2}}(x_2 + \frac{y_2}{\sqrt{\tilde{a}(y,y)}})\Big) \\ +y_4\Big((x_4 + \frac{y_4}{\sqrt{\tilde{a}(y,y)}}) - \frac{2\beta}{\sqrt{1-\beta^2}}(x_1 + \frac{y_1}{\sqrt{\tilde{a}(y,y)}})\Big) = 0. \end{cases} \tag{4.9}$$

If $y_5 = 0 \neq a$, Then we get $y_1 = -x_1$, $y_2 = -x_2$, $y_3 = -x_3$, $y_4 = -x_4$. So $Y = \sum_{i=1}^{4} y_i e_i + aA$ is geodesic vector. For $y_5 \neq 0 = a$, we get $y_1 = -x_1$, $y_2 = -x_2$, $y_3 = -x_3$, $y_4 = -x_4$, Therefore $Y = \sum_{i=1}^{5} y_i e_i$ is geodesic vector. If $y_5 \neq 0$, $a \neq 0$ and $\tilde{a}(y, y) = 1$, (4.9) gives $y_1 = -x_1$, $y_2 = -x_2$, $y_3 = -x_3$, $y_4 = -x_4$. For $y_5 = 0 = a$, $X = x_5 e_5$ and $\tilde{a}(y, y) = 1$, (4.9) reduces to the last equation, which gives

$$y_1(y_1 + \frac{2\beta}{\sqrt{1-\beta^2}}y_4) + y_2^2 = y_3(y_3 + \frac{2\beta}{\sqrt{1-\beta^2}}y_2) + y_4^2. \tag{4.10}$$

Hence, we proved that $Y$ is a geodesic vector of a generalized symmetric space of type $(2b)$ equipped with an invariant Randers metric defined by the Riemannian metric $\tilde{a}$ and vector field $X$ if and only if

$$Y_{\mathfrak{m}} = -\sum_{i=1}^{4} x_i e_i + y_5 e_5, or$$
$$Y_{\mathfrak{m}} = \sum_{i=1}^{4} y_i e_i \quad and \ (4.10) \ hold.$$

**Theorem 4.2.** *Let $(M, F)$ be a five-dimensional generalized symmetric spaces of type $(2b)$ equipped with an invariant Randers metric defined by the Riemannian metric $\tilde{a}$ and vector field $X = x_5 e_5$. Then $Y \in \mathfrak{g}$ is a geodesic vector of $(M, F)$ if and only if $Y$ is a geodesic vector of $(M, \tilde{a})$.*

*Proof.* The proof is the same as before. □

**2c)** $\lambda_1 > 0$ , $\lambda_2 = 0$ , $\alpha = 0$ and $0 < \beta < 1$
In this case, $\mathfrak{h} = \mathfrak{so}(2) \oplus \mathfrak{so}(2) = Span(A_1, A_2)$, where $A_1 = A$ of case (**2b**), while $A_2$ is determined by $A_2X_1 = X_2$, $A_2X_2 = -X_1$, $A_2Y_1 = -Y_2$, $A_2Y_2 = Y_1$, $A_2W = 0$. (see [9]). The Lie bracket [ , ] and the Riemannian metric $\tilde{a}$ are respectively determined by

| [ , ] | $X_1$ | $X_2$ | $Y_1$ | $Y_2$ | $W$ | $A_1$ | $A_2$ |
|---|---|---|---|---|---|---|---|
| $W$ | $\lambda_1 X_1$ | 0 | $-\lambda_1 Y_1$ | 0 | 0 | 0 | 0 |
| $A_1$ | $X_2$ | $-X_1$ | $Y_2$ | $-Y_1$ | 0 | − | − |
| $A_2$ | $X_2$ | $-X_1$ | $-Y_2$ | $Y_1$ | 0 | − | − |

$$\tilde{a}(X_1, X_1) = \tilde{a}(X_2, X_2) = \tilde{a}(Y_1, Y_1) = \tilde{a}(Y_2, Y_2) = \tilde{a}(W, W) = 1, \quad \tilde{a}(X_1, Y_2) = -\beta, \quad \tilde{a}(X_2, Y_1) = \beta.$$

we construct an orthonormal frame field $\{e_1, e_2, e_3, e_4, e_5\}$, by setting

$$e_1 = X_1, \quad e_2 = X_2, \quad e_3 = \frac{Y_1 - \beta X_2}{\sqrt{1-\beta^2}}, \quad e_4 = \frac{Y_2 + \beta X_1}{\sqrt{1-\beta^2}}, \quad e_5 = W. \tag{4.11}$$

We use the above table of Lie bracket and (4.11) to calculate the Lie brackets $[e_i, e_j]$ for all $i, j$.

$$[e_1, e_5] = -\lambda_1 e_1, \quad [e_2, e_5] = 0, \quad [e_3, e_5] = \lambda_1 e_3 + \frac{\beta \lambda_1 e_2}{\sqrt{1-\beta^2}}, \quad [e_4, e_5] = -\frac{\beta \lambda_1 e_1}{\sqrt{1-\beta^2}},$$
$$[e_1, A_1] = -e_2, \quad [e_2, A_1] = e_1, \quad [e_3, A_1] = -e_4, \quad [e_4, A_1] = e_3, \quad [e_1, A_2] = -e_2,$$
$$[e_2, A_2] = e_1, \quad [e_3, A_2] = e_4 - \frac{2\beta e_1}{\sqrt{1-\beta^2}}, \quad [e_4, A_2] = -e_3 - \frac{2\beta e_2}{\sqrt{1-\beta^2}}.$$

In the following we consider homogeneous geodesics of invariant Randers metrics defined by the Riemannian metric $\tilde{a}$ and vector field $X$ on five dimensional generalized symmetric spaces of type $(2c)$. By using Lemma (3.1) and equation (4.3), a vector $Y \in \mathfrak{g}$ is a geodesic vector of $(M, F)$ if and only if

$$\tilde{a}\Big(\sum_{i=1}^{5} x_i e_i + \frac{\sum_{i=1}^{5} y_i e_i}{\sqrt{\tilde{a}(y, y)}} \ , \ [\sum_{i=1}^{5} y_i e_i + aA_1 + bA_2, e_i]\Big) = 0. \tag{4.12}$$

for each $i = 1,2,3,4,5$. So we get

$$\begin{cases} y_5\lambda_1(x_1+\frac{y_1}{\sqrt{\tilde{a}(y,y)}})+(a+b)(x_2+\frac{y_2}{\sqrt{\tilde{a}(y,y)}})=0,\\ (a+b)(x_1+\frac{y_1}{\sqrt{\tilde{a}(y,y)}})=0,\\ y_5(-\lambda_1(x_3+\frac{y_3}{\sqrt{\tilde{a}(y,y)}})-\frac{\beta\lambda_1}{\sqrt{1-\beta^2}}(x_2+\frac{y_2}{\sqrt{\tilde{a}(y,y)}}))+(a-b)(x_4+\frac{y_4}{\sqrt{\tilde{a}(y,y)}})+\frac{2\beta b}{\sqrt{1-\beta^2}}(x_1+\frac{y_1}{\sqrt{\tilde{a}(y,y)}})=0,\\ \frac{\beta\lambda_1}{\sqrt{1-\beta^2}}(x_1+\frac{y_1}{\sqrt{\tilde{a}(y,y)}})y_5-(a-b)(x_3+\frac{y_3}{\sqrt{\tilde{a}(y,y)}})+\frac{2\beta b}{\sqrt{1-\beta^2}}(x_2+\frac{y_2}{\sqrt{\tilde{a}(y,y)}})=0,\\ -y_1(x_1+\frac{y_1}{\sqrt{\tilde{a}(y,y)}})+y_3(x_3+\frac{y_3}{\sqrt{\tilde{a}(y,y)}})+\frac{\beta}{\sqrt{1-\beta^2}}y_3(x_2+\frac{y_2}{\sqrt{\tilde{a}(y,y)}})-\frac{\beta}{\sqrt{1-\beta^2}}y_4(x_1+\frac{y_1}{\sqrt{\tilde{a}(y,y)}})=0. \end{cases} \tag{4.13}$$

We suppose that $X = x_3(e_3 - e_2)$, $\tilde{a}(y,y) = 1$, $\lambda_1 = 1$ and $\beta = \frac{\sqrt{2}}{2}$, Then the above system of equation take the form

$$y_5y_1 + (a+b)(y_2 - x_3) = 0, \tag{4.14}$$

$$(a+b)y_1 = 0, \tag{4.15}$$

$$-y_5(y_2+y_3) + (a-b)y_4 + 2by_1 = 0, \tag{4.16}$$

$$y_5y_1 - (a-b)(x_3+y_3) + 2b(y_2 - x_3) = 0, \tag{4.17}$$

$$-y_1(y_1+y_4) + y_3(y_3+y_2) = 0. \tag{4.18}$$

If $a \neq \pm b$, From (4.15), we have $y_1 = 0$. from (4.14) we get $y_2 = x_3$ and (4.17) gives $y_3 = -x_3$ and (4.16) gives $y_4 = 0$. So we get $Y = y_2(e_2 - e_3) + y_5e_5 + aA_1 + bA_2$.
If $a = b = 0$, from (4.14) we get $y_1 = 0$ or $y_5 = 0$. Also (4.16) gives $y_5 = 0$ or $y_2 = -y_3$. If $y_1 = 0$, we get $Y = y_2(e_2 - e_3) + y_4e_4 + y_5e_5$. If $y_1 \neq 0$, The above system of equation reduce the last equation $-y_1(y_1 + y_4) + y_3(y_3 + y_2) = 0$. Therefor $Y = \sum_{i=1}^{4} y_ie_1$ is geodesic vector and $-y_1(y_1 + y_4) + y_3(y_3 + y_2) = 0$ hold. For $a - b = 0$, $a + b \neq 0$, $b \neq 0$ and $y_5 \neq 0$, we get $y_1 = 0$ and $y_3 = -y_2$. we conclude $Y = y_2(e_2 - e_3) + y_4e_4 + y_5e_5 + a(A_1 + A_2)$. Also for $a + b = 0$, $b \neq 0$, we get $y_1 \neq 0$ and $y_1 = y_4$. So we conclude $Y = y_1(e_1 + e_4) + y_2(e_2 - e_3) + b(A_2 - A_1)$.

**Theorem 4.3.** *Let $(M, F)$ be a five-dimensional generalized symmetric spaces of type $(2c)$ equipped with an invariant Randers metric defined by the Riemannian metric $\tilde{a}$ and vector field $X = x_5e_5$. Then $Y \in \mathfrak{g}$ is a geodesic vector of $(M, F)$ if and only if $Y$ is a geodesic vector of $(M, \tilde{a})$.*

*Proof.* The proof is the same as before. □

From the study of cases $(2a)$, $(2b)$ and $(2c)$ and taking into account Proposition (2.2) in [2], we can conclude that

**Theorem 4.4.** *Let $(M, F)$ be a five-dimensional Randers manifold defined by an invariant Riemannian metric $\tilde{a}$ and invariant vector field $X = x_5e_5$ on generalized symmetric spaces of type $(2a, 2b, 2c)$. Then $(M, F)$ admit five mutually orthogonal homogeneous geodesics through the origin.*

## 5. Homogeneous geodesics of generalized symmetric spaces of type 3

A five-dimensional generalized symmetric space $M$ of type $(3)$ is the homogeneous spaces $M = SO(3,\mathbb{C})/SO(2)$. where $SO(3,\mathbb{C})$ is the special complex orthogonal group and the Riemannian metric of $M$ is induced by a real invariant positive semi-definite form of $GL(3,C)$ (see [9]). According to a result due to O. Kowalski in [9] $\mathfrak{h} = \mathfrak{so}(2) = Span(A)$, where $A$ is determined by $AX_1 = X_2, AX_2 = -X_1, AY_1 = Y_2, AY_2 = -Y_1, AW = 0$, with respect to a basis $\{X_1, X_2, Y_1, Y_2, W\}$ of $\mathfrak{m}$ . The Lie bracket [ , ] and the Riemannian metric $\tilde{a}$ are respectively given by

| [ , ] | $X_1$ | $X_2$ | $Y_1$ | $Y_2$ | $W$ | $A$ |
|---|---|---|---|---|---|---|
| $X_1$ | 0 | 0 | 0 | $-W$ | $-X_1$ | $-X_2$ |
| $X_2$ | 0 | 0 | $W$ | 0 | $-X_2$ | $X_1$ |
| $Y_1$ | 0 | $-W$ | 0 | 0 | $Y_1$ | $-Y_2$ |
| $Y_2$ | $W$ | 0 | 0 | 0 | $Y_2$ | $Y_1$ |
| $W$ | $X_1$ | $X_2$ | $-Y_1$ | $-Y_2$ | 0 | 0 |
| $A$ | $X_2$ | $-X_1$ | $Y_2$ | $Y_1$ | 0 | 0 |

$$\tilde{a}(X_1,X_1)=\tilde{a}(X_2,X_2)=\tilde{a}(Y_1,Y_1)=\tilde{a}(Y_2,Y_2)=a^2,\quad \tilde{a}(W,W)=b^2,\quad \tilde{a}(X_1,Y_2)=-\gamma,\quad \tilde{a}(X_2,Y_1)=\gamma.$$

where $a,b>0$, $\gamma$ are real numbers, $a^2 >|\gamma|$. We construct an orthonormal frame field $\{e_1,e_2,e_3,e_4,e_5\}$, by setting

$$e_1=\frac{X_1}{a},\quad e_2=\frac{X_2}{a},\quad e_3=\frac{a^2Y_1-\gamma X_2}{a\sqrt{a^4-\gamma^2}},\quad e_4=\frac{a^2Y_2+\gamma X_1}{a\sqrt{a^4-\gamma^2}},\quad e_5=\frac{W}{b}. \tag{5.1}$$

We suppose $b=1$ and use the above table of Lie bracket and (5.1) to calculate the Lie brackets $[e_i,e_j]$ for all $i,j$.

$$[e_1,e_4]=-\frac{e_5}{\sqrt{a^4-\gamma^2}},\quad [e_2,e_3]=\frac{e_5}{\sqrt{a^4-\gamma^2}},\quad [e_1,e_5]=-e_1,\quad [e_2,e_5]=-e_2,\quad [e_3,e_5]=e_3+\frac{2\gamma e_2}{\sqrt{a^4-\gamma^2}},$$
$$[e_4,e_5]=e_4-\frac{2\gamma e_1}{\sqrt{a^4-\gamma^2}},\quad [e_1,A]=-e_2,\ [e_2,A]=e_1,\quad [e_3,A]=-e_4,\quad [e_4,A]=e_3.$$

In the following we consider homogeneous geodesics of invariant Randers metrics defined by the Riemannian metric $\tilde{a}$ and vector field $X$ on five dimensional generalized symmetric spaces of type (3). By using Lemma (3.1) and equation (4.8), a vector $Y\in\mathfrak{g}$ is a geodesic vector of $(M,F)$ if and only if

$$\begin{cases} y_5(x_1+\frac{y_1}{\sqrt{\tilde{a}(y,y)}})+\frac{y_4}{\sqrt{a^4-\gamma^2}}(x_5+\frac{y_5}{\sqrt{\tilde{a}(y,y)}})+r(x_2+\frac{y_2}{\sqrt{\tilde{a}(y,y)}})=0,\\ -\frac{y_3}{\sqrt{a^4-\gamma^2}}(x_5+\frac{y_5}{\sqrt{\tilde{a}(y,y)}})+y_5(x_2+\frac{y_2}{\sqrt{\tilde{a}(y,y)}})-r(x_1+\frac{y_1}{\sqrt{\tilde{a}(y,y)}})=0,\\ \frac{y_2}{\sqrt{a^4-\gamma^2}}(x_5+\frac{y_5}{\sqrt{\tilde{a}(y,y)}})-y_5\Big((x_3+\frac{y_3}{\sqrt{\tilde{a}(y,y)}})+\frac{2\gamma}{\sqrt{a^4-\gamma^2}}(x_2+\frac{y_2}{\sqrt{\tilde{a}(y,y)}})\Big)+r(x_4+\frac{y_4}{\sqrt{\tilde{a}(y,y)}})=0,\\ -\frac{y_1}{\sqrt{a^4-\gamma^2}}(x_5+\frac{y_5}{\sqrt{\tilde{a}(y,y)}})-y_5\Big((x_4+\frac{y_4}{\sqrt{\tilde{a}(y,y)}})+\frac{2\gamma}{\sqrt{a^4-\gamma^2}}(x_1+\frac{y_1}{\sqrt{\tilde{a}(y,y)}})\Big)-r(x_3+\frac{y_3}{\sqrt{\tilde{a}(y,y)}})=0,\\ -y_1(x_1+\frac{y_1}{\sqrt{\tilde{a}(y,y)}})-y_2(x_2+\frac{y_2}{\sqrt{\tilde{a}(y,y)}})+y_3(x_3+\frac{y_3}{\sqrt{\tilde{a}(y,y)}})+y_4(x_4+\frac{y_4}{\sqrt{\tilde{a}(y,y)}})\\ +\frac{2\gamma}{\sqrt{a^4-\gamma^2}}\Big(y_3(x_2+\frac{y_2}{\sqrt{\tilde{a}(y,y)}})-y_4(x_1+\frac{y_1}{\sqrt{\tilde{a}(y,y)}})\Big)=0. \end{cases} \tag{5.2}$$

For $X=x_2(e_2-2e_3)$, $\tilde{a}(y,y)=1$ and $\frac{\gamma}{B}=1$ where $B=\sqrt{a^4-\gamma^2}$, we have:

$$\begin{cases} y_5(y_1+\frac{y_4}{A})+r(x_2+y_2)=0,\\ y_5(y_2+x_2-\frac{y_3}{A})-ry_1=0,\\ y_5(-2y_2-y_3+\frac{y_2}{A})+ry_4=0,\\ -y_5(y_4+2y_1+\frac{y_1}{A})-r(y_3-2x_2)=0,\\ -y_1^2-y_2^2+y_3^2+y_4^2+y_2(2y_3-x_2)-2y_4y_1=0. \end{cases} \tag{5.3}$$

If $r=0\neq y_5$, (5.3) gives $y_1=y_4=0$, $y_3=\frac{y_2(1-2B)}{B}$ and $y_2=-\frac{B^2x_2}{B^2+2B-1}$. So we get $Y=y_2e_2+y_3e_3+y_5e_5$. If $y_5=0\neq r$, Then we get $Y=y_2(e_2-2e_3)+rA$.
For $X=x_1(e_1+2e_4)$ and $r=0\neq y_5$, (5.2) gives $y_2=y_3=0$, $x_1+y_1=-\frac{y_4}{B}$ and $y_4=-\frac{Bx_1(1+2B)}{1-B^2+2B}$. Therefore we get $Y=y_1e_1+y_4e_4+y_5e_5$. If $y_5=0\neq r$, Then we get $Y=y_1(e_1+2e_4)+rA$.
If $X=e_3x_3$ and $r=0\neq y_5$, Then (5.2) gives $y_1=y_4=0$, $y_2=\frac{y_3}{B}$ and $y_3=-\frac{B^2x_3}{B^2+2B-1}$. So we conclude $Y=y_2e_2+y_3e_3+y_5e_5$. If $y_5=0\neq r$, Then we get $Y=y_3e_3+rA$. The case $r\neq 0$

and $y_5 \neq 0$ is much more complicated. After some standard but quite long calculations, we eventually find that, when $r \neq 0$ the solutions of (5.2) are given by

$$y_1 = -\frac{ry_5}{B(y_5^2+r^2)}y_3, \quad y_2 = \frac{y_5^2}{B(y_5^2+r^2)}y_3, \quad y_3 = -\frac{B^2(y_5^2+r^2)}{(1-2B)y_5^2 - B^2(y_5^2+r^2)}x_3,$$
$$\text{with} \quad r^2 = \frac{(1+2B)y_5^2}{B^2} - y_5^2 \neq 0.$$

So we have $Y = \sum_{i=1}^{5} y_i e_i + rA$.
For $X = e_4 x_4$ and $r = 0 \neq y_5$, Then (5.2) gives $y_3 = y_2 = 0$, $y_1 = -\frac{y_4}{B}$ and $y_4 = -\frac{B^2 x_4}{B^2-2B-1}$. So we have $Y = y_1e_1 + y_4e_4 + y_5e_5$. If $y_5 = 0 \neq r$, Then we get $Y = y_4e_4 + rA$. Therefore we have the following:

**Corollary 5.1.** *Let $(M, F)$ be a five-dimensional generalized symmetric spaces of type (3) equipped with an invariant Randers metric F defined by the Riemannian metric $\tilde{a}$ and the vector field X. Then geodesic vectors depending only on $x_1$, $x_2$, $x_3$, $x_4$, $x_5$ and $r, y_5$.*

## 6. Homogeneous geodesics of generalized symmetric spaces of type 7

As homogeneous spaces, five-dimensional generalized symmetric spaces $M$ of type 7 are real matrix groups

$$\begin{pmatrix} e^{\lambda t} & 0 & 0 & 0 & x \\ 0 & e^{-\lambda t} & 0 & 0 & y \\ te^{\lambda t} & 0 & e^{\lambda t} & 0 & u \\ 0 & -te^{-\lambda t} & 0 & e^{-\lambda t} & v \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

$M$ is also $\mathbb{R}^5(x, y, u, v, t)$, equipped with a Riemannian metric

$$\tilde{a} = dt^2 + e^{-2\lambda t}(tdx - du)^2 + e^{2\lambda t}(tdy + dv)^2 + a^2(e^{-2\lambda t}dx^2 + e^{2\lambda t}dy^2) + 2\gamma(dydu - dxdv),$$

where $\lambda, a, \gamma \in \mathbb{R}$, $\lambda \geq 0$, $a > 0$ and $\gamma^2 < a^2$.
**7a**) $\lambda \neq 0$, In this case $\mathfrak{h} = 0$, [9]. Following [9], there exists a basis $\{X_1, X_2, Y_1, Y_2, W\}$ of $\mathfrak{g}$ such

that

| [ , ] | $X_1$ | $X_2$ | $Y_1$ | $Y_2$ | $W$ |
|---|---|---|---|---|---|
| $W$ | $\lambda X_1 + X_2$ | $\lambda X_2$ | $-\lambda Y_1 - Y_2$ | $-\lambda Y_2$ | 0 |

It is considered that the other multiplication are zero.

$$\tilde{a}(X_1, X_1) = \tilde{a}(Y_1, Y_1) = a^2, \quad \tilde{a}(X_2, X_2) = \tilde{a}(Y_2, Y_2) = \tilde{a}(W, W) = 1, \quad \tilde{a}(X_1, Y_2) = -\gamma, \quad \tilde{a}(X_2, Y_1) = \gamma.$$

we construct an orthonormal frame field $\{e_1, e_2, e_3, e_4, e_5\}$, by setting

$$e_1 = \frac{X_1}{a}, \quad e_2 = X_2, \quad e_3 = \frac{Y_1 - \gamma X_2}{\sqrt{a^2-\gamma^2}}, \quad e_4 = \frac{a^2 Y_2 + \gamma X_1}{a\sqrt{a^2-\gamma^2}}, \quad e_5 = W. \tag{6.1}$$

We use the above table of Lie bracket and (6.1) to calculate the Lie brackets $[e_i, e_j]$ for all $i, j$.

$$[e_1, e_5] = -\lambda e_1 - \frac{e_2}{a}, \quad [e_2, e_5] = -\lambda e_2, \quad [e_3, e_5] = -\frac{\gamma e_1}{a\sqrt{a^2-\gamma^2}} + \frac{2\gamma\lambda e_2}{\sqrt{a^2-\gamma^2}} + \lambda e_3 + \frac{e_4}{a}, \tag{6.2}$$
$$[e_4, e_5] = -\frac{2\gamma\lambda e_1}{\sqrt{a^2-\gamma^2}} - \frac{\gamma e_2}{a\sqrt{a^2-\gamma^2}} + \lambda e_4.$$

In the following we consider homogeneous geodesics of left invariant Randers metrics defined by the Riemannian metric $\tilde{a}$ and vector field $X$ on five dimensional generalized symmetric spaces of type $(7a)$. By using Lemma (3.1) and equation (4.4), a vector $Y \in \mathfrak{g}$ is a geodesic vector of $(M, F)$ if and only if

$$\begin{cases} a\lambda y_5(x_1 + \frac{y_1}{\sqrt{\tilde{a}(y,y)}}) + y_5(x_2 + \frac{y_2}{\sqrt{\tilde{a}(y,y)}}) = 0, \\ \lambda y_5(x_2 + \frac{y_2}{\sqrt{\tilde{a}(y,y)}}) = 0, \\ y_5\Big(\lambda(x_3 + \frac{y_3}{\sqrt{\tilde{a}(y,y)}}) + \frac{2\gamma\lambda}{\sqrt{a^2-\gamma^2}}(x_2 + \frac{y_2}{\sqrt{\tilde{a}(y,y)}}) + \frac{1}{a}(x_4 + \frac{y_4}{\sqrt{\tilde{a}(y,y)}}) - \frac{\gamma}{a\sqrt{a^2-\gamma^2}}(x_1 + \frac{y_1}{\sqrt{\tilde{a}(y,y)}})\Big) = 0, \\ y_5\Big(\lambda(x_4 + \frac{y_4}{\sqrt{\tilde{a}(y,y)}}) - \frac{2\gamma\lambda}{\sqrt{a^2-\gamma^2}}(x_1 + \frac{y_1}{\sqrt{\tilde{a}(y,y)}}) - \frac{\gamma}{a\sqrt{a^2-\gamma^2}}(x_2 + \frac{y_2}{\sqrt{\tilde{a}(y,y)}})\Big) = 0, \\ -y_1\Big(\lambda(x_1 + \frac{y_1}{\sqrt{\tilde{a}(y,y)}}) + \frac{1}{a}(x_2 + \frac{y_2}{\sqrt{\tilde{a}(y,y)}})\Big) - \lambda y_2(x_2 + \frac{y_2}{\sqrt{\tilde{a}(y,y)}}) \\ +y_3\Big(\lambda(x_3 + \frac{y_3}{\sqrt{\tilde{a}(y,y)}}) + \frac{2\gamma\lambda}{\sqrt{a^2-\gamma^2}}(x_2 + \frac{y_2}{\sqrt{\tilde{a}(y,y)}}) + \frac{1}{a}(x_4 + \frac{y_4}{\sqrt{\tilde{a}(y,y)}}) - \frac{\gamma}{a\sqrt{a^2-\gamma^2}}(x_1 + \frac{y_1}{\sqrt{\tilde{a}(y,y)}})\Big) + \\ y_4\Big(\lambda(x_4 + \frac{y_4}{\sqrt{\tilde{a}(y,y)}}) - \frac{2\gamma\lambda}{\sqrt{a^2-\gamma^2}}(x_1 + \frac{y_1}{\sqrt{\tilde{a}(y,y)}}) - \frac{\gamma}{a\sqrt{a^2-\gamma^2}}(x_2 + \frac{y_2}{\sqrt{\tilde{a}(y,y)}})\Big) = 0. \end{cases} \tag{6.3}$$

For $\tilde{a}(y, y) = 1$ and $y_5 \neq 0$, we have $y_1 = -x_1$ $y_2 = -x_2$ $y_3 = -x_3$ $y_4 = -x_4$. If $y_5 = 0$ and $X = x_5 e_5$ then we have

$$y_1(\lambda y_1 + \frac{1}{a}y_2) + \lambda y_2^2 - y_3\Big(\lambda y_3 + \frac{2\gamma\lambda}{\sqrt{a^2-\gamma^2}}y_2 + \frac{1}{a}y_4 - \frac{\gamma}{a\sqrt{a^2-\gamma^2}}y_1\Big) \tag{6.4}$$
$$- y_4\Big(\lambda y_4 - \frac{2\gamma\lambda}{\sqrt{a^2-\gamma^2}}y_1 - \frac{\gamma}{a\sqrt{a^2-\gamma^2}}y_2\Big) = 0.$$

Hence, we proved that $Y$ is a geodesic vector of a generalized symmetric space of type $(7a)$ equipped with a left invariant Randers metric defined by the Riemannian metric $\tilde{a}$ and vector field $X$ if and only if

$$Y = -\sum_{i=1}^{4} x_i e_i + y_5 e_5, or$$
$$Y = \sum_{i=1}^{4} y_i e_i \quad and \quad (6.4) \quad hold.$$

**Theorem 6.1.** *Let $(M, F)$ be a five-dimensional generalized symmetric spaces of type $(7a)$ equipped with a left invariant Randers metric defined by the Riemannian metric $\tilde{a}$ and vector field $X = x_5 e_5$. Then $Y \in \mathfrak{g}$ is a geodesic vector of $(M, F)$ if and only if $Y$ is a geodesic vector of $(M, \tilde{a})$.*

*Proof.* The proof is the same as before. □

**7b)**: $\lambda = 0$, In this case $\mathfrak{h} = \mathfrak{so}(2) = Span(A)$, where $A$ is determined by $AX_1 = -Y_1$, $AX_2 = Y_2$, $AY_1 = X_1$, $AY_2 = -X_2$, $AW = 0$, also with respect to the orthogonal basis of $\mathfrak{g}$, the lie bracket and Riemannian metric $\tilde{a}$ are the following

| [ , ] | $X_1$ | $X_2$ | $Y_1$ | $Y_2$ | $W$ | $A$ |
|---|---|---|---|---|---|---|
| $W$ | $X_2$ | 0 | $-Y_2$ | 0 | 0 | 0 |
| $A$ | $-Y_1$ | $Y_2$ | $X_1$ | $-X_2$ | 0 | 0 |

$$\tilde{a}(X_1, X_1) = \tilde{a}(Y_1, Y_1) = a^2,\ a > 0 \qquad \tilde{a}(X_2, X_2) = \tilde{a}(Y_2, Y_2) = \tilde{a}(W, W) = 1.$$

we construct an orthonormal frame field $\{e_1, e_2, e_3, e_4, e_5\}$, by setting

$$e_1 = \frac{X_1}{a}, \quad e_2 = X_2, \quad e_3 = \frac{Y_1}{a}, \quad e_4 = Y_2, \quad e_5 = W. \tag{6.5}$$

We use the above table of Lie bracket and (7.6) to calculate the Lie brackets $[e_i, e_j]$ for all $i, j$.

$$[e_1, e_5] = -\frac{e_2}{a},\ [e_3, e_5] = \frac{e_4}{a},\ [e_1, A] = e_3,\ [e_2, A] = -e_4,\ [e_3, A] = -e_1,\ [e_4, A] = e_2. \tag{6.6}$$

In the following we consider homogeneous geodesics of invariant Randers metrics defined by the Riemannian metric $\tilde{a}$ and vector field $X$ on five dimensional generalized symmetric spaces of type $(7b)$. By using Lemma (3.1) and equation (4.8), a vector $Y \in \mathfrak{g}$ is a geodesic vector of $(M, F)$ if and only if

$$\begin{cases} y_5(x_2 + \frac{y_2}{\sqrt{\tilde{a}(y,y)}}) - aa^{'}(x_3 + \frac{y_3}{\sqrt{\tilde{a}(y,y)}}) = 0, \\ a^{'}(x_4 + \frac{y_4}{\sqrt{\tilde{a}(y,y)}}) = 0, \\ y_5(x_4 + \frac{y_4}{\sqrt{\tilde{a}(y,y)}}) - aa^{'}(x_1 + \frac{y_1}{\sqrt{\tilde{a}(y,y)}}) = 0, \\ a^{'}(x_2 + \frac{y_2}{\sqrt{\tilde{a}(y,y)}}) = 0, \\ y_1(x_2 + \frac{y_2}{\sqrt{\tilde{a}(y,y)}}) - y_3(x_4 + \frac{y_4}{\sqrt{\tilde{a}(y,y)}}) = 0. \end{cases} \tag{6.7}$$

If $a^{'} \neq 0$ and $\tilde{a}(y, y) = 1$, then $y_1 = -x_1$, $y_2 = -x_2$, $y_3 = -x_3$, $y_4 = -x_4$. If $a^{'} = 0 \neq y_5$, we have $y_2 = -x_2$, $y_4 = -x_4$. For $a^{'} = 0 = y_5$ and $X = e_5x_5$ then (6.7) reduce to $y_1y_2 - y_3y_4 = 0$. Hence, we proved that $Y$ is a geodesic vector of a generalized symmetric space of type $(7b)$ equipped with an invariant Randers metric defined by the Riemannian metric $\tilde{a}$ and vector field $X$ if and only if

$$Y_{\mathfrak{m}} = -\sum_{i=1}^{4} x_ie_i + y_5e_5, or$$

$$Y_{\mathfrak{m}} = \sum_{i=1}^{4} y_ie_i \quad and \quad y_1y_2 - y_3y_4 = 0\ holds.$$

**Theorem 6.2.** *Let $(M, F)$ be a five-dimensional generalized symmetric spaces of type $(7b)$ equipped with an invariant Randers metric defined by the Riemannian metric $\tilde{a}$ and vector field $X = x_5e_5$. Then $Y \in \mathfrak{g}$ is a geodesic vector of $(M, F)$ if and only if $Y$ is a geodesic vector of $(M, \tilde{a})$.*

*Proof.* The proof is the same as before. □

## REFERENCES


[1] D. Bao, S. S. Chern and Z. Shen, *An Introduction to Riemann-Finsler Geometry*. Springer-Verlag, New York, 2000.

[2] G. Calvaruso and R. A. Marinosci, *Homogeneous geodesics in Five dimensional generalized symmetric spaces*. Balkan Journal of and Its Applications: Vol.8, No.1, (2003): 1-19.

[3] R. Chavosh Khatamy, A. Sadighi, *Homogeneous Geodesics on the Solvable Homogeneous Principal Bundles*. STM Journals, (2016): 26-37.

[4] Z. Du**V**sek, O. Kowalski, *Light-like homogeneous geodesics and the geodesic lemma for any signature*. Publ. Math. Debrecen, 71/1-2 (2007): 245-252.

[5] Z. Dus**V**sek, *The existence of homogeneous geodesics in homogeneous pseudo-Riemannian and affine manifolds*. J. Geom. Phys, 60 (2010): 687-689.

[6] M. Ebrahimi, D. Latifi,*On Flag Curvature and Homogeneous Geodesics of Left Invariant Randers Metrics on the Semi-Direct Product $a \oplus_p r$* . Journal of Lie Theory. 29 (2019): 619-627. ——————————————————————————————

[7] J. Figueroa-O'Farrill, P. Meessen, S. Philip, *Homogeneity and plane-wave limits*. J. High Energy Physics 05 (2005): 0-50.

[8] B. Kostant, *Holonomy and Lie algebra of motions in Riemannian manifolds*, Trans. Amer.Math. Soc.,80 (1955): 520-542.

[9] O. Kowalski, *Generalized symmetric spaces Lectures Notes in Math*. Springer- Verlag, Berlin, Heidelberg, New York, 805 (1980).

[10] O. Kowalski and J. Szenthe, *On the Existence of Homogeneous Geodesics in Homogeneous Riemannian manifolds*. Geom. Dedicata, 81 (2000): 209-214.
[11] O. Kowalski and L. Vanhecke, *Riemannian manifolds with homogeneous geodesics*, Boll. Un. Mat. Ital. **5** (1991) 189-246.
[12] D. Latifi, *Homogeneous geodesics in homogeneous Finsler spaces*. J. Geom. Phys. 57 (2007): 1421-1433.
[13] G. Randers, *On an asymmetrical metric in the four-space of general relativity*. Phys. Rev, 59 (1941): 195-199.
[14] E. B. Vinberg, *Invariant linear connection in a homogeneous manifold*, Trudy MMO 9 (1960): 191-210.
[15] Z.Yan, S. Deng, *Existence of homogeneous geodesics on homogeneous Randers spaces*. Houston J. Math, 44 (2018): 481–493.
[16] Z. Yan, L. Huang, *On the existence of homogeneous geodesic in homogeneous Finsler spaces*. J. Geom. Phys, 124 (2018): 264–267

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